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A NOTE ON A SYMMETRY RESULT FOR TRAVELING WAVES IN CYLINDERS

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Abstract. We prove in this note that all bounded traveling waves, in cylinders, of some N-dimensional viscous conservation laws are symmetric.

I. The main result

In this note, we consider traveling wave solutions of the equation

$$\frac{\partial U}{\partial t} = \triangle U - \sum_{i=1}^{N} \frac{\partial}{\partial_{x_i}}(f_i(U)) \tag{1.1}$$

for $(x_1, x') \in \mathbb{R} \times T^{N-1}$ and $f_i \in C^2(\mathbb{R}, \mathbb{R})$. That is, we consider solutions of the form

$$U(t, x) = u(x_1 - ct, x'),$$

where u satisfies

$$\triangle u - (f_1(u) - cu)_{x_1} - \sum_{i=2}^{N} (f_i(u))_{x_i} = 0 \tag{1.2}$$

for $(x_1, x') \in \mathbb{R} \times T^{N-1}$.

The main result of this note is the following.

Theorem 1. *Assume that u is an L^∞ solution of* (1.2).

i) *Then there are $a_\pm$ such that $\forall x' \in T^{N-1}, \lim_{x_1 \to \pm\infty} u, (x_1, x')$ is defined and equal to $a_\pm$. In addition, the limit is uniform in $x' \in T^{N-1}$.*

ii) *Assume in addition that*

$$f_1'(a_\pm) - c \neq 0.$$

Then $u(x_1, x') = v(x_1)$, where

$$v_{v_1 x_1} - (f_1(v) - cv)_{x_1} = 0 \qquad \text{for } x_1 \in \mathbb{R}.$$

Remark. If $f_i \equiv 0$ for $i \geq 2$, or in some sense the degeneracy of f_i at $a_\pm$ is of higher order than the one of f_1, the degeneracy condition in part (ii) of the theorem can be relaxed.

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Under the assumptions of Theorem 1, we have the following.

Theorem 2 (Liouville Theorem for (1.1))**.** *Let $u(x)$ be a solution of equation* (1.2)*. Consider $U(t,x)$ to be a solution of equation* (1.1) *defined for all time $t \in \mathbb{R}$ such that for a constant $C_0 > 0$,*

$$\forall t \in \mathbb{R}, \quad \| U(t,x) - u(x_1 - ct) \|_{L^1} \leq C_0.$$

Then there is an $x_0 \in \mathbb{R}$ such that

$$\forall t \in \mathbb{R},\ \forall x \in \mathbb{R} \times T^{N-1}, \quad U(t,x) = u(x_1 - ct + x_0).$$

From this result, we can then derive the following asymptotic stability result for equation (1.1).

Theorem 3 (Asymptotic Stability for Traveling Waves)**.** *Let $U(t,x)$ be a solution of equation* (1.1) *for $t > 0$, $x \in \mathbb{R} \times T^{N-1}$ and initial data*

$$U(0,x) = U_0(x).$$

Assume in addition that for $A > 0$ and for a travelling wave u,

$$\| U_0(x_1, x') - u(x_1) \|_{L^1(\mathbb{R} \times T^{N-1})} \leq A.$$

Then there exists a function $g_A(t)$ depending only on A, u with $g_A(t) \to 0$ as $t \to +\infty$, such that, for all such u, we have

$$\forall t > 1, Inf_{x_{10} \in \mathbb{R}} \| U(t,.) - u(. + x_{10} - ct) \|_{L^\infty(\mathbb{R} \times T^{N-1})} \leq g_A(t).$$

The proofs of Theorems 2, 3 are completely similar to the corresponding ones of [1] as soon as Theorem 1 is proved. We therefore devote the rest of this paper to the proof of Theorem 1.

II. Proof of Theorem 1

Let u be an L^∞ solution of (1.2) and let $C = \mathbb{R} \times T^{n-1}$. We first prove the following.

Lemma 1. *We have*

$$\lim_{x_1 \to +\infty} u(x_1, x') = a_+,$$
$$\lim_{x_1 \to -\infty} u(x_1, x') = a_-,$$

uniformly in $x' \in T^{N-1}$, where (a_+, a_-) are $(Sup_C u, Inf_C u)$ or $(Inf_C u, Sup_C u)$.

Proof. We first remark from standard elliptic theory that

$$u \in C^3(\mathbb{R} \times T^{N-1}, \mathbb{R}) \ \text{ and } \ |u|_{C^3} \leq M.$$

Moreover, if u achieves a local maximum or a local minimum, from the strong maximum principle we have

$$u(x_1, x') \equiv \text{ constant } \equiv a,$$

and all the conclusions of Theorem 1 hold. We now assume that $u(x_1, x')$ is different from a constant solution.

From the fact that there are no local extrema, Inf_C and Sup_C are not achieved and there are $(y_{1n}, y'_n) \in \mathbb{R} \times T^{N-1}$ (respectively $(z_{1n}, z'_n) \in \mathbb{R} \times T^{N-1}$) such that

$$u(y_{1n}, y'_n) \underset{n\to+\infty}{\longrightarrow} Sup_C u = a_+, \tag{2.3}$$

$$u(z_{1n}, z'_n) \underset{n\to+\infty}{\longrightarrow} Inf_C u = a_-, \tag{2.4}$$

with $|y_{1n}| \to +\infty$ and $|z_{1n}| \to +\infty$.

One can assume, eventually extracting a subsequence, that

$$y_{1n} \to +\infty \quad \text{and} \quad y_{1n} < y_{1n+1}. \tag{2.5}$$

(The proof in the other case is identical.)

Let us now prove

$$Inf_{x'\in T^{N-1}} u(y_{1n}, x') \underset{n\to+\infty}{\longrightarrow} a_+. \tag{2.6}$$

Indeed, let us consider

$$u_n(x_1, x') = u(y_{1n} + x_1, x').$$

We have

- $|u_n|_{C^3} \le C$,
- u_n is solution of equation (1.2),
- $u_n(0, y'_n) \underset{n\to+\infty}{\longrightarrow} a_+ = Sup\ u = Sup\ u_n$.

Extracting a subsequence, we have for $W : (x_1, x') \mapsto W(x_1, x')$ and $y' \in T^{N-1}$,

$$u_n(x_1, x') \underset{C^2_{loc}}{\longrightarrow} W(x_1, x') \quad \text{and} \quad y'_n \to y'$$

where

$$W \quad \text{is solution of (1.2)},\ SupW \le a_+\ ,\ W(0, y') = a_+.$$

Therefore,

$$W \equiv a_+$$

and since the result is true for all subsequences

$$u(y_{1n} + x_1, x') \underset{C^2_{loc}}{\longrightarrow} a_+,$$

then (1.7) follows.

From the fact that u does not have local minimum, we have

$$a_+ \ge \underset{\substack{x_1\in(y_{1n}, y_{1(n+1)})\\ x'\in T^{N-1}}}{Inf} u(x_1, x) \ge \underset{\substack{x_1=y_{1n}, y_{1(n+1)}\\ x'\in T^{N-1}}}{Inf} u(x_1, x) \underset{n\to+\infty}{\longrightarrow} a_+.$$

Since $y_{1n} < y_{1(n+1)}$ and $y_{1n} \longrightarrow_{n\to+\infty} +\infty$, then

$$\lim_{x_1\to+\infty} \{Inf_{x'\in T^{N-1}} u(x_1, x')\} \text{ exists and equals } a_+.$$

It follows that $z_{1n} \to -\infty$ and by the same procedure,

$$\lim_{x_1\to-\infty} \{Sup_{x'\in T^{N-1}} u(x_1, x')\} \text{ exists and is equal to } a_-.$$

This concludes the proofs of Lemma 1 and Theorem 1 part (i).

We now assume in addition a nondegeracy condition at a_+, a_- for $f_1(u)$, namely

$$f'_1(a_+) - c \ne 0 \quad \text{and} \quad f'_1(a_-) - c \ne 0,$$

and, for example, $a_- < a_+$. □

We first have the following estimates.

Lemma 2. *There exist* $\alpha > 0$ *and* $C_0 > 0$ *such that*

$$|u(x_1, x') - a_+| \le C_0 e^{-\alpha x_1}, \tag{2.7}$$

$$|u(x_1, x') - a_-| \le C_0 e^{+\alpha x_1}. \tag{2.8}$$

Proof. Let us prove the first one for example. The only question is when $x_1 \to +\infty$.

(i) Let us first introduce

$$w(x_1) = \frac{1}{|\mathrm{vol}(T^{N-1})|} \int_{T^{N-1}} (a_+ - u(x_1, x'))\, dx'. \tag{2.9}$$

We have

$$w \ge 0.$$

Averaging the equation (1.2) over T^{N-1} and using the periodic boundary conditions, we obtain that w satisfies the following equation $\forall x_1 \in \mathbb{R}$:

$$w_{x_1 x_1} - \frac{\partial}{\partial x_1} \left[\frac{1}{\mathrm{vol}(T^{N-1})} \int_{T^{N-1}} (f_1(u(x_1, x')) - cu(x_1, x'))\, dx' \right] = 0$$

or equivalently

$$w_{x_1} - \left[\frac{1}{\mathrm{vol}(T^{N-1})} \int_{T^{N-1}} (f_1(u(x_1, x')) - cu(x_1, x')) dx' \right] \equiv C_0. \tag{2.10}$$

Define $\beta = f_1'(a_+) - c$ and $\gamma = f_1(a_+) - ca_+$. We then have by linearization of the nonlinear term at a_+ in (2.10):

$$\forall x_1 \in \mathbb{R}, \quad |w_{x_1} - \beta w - C_0 + \gamma| \le C Sup_{x' \in T^{N-1}} |u(x_1, x') - a_+|^2.$$

Since $w(x_1, x') \to 0$, as $x_1 \to +\infty$, we have $C_0 - \gamma = 0$, and

$$\forall x_1 \in \mathbb{R}, |w_{x_1} - \beta w| \le C(Sup_{x' \in T^{N-1}} |u(x_1, x') - a_+|^2). \tag{2.11}$$

(ii) Relation between u and w: We now apply the Harnack principle as $x_1 \to +\infty$ to u, and we get: there is a $C > 0$ such that for $x_1 \ge 0$,

$$\underset{\substack{x' \in T^{N-1} \\ \tilde{x}_1 \in (x_1 - 1, x_1 + 1)}}{Sup} (a_+ - u(\tilde{x}_1, x')) \le C \underset{\substack{x' \in T^{N-1} \\ \tilde{x}_1 \in (x_1 - 1, x_1 + 1)}}{Inf} (a_+ - u(\tilde{x}_1, x')),$$

and, in particular, $\forall x_1 \ge 0$,

$$Sup_{x' \in T^{N-1}} (a_+ - u(x_1, x')) \le C w(x_1). \tag{2.12}$$

In particular, it is enough to prove the exponential decay for $w(x_1)$ to reach the conclusion of the lemma.

(iii) Exponential decay of w: We have from (2.11) and (2.12),

$$\forall x_1 \ge 0, \quad |w_{x_1} - \beta w| \le C w^2.$$

Since $w \to 0$ as $x_1 \to +\infty$ and $w > 0$ we have $\beta < 0$ and for x_1 large,

$$-\frac{3}{2} \beta w \le w_{x_1} \le -\frac{\beta}{2} w,$$

which concludes the proof of Lemma 2, by integration in x_1.

We are now able to conclude the proof of Theorem 1 part (ii). We argue by contradiction: Assume there is $x_1^0 \in \mathbb{R}$ such that for some $x_0', x_1' \in T^{N-1}$,

$$u(x_1^0, x_0') \ne u(x_1^0, x_1').$$

Then using the periodicity, there are $x'_2, x'_3 \in T^{N-1}$ such that

$$u(x_1^0, x'_2) = u(x_1^0, x'_3)$$

and

$$\nabla_{x'} u(x_1^0, x_2^1) \neq \nabla_{x'} u(x_1^0, x'_3).$$

Then

$$H(t,x) \;=\; u(x_1 - ct, x'_2 + x') \quad - \quad u(x_1 - ct, x'_3 + x')$$

is the difference of two solutions of equation (1.1) such that

i) $\forall t \in \mathbb{R}, \quad \| H(t) \|_{L^1(C)} \equiv$ constant,

ii) $H(0,(0,0)) = 0$ and $\nabla_{x'} H(0,(0,0)) \neq 0$.

Applying Lemma 2.9 of [1], we obtain

$$\| H(1) \|_{L^1(C)} < \| H(0) \|_{L^1(C)},$$

which is a contradiction.

(Of course, multiplying $u(x_1, x_2 + x') - u(x_1, x_3 + x')$ by sign $[u(x_1, x_2 + x') - u(x_1, x_3 + x')]$ yields a contradiction, by the same calculation as in the proof of Lemma 2.9 of [1].) This concludes the proof of Theorem 1. □

References

[1] C. Kenig and F. Merle. Asymptotic stability and Liouville theorem for scalar viscous conservation laws in cylinder, to appear in Comm. Pure Appl. Math.

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