PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 3, Pages 697–701
S 0002-9939(05)08412-1
Article electronically published on October 17, 2005

A NOTE ON A SYMMETRY RESULT FOR TRAVELING WAVES IN CYLINDERS

C. E. KENIG AND F. MERLE

(Communicated by David S. Tartakoff)

Abstract. We prove in this note that all bounded traveling waves, in cylinders, of some N-dimensional viscous conservation laws are symmetric.

I. The main result

In this note, we consider traveling wave solutions of the equation

$$\frac{\partial U}{\partial t} = \triangle U - \sum_{i=1}^{N} \frac{\partial}{\partial_{x_i}}(f_i(U)) \tag{1.1}$$

for $(x_1, x') \in \mathbb{R} \times T^{N-1}$ and $f_i \in C^2(\mathbb{R}, \mathbb{R})$. That is, we consider solutions of the form

$$U(t, x) = u(x_1 - ct, x'),$$

where u satisfies

$$\triangle u - (f_1(u) - cu)_{x_1} - \sum_{i=2}^{N} (f_i(u))_{x_i} = 0 \tag{1.2}$$

for $(x_1, x') \in \mathbb{R} \times T^{N-1}$.

The main result of this note is the following.

Theorem 1. *Assume that u is an L^∞ solution of* (1.2).

i) *Then there are $a_\pm$ such that $\forall x' \in T^{N-1}, \lim_{x_1 \to \pm\infty} u, (x_1, x')$ is defined and equal to $a_\pm$. In addition, the limit is uniform in $x' \in T^{N-1}$.*

ii) *Assume in addition that*

$$f_1'(a_\pm) - c \neq 0.$$

Then $u(x_1, x') = v(x_1)$, where

$$v_{v_1 x_1} - (f_1(v) - cv)_{x_1} = 0 \qquad \text{for } x_1 \in \mathbb{R}.$$

Remark. If $f_i \equiv 0$ for $i \geq 2$, or in some sense the degeneracy of f_i at $a_\pm$ is of higher order than the one of f_1, the degeneracy condition in part (ii) of the theorem can be relaxed.

Received by the editors April 24, 2004.
2000 *Mathematics Subject Classification.* Primary 35B40, 35B99, 35L65.
The first author was partially supported by the NSF, and at IAS by the von Neumann Fund, the Weyl Fund, the Oswald Veblen Fund and the Bell Companies Fellowship.

Under the assumptions of Theorem 1, we have the following.

Theorem 2 (Liouville Theorem for (1.1))**.** *Let $u(x)$ be a solution of equation* (1.2)*. Consider $U(t,x)$ to be a solution of equation* (1.1) *defined for all time $t \in \mathbb{R}$ such that for a constant $C_0 > 0$,*

$$\forall t \in \mathbb{R}, \quad \| U(t,x) - u(x_1 - ct) \|_{L^1} \leq C_0.$$

Then there is an $x_0 \in \mathbb{R}$ such that

$$\forall t \in \mathbb{R},\ \forall x \in \mathbb{R} \times T^{N-1}, \quad U(t,x) = u(x_1 - ct + x_0).$$

From this result, we can then derive the following asymptotic stability result for equation (1.1).

Theorem 3 (Asymptotic Stability for Traveling Waves)**.** *Let $U(t,x)$ be a solution of equation* (1.1) *for $t > 0$, $x \in \mathbb{R} \times T^{N-1}$ and initial data*

$$U(0,x) = U_0(x).$$

Assume in addition that for $A > 0$ and for a travelling wave u,

$$\| U_0(x_1, x') - u(x_1) \|_{L^1(\mathbb{R} \times T^{N-1})} \leq A.$$

Then there exists a function $g_A(t)$ depending only on A, u with $g_A(t) \to 0$ as $t \to +\infty$, such that, for all such u, we have

$$\forall t > 1, Inf_{x_{10} \in \mathbb{R}} \| U(t,.) - u(. + x_{10} - ct) \|_{L^\infty(\mathbb{R} \times T^{N-1})} \leq g_A(t).$$

The proofs of Theorems 2, 3 are completely similar to the corresponding ones of [1] as soon as Theorem 1 is proved. We therefore devote the rest of this paper to the proof of Theorem 1.

II. Proof of Theorem 1

Let u be an L^∞ solution of (1.2) and let $C = \mathbb{R} \times T^{n-1}$. We first prove the following.

Lemma 1. *We have*

$$\lim_{x_1 \to +\infty} u(x_1, x') = a_+,$$
$$\lim_{x_1 \to -\infty} u(x_1, x') = a_-,$$

uniformly in $x' \in T^{N-1}$, where (a_+, a_-) are $(Sup_C u, Inf_C u)$ or $(Inf_C u, Sup_C u)$.

Proof. We first remark from standard elliptic theory that

$$u \in C^3(\mathbb{R} \times T^{N-1}, \mathbb{R}) \ \text{ and } \ |u|_{C^3} \leq M.$$

Moreover, if u achieves a local maximum or a local minimum, from the strong maximum principle we have

$$u(x_1, x') \equiv \text{ constant } \equiv a,$$

and all the conclusions of Theorem 1 hold. We now assume that $u(x_1, x')$ is different from a constant solution.

From the fact that there are no local extrema, Inf_C and Sup_C are not achieved and there are $(y_{1n}, y'_n) \in \mathbb{R} \times T^{N-1}$ (respectively $(z_{1n}, z'_n) \in \mathbb{R} \times T^{N-1}$) such that

$$u(y_{1n}, y'_n) \underset{n\to+\infty}{\longrightarrow} Sup_C u = a_+, \tag{2.3}$$

$$u(z_{1n}, z'_n) \underset{n\to+\infty}{\longrightarrow} Inf_C u = a_-, \tag{2.4}$$

with $|y_{1n}| \to +\infty$ and $|z_{1n}| \to +\infty$.

One can assume, eventually extracting a subsequence, that

$$y_{1n} \to +\infty \quad \text{and} \quad y_{1n} < y_{1n+1}. \tag{2.5}$$

(The proof in the other case is identical.)

Let us now prove

$$Inf_{x' \in T^{N-1}} u(y_{1n}, x') \underset{n\to+\infty}{\longrightarrow} a_+. \tag{2.6}$$

Indeed, let us consider

$$u_n(x_1, x') = u(y_{1n} + x_1, x').$$

We have

- $|u_n|_{C^3} \leq C$,
- u_n is solution of equation (1.2),
- $u_n(0, y'_n) \xrightarrow[n\to+\infty]{} a_+ = Sup\ u = Sup\ u_n$.

Extracting a subsequence, we have for $W : (x_1, x') \mapsto W(x_1, x')$ and $y' \in T^{N-1}$,

$$u_n(x_1, x') \underset{C^2_{loc}}{\longrightarrow} W(x_1, x') \quad \text{and} \quad y'_n \to y'$$

where

$$W \ \text{ is solution of (1.2), } SupW \leq a_+ \ , \ W(0, y') = a_+.$$

Therefore,

$$W \equiv a_+$$

and since the result is true for all subsequences

$$u(y_{1n} + x_1, x') \underset{C^2_{loc}}{\longrightarrow} a_+,$$

then (1.7) follows.

From the fact that u does not have local minimum, we have

$$a_+ \geq \underset{\substack{x_1 \in (y_{1n}, y_{1(n+1)}) \\ x' \in T^{N-1}}}{Inf} u(x_1, x) \geq \underset{\substack{x_1 = y_{1n}, y_{1(n+1)} \\ x' \in T^{N-1}}}{Inf} u(x_1, x) \underset{n\to+\infty}{\longrightarrow} a_+.$$

Since $y_{1n} < y_{1(n+1)}$ and $y_{1n} \longrightarrow_{n\to+\infty} +\infty$, then

$$\lim_{x_1 \to +\infty} \{Inf_{x' \in T^{N-1}} u(x_1, x')\} \text{ exists and equals } a_+.$$

It follows that $z_{1n} \to -\infty$ and by the same procedure,

$$\lim_{x_1 \to -\infty} \{Sup_{x' \in T^{N-1}} u(x_1, x')\} \text{ exists and is equal to } a_-.$$

This concludes the proofs of Lemma 1 and Theorem 1 part (i).

We now assume in addition a nondegeracy condition at a_+, a_- for $f_1(u)$, namely

$$f'_1(a_+) - c \neq 0 \quad \text{and} \quad f'_1(a_-) - c \neq 0,$$

and, for example, $a_- < a_+$. □

We first have the following estimates.

Lemma 2. *There exist $\alpha > 0$ and $C_0 > 0$ such that*

$$|u(x_1, x') - a_+| \leq C_0 e^{-\alpha x_1}, \tag{2.7}$$

$$|u(x_1, x') - a_-| \leq C_0 e^{+\alpha x_1}. \tag{2.8}$$

Proof. Let us prove the first one for example. The only question is when $x_1 \to +\infty$.

(i) Let us first introduce

$$w(x_1) = \frac{1}{|\mathrm{vol}(T^{N-1})|} \int_{T^{N-1}} (a_+ - u(x_1, x'))\, dx'. \tag{2.9}$$

We have

$$w \geq 0.$$

Averaging the equation (1.2) over T^{N-1} and using the periodic boundary conditions, we obtain that w satisfies the following equation $\forall x_1 \in \mathbb{R}$:

$$w_{x_1 x_1} \; - \; \frac{\partial}{\partial x_1} \left[\frac{1}{\mathrm{vol}(T^{N-1})} \int_{T^{N-1}} (f_1(u(x_1, x')) - cu(x_1, x'))\, dx' \right] = 0$$

or equivalently

$$w_{x_1} - \left[\frac{1}{\mathrm{vol}(T^{N-1})} \int_{T^{N-1}} (f_1(u(x_1, x')) - cu(x_1, x')) dx' \right] \equiv C_0. \tag{2.10}$$

Define $\beta = f_1'(a_+) - c$ and $\gamma = f_1(a_+) - ca_+$. We then have by linearization of the nonlinear term at a_+ in (2.10):

$$\forall x_1 \in \mathbb{R}, \quad |w_{x_1} - \beta w - C_0 + \gamma| \leq C Sup_{x' \in T^{N-1}} |u(x_1, x') - a_+|^2.$$

Since $w(x_1, x') \to 0$, as $x_1 \to +\infty$, we have $C_0 - \gamma = 0$, and

$$\forall x_1 \in \mathbb{R}, |w_{x_1} - \beta w| \leq C(Sup_{x' \in T^{N-1}} |u(x_1, x') - a_+|^2). \tag{2.11}$$

(ii) Relation between u and w: We now apply the Harnack principle as $x_1 \to +\infty$ to u, and we get: there is a $C > 0$ such that for $x_1 \geq 0$,

$$\underset{\substack{x' \in T^{N-1} \\ \tilde{x}_1 \in (x_1 - 1, x_1 + 1)}}{Sup} (a_+ - u(\tilde{x}_1, x')) \leq C \underset{\substack{x' \in T^{N-1} \\ \tilde{x}_1 \in (x_1 - 1, x_1 + 1)}}{Inf} (a_+ - u(\tilde{x}_1, x')),$$

and, in particular, $\forall x_1 \geq 0$,

$$Sup_{x' \in T^{N-1}} (a_+ - u(x_1, x')) \leq C w(x_1). \tag{2.12}$$

In particular, it is enough to prove the exponential decay for $w(x_1)$ to reach the conclusion of the lemma.

(iii) Exponential decay of w: We have from (2.11) and (2.12),

$$\forall x_1 \geq 0, \quad |w_{x_1} - \beta w| \leq C w^2.$$

Since $w \to 0$ as $x_1 \to +\infty$ and $w > 0$ we have $\beta < 0$ and for x_1 large,

$$-\frac{3}{2} \beta w \leq w_{x_1} \leq -\frac{\beta}{2} w,$$

which concludes the proof of Lemma 2, by integration in x_1.

We are now able to conclude the proof of Theorem 1 part (ii). We argue by contradiction: Assume there is $x_1^0 \in \mathbb{R}$ such that for some $x_0', x_1' \in T^{N-1}$,

$$u(x_1^0, x_0') \neq u(x_1^0, x_1').$$

Then using the periodicity, there are $x_2', x_3' \in T^{N-1}$ such that

$$u(x_1^0, x_2') = u(x_1^0, x_3')$$

and

$$\nabla_{x'} u(x_1^0, x_2^1) \neq \nabla_{x'} u(x_1^0, x_3').$$

Then

$$H(t,x) \;=\; u(x_1 - ct, x_2' + x') \;\; - \;\; u(x_1 - ct, x_3' + x')$$

is the difference of two solutions of equation (1.1) such that

i) $\forall t \in \mathbb{R}, \quad \| H(t) \|_{L^1(C)} \equiv$ constant,

ii) $H(0,(0,0)) = 0$ and $\nabla_{x'} H(0,(0,0)) \neq 0$.

Applying Lemma 2.9 of [1], we obtain

$$\| H(1) \|_{L^1(C)} < \| H(0) \|_{L^1(C)},$$

which is a contradiction.

(Of course, multiplying $u(x_1, x_2 + x') - u(x_1, x_3 + x')$ by sign $[u(x_1, x_2 + x') - u(x_1, x_3 + x')]$ yields a contradiction, by the same calculation as in the proof of Lemma 2.9 of [1].) This concludes the proof of Theorem 1. □

References

[1] C. Kenig and F. Merle. Asymptotic stability and Liouville theorem for scalar viscous conservation laws in cylinder, to appear in Comm. Pure Appl. Math.

Department of Mathematics, University of Chicago, Chicago, Illinois 60637-1514 – and – Institute for Advanced Study, Princeton, New Jersey 08540

Université de Cergy-Pontoise, Institut Universitaire de France, 33, boulevard du Port, 95011 Cergy-Pontoise Cedex, France – and – Institute for Advanced Study, Princeton, New Jersey 08540