

STRONG PARACOMPACTNESS AND W - $\delta\theta$ -REFINABILITY OF INVERSE LIMITS

KEIKO CHIBA

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ABSTRACT. In this paper we construct inverse systems of which each space is strongly paracompact and each bonding map and each projection are open and onto maps, and the limit space is paracompact and not strongly paracompact and we investigate $\delta\theta$ -refinability-like properties of limit spaces of inverse systems.

1. INTRODUCTION

Throughout this paper we assume that each space is a topological space having at least two points. No separation axioms are assumed. Each map is assumed to be continuous.

Let $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ be an inverse system of spaces X_α with bonding maps $\pi_\beta^\alpha : X_\alpha \rightarrow X_\beta$ ($\alpha \geq \beta$) and let $|\Lambda|$ denote the cardinal number of Λ . Let X be the limit space of this system. Then a map $\pi_\alpha : X \rightarrow X_\alpha$ denotes the projection for each $\alpha \in \Lambda$ (for the definitions of inverse systems and limits, see [7, 2.5]). We denote $X = \varprojlim \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$. An inverse system $\{X_n, \pi_n^m, \mathbf{N}\}$, where \mathbf{N} is the set of natural numbers directed by its natural order, is called an inverse sequence. An onto map $f : X \rightarrow Y$ is called “pseudo-open” if for any $y \in Y$ and any open set U in X such that $f^{-1}(y) \subset U$, $y \in \text{Int } f(U)$. Here $\text{Int } f(U)$ denotes the interior of $f(U)$. Every open and onto map is a pseudo-open map, and every closed and onto map is a pseudo-open map. For an infinite cardinal κ , a space X is called “ κ -paracompact” if every open cover \mathcal{U} of X with $|\mathcal{U}| \leq \kappa$ has a locally finite open refinement.

In previous papers [1], [2], [4], [5], [6], the following question was investigated.

Question. Let \mathcal{P} be a topological property and $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ an inverse system. If $X = \varprojlim \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ is $|\Lambda|$ -paracompact and each projection π_α is pseudo-open, must \bar{X} have property \mathcal{P} if each X_α has \mathcal{P} ?

A positive answer to the above question has been proved for many topological properties \mathcal{P} (paracompactness, collectionwise normality, normality and many other covering properties and various expandabilities; cf. [1], [2], [4], [5], [6], [11]). However, for the strong paracompactness and $\delta\theta$ -refinability, the questions are not yet solved.

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In this note we shall give a negative answer for the case of strong paracompactness. This is the first case for which the answer to this question is “no”. Let \mathcal{P} be a topological property which is stronger than paracompactness. Then, if $X = \varprojlim \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ is $|\Lambda|$ -paracompact and each projection π_α is a pseudo-open map and each X_α has \mathcal{P} , then X is paracompact. Therefore, for strong paracompactness, the question is: Suppose $X = \varprojlim \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ is paracompact and each projection π_α is a pseudo-open map and each X_α is strongly paracompact. Is X strongly paracompact?

We cannot answer this question for the case of $\delta\theta$ -refinability. We shall investigate $\delta\theta$ -refinability-like properties.

2. STRONG PARACOMPACTNESS

A space X is “strongly paracompact” if for every open cover \mathcal{G} of X , there is a star finite open cover \mathcal{H} of X such that $\mathcal{H} \prec \mathcal{G}$. Here $\mathcal{H} \prec \mathcal{G}$ means \mathcal{H} is a refinement of \mathcal{G} . A cover \mathcal{H} is called “star finite” (resp. “star countable”) if for each $H \in \mathcal{H}$, $\{H' | H' \in \mathcal{H}, H \cap H' \neq \emptyset\}$ is finite (resp. countable). It is known that a space X is strongly paracompact if and only if every open cover of X has a star countable open refinement. Therefore, every Lindelöf space is strongly paracompact, in particular, the real line \mathbf{R} with the usual topology is strongly paracompact.

In this paper we shall prove the following.

Theorem 1. *For each infinite cardinal κ , there exists an inverse system $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ with $|\Lambda| = \kappa$ such that each X_α is strongly paracompact, each π_β^α and each projection π_α are open and onto maps, and $X = \varprojlim \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ is paracompact, is not strongly paracompact.*

First we shall prove the case of $\kappa = \omega$.

Theorem 2. *There exists an inverse sequence $\{X_n, \pi_n^m, \mathbf{N}\}$ such that each X_n is strongly paracompact, each π_n^m and each projection π_n are open and onto maps, and $X = \varprojlim \{X_n, \pi_n^m, \mathbf{N}\}$ is paracompact and is not strongly paracompact.*

Definition 1. Let κ be an infinite ordinal and denote by $T(\kappa)$ the space $\{\alpha | \alpha \text{ is an ordinal, } 0 \leq \alpha \leq \kappa\}$ with the topology as follows: $\{\alpha\}$ is open for each $\alpha < \kappa$, $\{(\alpha, \kappa] | \alpha < \kappa\}$ is a neighborhood base of κ . Here $(\alpha, \kappa] = \{\beta | \beta \text{ is an ordinal, } \alpha < \beta \leq \kappa\}$.

Example 1 ([3, Example 2]). $\mathbf{R}^+ \times T(\omega_\omega)$ is not strongly paracompact. Here $\mathbf{R}^+ = \{x \in \mathbf{R} | x \geq 0\}$.

To prove Theorem 2, we shall show the following theorem.

Theorem 3. *There exists an inverse sequence $\{A_n, f_n^m, \mathbf{N}\}$ such that each A_n is a discrete space, each f_n^m and each projection f_n are onto maps, and $\varprojlim \{A_n, f_n^m, \mathbf{N}\} = T(\omega_\omega)$.*

Theorem 3 follows from Theorem 4 when $\kappa = \omega_\omega$ by using Lemma 1 below, since ω_ω has a cofinal subset of order type \mathbf{N} , and hence the limit of the inverse system corresponding to ω_ω is homeomorphic to the limit of an inverse sequence (cf. the comments after Corollary 2.5.12 in [7]).

Lemma 1 ([7, Corollary 2.5.11]). *Let $\mathcal{S} = \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ be an inverse system and Λ' a subset cofinal in Λ . Then the map $f : X = \varprojlim \mathcal{S} \rightarrow X' = \varprojlim \mathcal{S}'$ defined by $f(x)_\alpha = x_\alpha$ for each $\alpha \in \Lambda'$ ($x = (x_\alpha)_{\alpha \in \Lambda} \in X$) is a homeomorphism where $\mathcal{S}' = \{X_\alpha, \pi_\beta^\alpha, \Lambda'\}$.*

Theorem 4. *For each infinite cardinal κ , there exists an inverse system $\mathcal{S} = \{A_\lambda, f_\lambda^\mu, \Lambda\}$ such that each A_λ is a discrete space, each f_λ^μ and each projection f_λ are onto maps, and $\varprojlim \mathcal{S} = T(\kappa)$ and $|\Lambda| = \kappa$.*

Proof. This theorem was essentially proved in [10, Theorem 2]. But, for the reader's convenience, we shall write this proof briefly. Let $X = T(\kappa)$. Let $\Lambda = \{\lambda \mid \lambda < \kappa\}$ be the set of ordinals and for each $\lambda \in \Lambda$, let $A_\lambda = (\lambda + 1) \cup \{\kappa\} = \{\alpha \mid \alpha \leq \lambda\} \cup \{\kappa\}$ be the space of ordinals with the discrete topology.

Then Λ , with the usual order of ordinals, is a directed set. For each $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$, define $f_\lambda^\mu : A_\mu \rightarrow A_\lambda$ by

$$f_\lambda^\mu(\alpha) = \begin{cases} \alpha & \text{if } \alpha \leq \lambda, \\ \kappa & \text{if } \alpha > \lambda. \end{cases}$$

Then

(I) $\mathcal{S} = \{A_\lambda, f_\lambda^\mu, \Lambda\}$ is an inverse system.

(II) $\varprojlim \mathcal{S}$ is homeomorphic to X .

To prove (II), let us define $g : X \rightarrow \prod_{\lambda \in \Lambda} A_\lambda$ by

$$g(\alpha)_\lambda = \begin{cases} \alpha & \text{if } \alpha \leq \lambda, \\ \kappa & \text{if } \alpha > \lambda. \end{cases}$$

Then $g(\alpha) = (g(\alpha)_\lambda)_{\lambda \in \Lambda} \in \varprojlim \mathcal{S}$ for each $\alpha \in X$. □

Claim. g is a homeomorphism from X onto $\varprojlim \mathcal{S}$.

Proof. 1) It is clear that g is one-to-one.

2) g is onto. To prove this, let $a \in \varprojlim \mathcal{S}$ be an arbitrary element.

(i) In case $a_\lambda = \kappa$ for each $\lambda \in \Lambda$, then $g(\kappa) = a$.

(ii) In case there exists $\lambda \in \Lambda$ such that $a_\lambda \neq \kappa$, let $\lambda_0 = \min\{\lambda \mid a_\lambda \neq \kappa\}$ and put $\alpha = a_{\lambda_0}$.

Then

$$a_\lambda = \begin{cases} \alpha & \text{if } \lambda \geq \lambda_0, \\ \kappa & \text{if } \lambda < \lambda_0. \end{cases}$$

Moreover, we have $\alpha = a_{\lambda_0}$ and we have $g(\alpha) = a$.

3) g is continuous. Since $f_\lambda \circ g$ is continuous for each $\lambda \in \Lambda$, g is continuous.

4) g is an open map. □

Proof. Let $\lambda \in X$. If $\lambda \neq \kappa$, then $g(\lambda) = f_\lambda^{-1}(\{\lambda\})$. Moreover, $g((\lambda, \kappa]) = \{g(\kappa)\} \cup \{g(\mu) \mid \lambda < \mu < \kappa\} = \{g(\kappa)\} \cup \{f_\mu^{-1}(\{\mu\}) \mid \lambda < \mu < \kappa\} = f_\lambda^{-1}(\{\kappa\})$.

(III) It is easy to see that f_λ is onto for each $\lambda \in \Lambda$. □

Proof of Theorem 2. Let \mathcal{S} be the inverse sequence in Theorem 3. For each $n \in \mathbf{N}$, let $X_n = \mathbf{R} \times A_n$ and for each $n, m \in \mathbf{N}$ with $n \leq m$, define $\pi_n^m = 1_{\mathbf{R}} \times f_n^m$. Then $\{X_n, \pi_n^m, \mathbf{N}\}$ is an inverse sequence and $\varprojlim \{X_n, \pi_n^m, \mathbf{N}\} = \mathbf{R} \times T(\omega_\omega)$. Since $\mathbf{R} \times T(\omega_\omega)$ is metrizable, it is paracompact. Each X_n is strongly paracompact and

each π_n^m is an open and onto map and all projections π_n are open and onto maps. Thus $\varprojlim\{X_n, \pi_n^m, \mathbf{N}\}$ is not strongly paracompact by Example 1 because \mathbf{R}^+ is closed in \mathbf{R} . \square

Proposition. *If π_α is a closed (resp. an open) and onto map for each $\alpha, \beta \in \Lambda$, then π_α^β is a closed (resp. an open) and onto map for each $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$.*

Proof. By [6, Lemma 2.1], if the projection π_α is onto, then $\pi_\beta^\alpha(A) = \pi_\beta\pi_\alpha^{-1}(A)$ for each β with $\beta \leq \alpha$ and for each $A \subset X_\alpha$. This implies the Proposition. \square

Remark 1. In Theorem 2, if $n < m$, then π_n^m is not a closed map. To prove this, let us choose $\alpha_i, i = 1, 2, \dots$, such that $\omega_n < \alpha_1 < \alpha_2 < \dots < \alpha_i < \dots < \omega_m$ and put $F = \bigcup_{i \in \mathbf{N}}(\{1/i\} \times \{\alpha_i\})$. Then F is a closed subset of $\mathbf{R} \times A_m$ because $\{\alpha_i | i = 1, 2, \dots\}$ is a closed and discrete subset of A_m . But $\pi_n^m(F) = (\bigcup_{i \in \mathbf{N}}\{1/i\}) \times \{\omega_\omega\}$ is not closed in $\mathbf{R} \times A_n$.

By the Proposition, π_n is not a closed map. Thus the following question remains.

Question 1. Let $\mathcal{S} = \{X_n, \pi_n^m, \mathbf{N}\}$ be an inverse sequence with π_n^m (or π_n) a closed and onto map and each X_n is strongly paracompact. Suppose $\varprojlim \mathcal{S}$ is paracompact. Is $\varprojlim \mathcal{S}$ strongly paracompact?

Lemma 2. *If X is a paracompact T_2 -space and Y is a T_1 -space having only one non-isolated point, then $X \times Y$ is paracompact.*

Proof. Let \mathcal{G} be an arbitrary open cover of $X \times Y$ and y_0 the only non-isolated point of Y . For each $x \in X$, let us choose an open set $U(x)$ in X and an open set V_x in Y such that $(x, y_0) \in U(x) \times V_x$ and $\{\overline{U(x)} \times \overline{V_x} \mid x \in X\} \prec \mathcal{G}$ (partial refinement). Since X is paracompact, there is a locally finite open cover \mathcal{M} of X such that $\mathcal{M} \prec \{U(x) \mid x \in X\}$. For each $M \in \mathcal{M}$, let us choose an $x \in X$ such that $M \subset U(x)$ and put $x = x(M)$. Then $\mathcal{H} = \{M \times V_{x(M)} \mid M \in \mathcal{M}\}$ is a locally finite collection of open sets in $X \times Y$, $\overline{\mathcal{H}} = \{\overline{H} \mid H \in \mathcal{H}\} \prec \mathcal{G}$ and $\bigcup \mathcal{H} \supset X \times \{y_0\}$.

Since $X \times (Y \setminus \{y_0\})$ is paracompact and $X \times Y \setminus \bigcup \mathcal{H}$ is closed in $X \times (Y \setminus \{y_0\})$, $X \times Y \setminus \bigcup \mathcal{H}$ is paracompact. Therefore, there is a locally finite closed cover \mathcal{K} of $X \times Y \setminus \bigcup \mathcal{H}$ such that $\mathcal{K} \prec \mathcal{G}$. Then $\overline{\mathcal{H}} \cup \mathcal{K}$ is a locally finite closed refinement of \mathcal{G} . Since $X \times Y$ is a regular space, $X \times Y$ is paracompact. \square

Proof of Theorem 1. Let κ be an arbitrary infinite cardinal. Let $\mathcal{S} = \{A_n, f_n^m, \mathbf{N}\}$ be the inverse sequence in Theorem 3. By Theorem 4, there is an inverse system $\mathcal{S}' = \{B_\lambda, g_\lambda^\mu, \Lambda\}$ with $|\Lambda| = \kappa$, each B_λ is a discrete space such that $\varprojlim \mathcal{S}'$ is homeomorphic to $T(\kappa)$.

Let us define a partial order on $\mathbf{N} \times \Lambda$ as follows: Let $\langle n, \lambda \rangle, \langle m, \mu \rangle \in \mathbf{N} \times \Lambda$. Let us define $\langle n, \lambda \rangle \leq \langle m, \mu \rangle$ if $n \leq m$ and $\lambda \leq \mu$. Then $\mathbf{N} \times \Lambda$ is a directed set. For each $\langle n, \lambda \rangle \in \mathbf{N} \times \Lambda$, put $Z_{\langle n, \lambda \rangle} = \mathbf{R} \times A_n \times B_\lambda$, and for each $\langle n, \lambda \rangle, \langle m, \mu \rangle \in \mathbf{N} \times \Lambda$ with $\langle n, \lambda \rangle \leq \langle m, \mu \rangle$, put $h_{\langle n, \lambda \rangle}^{\langle m, \mu \rangle} = 1_{\mathbf{R}} \times f_n^m \times g_\lambda^\mu$. Then $\mathcal{S}'' = \{Z_{\langle n, \lambda \rangle}, h_{\langle n, \lambda \rangle}^{\langle m, \mu \rangle}, \mathbf{N} \times \Lambda\}$ is an inverse system with $|\mathbf{N} \times \Lambda| = \kappa$ and each $Z_{\langle n, \lambda \rangle}$ is strongly paracompact and each $h_{\langle n, \lambda \rangle}^{\langle m, \mu \rangle}$ is an open and onto map and each projection $h_{\langle n, \lambda \rangle}$ is an open and onto map, and $\varprojlim \mathcal{S}''$ is homeomorphic to $\mathbf{R} \times T(\omega_\omega) \times T(\kappa)$. Since $\mathbf{R} \times T(\omega_\omega)$ is paracompact and $T(\kappa)$ has only one non-isolated point, by Lemma 2, $\mathbf{R} \times T(\omega_\omega) \times T(\kappa)$ is paracompact. Therefore, $\varprojlim \mathcal{S}''$ is paracompact but is not strongly paracompact. \square

3. COMPACTNESS

It is well known that for T_2 -spaces, every limit space of inverse systems of compact spaces is compact (cf. [7]). But, for non- T_2 -spaces, limit spaces are not necessarily compact, e.g. A. H. Stone constructed the following example.

Example 2 ([12, Example 3]). There is an inverse sequence \mathcal{S} of compact spaces such that $\varprojlim \mathcal{S}$ is not compact.

Similarly, we obtain the following.

Example 3. For each cardinal $\kappa \geq \omega$, there is an inverse system \mathcal{S} of compact spaces such that $\varprojlim \mathcal{S}$ is not κ -Lindelöf, i.e., $L(\varprojlim \mathcal{S}) > \kappa$ where $L(X)$ is the Lindelöf number of X .

Proof. For each $\alpha < \kappa^+$, let $X_\alpha = \kappa^+$ with the topology as follows: For each $\beta \leq \alpha$, $\{\beta\}$ is an open set and for each $\beta > \alpha$, $\mathcal{V}_\beta = \{V \mid \beta \in V \subset X_\alpha, |X_\alpha \setminus V| \text{ finite}\}$ is a neighborhood base of β .

Then X_α is compact and is not T_2 -space. For each $\alpha \leq \beta$, define $\pi_\alpha^\beta : X_\beta \rightarrow X_\alpha$ to be the identity map. Then π_α^β is continuous. Put $\mathcal{S} = \{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ where $\Lambda = \kappa^+$. Then $X = \varprojlim \mathcal{S}$ is discrete.

To show this, let $p_\beta : \prod_{\alpha \in \Lambda} X_\alpha \rightarrow X_\beta$ be the projection for each $\beta \in \Lambda$.

For each $\beta \in \Lambda$, define x_β^* as follows: $x_\beta^* = (x_\alpha)_{\alpha \in \Lambda}$, $x_\alpha = \beta$ for each $\alpha \in \Lambda$. Then $X = \{x_\beta^* \mid \beta \in \Lambda\}$. For each $\beta \in \Lambda$, define $U_\beta = p_\beta^{-1}(\{\beta\}) \cap X$. Then U_β is open in X and $U_\beta = \{x_\beta^*\}$. □

Stone proved the following.

Theorem A ([12, Theorem 5]). *Let $\mathcal{S} = \{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ be an inverse system such that each X_α is a compact T_0 -space and each π_α^β is a closed map. Then $X = \varprojlim \mathcal{S}$ is compact.*

Remark 2. In the above example, π_α^β is not a pseudo-open map. Since π_α is onto, by the Proposition, π_α is not an open map. Therefore, the following question remains.

Question 2. Is there an inverse system $\mathcal{S} = \{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ such that each X_α is compact and each π_α^β (or each π_α) is an open and onto map and $X = \varprojlim \mathcal{S}$ is paracompact, but X is not compact?

4. W - $\delta\theta$ -REFINABILITY AND WW - $\delta\theta$ -REFINABILITY

In this section we investigate $\delta\theta$ -refinability-like properties.

Definition 2. A space X is called “ θ -refinable” (resp. “ $\delta\theta$ -refinable”) if every open cover \mathcal{G} of X has a θ -sequence (resp. $\delta\theta$ -sequence) $\{\mathcal{H}_n \mid n \in \mathbf{N}\}$ of X such that each \mathcal{H}_n is an open cover of X and $\mathcal{H}_n \prec \mathcal{G}$.

A sequence $\{\mathcal{H}_n \mid n \in \mathbf{N}\}$ of X is called a “ θ -sequence” (resp. “ $\delta\theta$ -sequence”) of X if for any $x \in X$ there is an $n_x \in \mathbf{N}$ such that $\text{ord}(x, \mathcal{H}_{n_x}) < \omega$ (resp. $\text{ord}(x, \mathcal{H}_{n_x}) \leq \omega$) where $\text{ord}(x, \mathcal{H}_{n_x}) = |\{H; x \in H \in \mathcal{H}_{n_x}\}|$.

Definition 3. Let \mathcal{U} and \mathcal{V} be open covers of X . \mathcal{V} is called a pointwise W -refinement of \mathcal{U} at x if there is a finite subfamily \mathcal{U}' of \mathcal{U} such that $\mathcal{V}(x) \prec \mathcal{U}'$. Here $\mathcal{V}(x) = \{V \in \mathcal{V} \mid x \in V\}$.

Let \mathcal{U} be a cover of X and $\langle \mathcal{V}_n \rangle_{n \in \mathbf{N}}$ a sequence of covers of X . A sequence $\langle \mathcal{V}_n \rangle_n$ is called a pointwise W -refining sequence for \mathcal{U} if for each x , there exists an n_x such that \mathcal{V}_{n_x} is a pointwise W -refinement of \mathcal{U} at x .

The next characterization of θ -refinable spaces was given by J. M. W. Worrell.

Theorem B ([13], or cf. [14, Theorem 3.4]). *A space X is θ -refinable (submetacompact) if and only if every open cover of X has a pointwise W -refining sequence by open covers.*

Definition 4 ([8]). Let \mathcal{L} and \mathcal{G} be covers of X . \mathcal{L} is called a “point-star \dot{F} -refinement” of \mathcal{G} at $x \in X$ if there is a finite subfamily \mathcal{G}' of \mathcal{G} such that $x \in \bigcap \mathcal{G}'$ and $\text{st}(x, \mathcal{L}) \subset \bigcup \mathcal{G}'$ where $\text{st}(x, \mathcal{L}) = \bigcup \{L \mid x \in L \in \mathcal{L}\}$.

A sequence $\langle \mathcal{L}_n \rangle_{n \in \mathbf{N}}$ of covers of X is called a “point-star \dot{F} -refining sequence” of \mathcal{G} if for each $x \in X$, there is an $n_x \in \mathbf{N}$ such that \mathcal{L}_{n_x} is a point-star \dot{F} -refinement of \mathcal{G} at x .

H. J. K. Junnila gave the next characterization of submetacompactness.

Theorem C ([8]). *A space X is θ -refinable (submetacompact) if and only if every open cover of X has a point star \dot{F} -refining sequence by open covers.*

Now we shall define new properties.

Definition 5. Let \mathcal{U} and \mathcal{V} be open covers of X . \mathcal{V} is called a pointwise countable W -refinement of \mathcal{U} at x if there is a countable subfamily \mathcal{U}' of \mathcal{U} such that $\mathcal{V}(x) \prec \mathcal{U}'$.

Let \mathcal{U} be a cover of X and $\langle \mathcal{V}_n \rangle_{n \in \mathbf{N}}$ a sequence of covers of X . A sequence $\langle \mathcal{V}_n \rangle_n$ is called a pointwise countable W -refining sequence for \mathcal{U} if for each x , there exists an n_x such that \mathcal{V}_{n_x} is a pointwise countable W -refinement of \mathcal{U} at x .

We shall say a space X is w - $\delta\theta$ -refinable if every open cover of X has a pointwise countable W -refining sequence by open covers.

Definition 6. Let \mathcal{L} and \mathcal{G} be covers of X . \mathcal{L} is called a “point-star \dot{C} -refinement” of \mathcal{G} at $x \in X$ if there is a countable subfamily \mathcal{G}' of \mathcal{G} such that $x \in \bigcap \mathcal{G}'$ and $\text{st}(x, \mathcal{L}) \subset \bigcup \mathcal{G}'$.

A sequence $\langle \mathcal{L}_n \rangle_{n \in \mathbf{N}}$ of covers of X is called a “point-star \dot{C} -refining sequence” of \mathcal{G} if for each $x \in X$, there is an $n_x \in \mathbf{N}$ such that \mathcal{L}_{n_x} is a point-star \dot{C} -refinement of \mathcal{G} at x .

We shall say a space X is ww - $\delta\theta$ -refinable if every open cover of X has a point star \dot{C} -refining sequence by open covers.

It is obvious that every $\delta\theta$ -refinable space is w - $\delta\theta$ -refinable and every w - $\delta\theta$ -refinable space is ww - $\delta\theta$ -refinable.

Lemma 3. (1) *Let \mathcal{G} be an open cover of X and $\langle \mathcal{V}_n \rangle_{n \in \mathbf{N}}$ a pointwise countable W -refining sequence of \mathcal{G} . Then there exists a pointwise countable W -refining sequence $\langle \mathcal{H}_n \rangle_{n \in \mathbf{N}}$ of \mathcal{G} satisfying the following conditions: For each $x \in X$, there exist an $n_x \in \mathbf{N}$ and a countable subfamily \mathcal{G}' of \mathcal{G} such that $\mathcal{H}_n(x) \prec \mathcal{G}'$ for each $n \geq n_x$.*

(2) *Let \mathcal{G} be an open cover of X and $\langle \mathcal{V}_n \rangle_{n \in \mathbf{N}}$ a point-star \dot{C} -refining sequence of \mathcal{G} . Then there exists a point-star \dot{C} -refining sequence $\langle \mathcal{H}_n \rangle_{n \in \mathbf{N}}$ of \mathcal{G} satisfying the following conditions: For each $x \in X$, there exist an $n_x \in \mathbf{N}$ and a countable subfamily \mathcal{G}' of \mathcal{G} such that $x \in \bigcap \mathcal{G}'$ and $\text{st}(x, \mathcal{H}_n) \subset \bigcup \mathcal{G}'$ for each $n \geq n_x$.*

Proof. Let us put $\mathcal{H}_n = \bigwedge_{i=1}^n \mathcal{V}_i (= \{\bigcap_{i=1}^n V_i \mid V_i \in \mathcal{V}_i \text{ for each } i = 1, 2, \dots, n\})$. Then $\langle \mathcal{H}_n \rangle_{n \in \mathbf{N}}$ is a desired one. \square

A cover \mathcal{U} of a space X is *directed* if for any $U, V \in \mathcal{U}$, there is $W \in \mathcal{U}$ with $U \cup V \subset W$.

Lemma 4 ([9]). *A space X is κ -paracompact if and only if for every directed open cover \mathcal{U} with cardinality $\leq \kappa$, there is a locally finite open cover \mathcal{V} of X such that $\{\overline{V} \mid V \in \mathcal{V}\}$ refines \mathcal{U} .*

Theorem 5. *Let $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ be an inverse system and $X = \varprojlim \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$. Suppose each projection π_α is a pseudo-open map and X is κ -paracompact where $|\Lambda| = \kappa$. Then:*

- (a) *If each X_α is $w\text{-}\delta\theta$ -refinable, then so is X .*
- (b) *If each X_α is $ww\text{-}\delta\theta$ -refinable, then so is X .*

Since the proofs of part (a) and part (b) are quite similar, we shall give the proof only for part (b).

Proof of part (b). Let $\mathcal{G} = \{G_\xi \mid \xi \in \Xi\}$ be an arbitrary open cover of X . For each $\alpha \in \Lambda$ and $\xi \in \Xi$, let $U_{\alpha,\xi}$ be the maximal open set in X_α satisfying $\pi_\alpha^{-1}(U_{\alpha,\xi}) \subset G_\xi$ and put $U_\alpha = \bigcup_{\xi \in \Xi} U_{\alpha,\xi}$. Then $\{\pi_\alpha^{-1}(U_\alpha) \mid \alpha \in \Lambda\}$ is a directed open cover of X . Since X is κ -paracompact, by Lemma 4, there exists an open cover $\{W_\alpha \mid \alpha \in \Lambda\}$ of X such that:

- (1) $\overline{W_\alpha} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$, and
- (2) $W_\alpha \subset W_\beta$ if $\alpha \leq \beta$.

Let us put $T_\alpha = X_\alpha \setminus \text{Int } \pi_\alpha(X \setminus \overline{W_\alpha})$. Then

- (3) T_α is closed in X_α and $T_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$.

Put $C_\alpha = \text{Int } \pi_\alpha^{-1}(T_\alpha)$ for each $\alpha \in \Lambda$. Then

- (4) $\mathcal{C} = \{C_\alpha \mid \alpha \in \Lambda\}$ is an open cover of X (cf. [2]).

Since X is κ -paracompact and $|\mathcal{C}| = \kappa$, there is a locally finite open cover $\mathcal{O} = \{O_\alpha \mid \alpha \in \Lambda\}$ such that:

- (5) $O_\alpha \subset C_\alpha$ for each $\alpha \in \Lambda$.

Let us put $\mathcal{U}_\alpha = \{U_{\alpha,\xi} \mid \xi \in \Xi\}$. Then \mathcal{U}_α is an open cover of U_α . Since X_α is $ww\text{-}\delta\theta$ -refinable, by (3), using Lemma 3, we can choose a sequence $\langle \mathcal{L}_{\alpha,m} \rangle_{m \in \mathbb{N}}$ of collections in X_α satisfying:

- (i) $_\alpha$. Each set of $\mathcal{L}_{\alpha,m}$ is an open set in X_α .
- (ii) $_\alpha$. $T_\alpha \subset \bigcup \mathcal{L}_{\alpha,m}$ for each m .
- (iii) $_\alpha$. For each $y \in T_\alpha$, there are an $m(y) \in \mathbb{N}$ and a countable subset $A(y)$ of Ξ such that $y \in \bigcap_{\xi \in A(y)} U_{\alpha,\xi}$ and $\text{st}(y, \mathcal{L}_{\alpha,m}) \subset \bigcup_{\xi \in A(y)} U_{\alpha,\xi}$ for each $m \geq m(y)$.

Put $\mathcal{L}_m = \{\pi_\alpha^{-1}(L) \cap O_\alpha \mid L \in \mathcal{L}_{\alpha,m}, \alpha \in \Lambda\}$ for each m . Then:

- (6) \mathcal{L}_m is an open cover of X for each m . It is obvious that each set of \mathcal{L}_m is an open set of X . Let $x = (x_\alpha)_{\alpha \in \Lambda} \in X$. Then $x \in O_\alpha$ for some α . Since $x_\alpha \in T_\alpha$, by (5), (ii) $_\alpha$, $x_\alpha \in L$ for some $L \in \mathcal{L}_{\alpha,m}$. Thus $x \in \pi_\alpha^{-1}(L) \cap O_\alpha \in \mathcal{L}_m$.
- (7) $\langle \mathcal{L}_m \rangle_{m \in \mathbb{N}}$ is a point-star \dot{C} -refining sequence of \mathcal{G} . □

Proof. Let $x \in X$. Then, since \mathcal{O} is locally finite, there exists a finite set $\{\alpha_i \mid i = 1, 2, \dots, k\}$ of Λ such that $x \in O_\alpha \iff \alpha = \alpha_i$ for some $i = 1, 2, \dots, k$. For each $i = 1, 2, \dots, k$, since $x_{\alpha_i} \in T_{\alpha_i}$, there exist an $m_i \in \mathbb{N}$ and a countable subset A_i of Ξ such that $x_{\alpha_i} \in \bigcap_{\xi \in A_i} U_{\alpha_i,\xi}$ and $\text{st}(x_{\alpha_i}, \mathcal{L}_{\alpha_i,m}) \subset \bigcup_{\xi \in A_i} U_{\alpha_i,\xi}$ for each $m \geq m_i$.

Let us put $m = \max\{m_i \mid i = 1, 2, \dots, k\}$ and $A = \bigcup_{i=1}^k A_i$. Then $m \in \mathbb{N}$ and A is a countable subset of Ξ . If $\xi \in A$, then $\xi \in A_i$ for some $i = 1, 2, \dots, k$. Hence $x_{\alpha_i} \in U_{\alpha_i,\xi}$ and so $x \in \pi_{\alpha_i}^{-1}(U_{\alpha_i,\xi})$. Thus $x \in G_\xi$. Hence $x \in \bigcap_{\xi \in A} G_\xi$.

Let $L' \in \mathcal{L}_m$ with $x \in L'$. Then $L' = \pi_\alpha^{-1}(L) \cap O_\alpha$ for some $\alpha \in \Lambda$ and some $L \in \mathcal{L}_{\alpha,m}$. Since $\alpha = \alpha_i$ for some i , $x \in \pi_{\alpha_i}^{-1}(L) \cap O_{\alpha_i}$. Thus $x_{\alpha_i} \in L \in \mathcal{L}_{\alpha_i,m}$. Since $m \geq m_i$, $L \subset \bigcup_{\xi \in A_i} U_{\alpha_i,\xi}$. Therefore, $\pi_{\alpha_i}^{-1}(L) \cap O_{\alpha_i} \subset \bigcup_{\xi \in A_i} \pi_{\alpha_i}^{-1}(U_{\alpha_i,\xi}) \cap O_{\alpha_i} \subset \bigcup_{\xi \in A_i} G_\xi$. Thus $\text{st}(x, \mathcal{L}_m) \subset \bigcup_{\xi \in A} G_\xi$. \square

Let κ be an infinite cardinal. A space X is called κ - θ -refinable (κ -submetacompact) if every open cover \mathcal{G} of X with $|\mathcal{G}| \leq \kappa$ has a θ -sequence $\{\mathcal{H}_n | n \in \mathbf{N}\}$ of open refinements of \mathcal{G} .

If \mathcal{P} is a topological property, X is called hereditarily \mathcal{P} if every subspace has \mathcal{P} .

It is easy to see that X is hereditarily \mathcal{P} if every open subspace has \mathcal{P} when \mathcal{P} is w - $\delta\theta$ -refinability or ww - $\delta\theta$ -refinability.

Theorem 6. *Let $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ be an inverse system and $X = \varprojlim \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$. Suppose X is hereditarily κ -submetacompact where $|\Lambda| = \kappa$.*

- (a) *If each X_α is hereditarily w - $\delta\theta$ -refinable, then so is X .*
- (b) *If each X_α is hereditarily ww - $\delta\theta$ -refinable, then so is X .*

We only give the proof of part (b).

Proof of part (b). Let G be an arbitrary open subspace of X and $\mathcal{G} = \{G_\xi | \xi \in \Xi\}$ an arbitrary open cover of G . For each $\alpha \in \Lambda$ and $\xi \in \Xi$, let $U_{\alpha,\xi}$ be the maximal open set in X_α satisfying $\pi_\alpha^{-1}(U_{\alpha,\xi}) \subset G_\xi$ and put $U_\alpha = \bigcup_{\xi \in \Xi} U_{\alpha,\xi}$. Then $\mathcal{U} = \{\pi_\alpha^{-1}(U_\alpha) | \alpha \in \Lambda\}$ is an open cover of G with $|\mathcal{U}| = \kappa$. Since G is κ -submetacompact, there is a sequence $\langle \mathcal{O}_n \rangle_{n \in \mathbf{N}}$ of open refinements of \mathcal{U} satisfying the condition: For each $x \in G$, there exists an $n_x \in \mathbf{N}$ such that $\text{ord}(x, \mathcal{O}_{n_x}) < \omega$. We can represent $\mathcal{O}_n = \{O_{\alpha,n} | \alpha \in \Lambda\}$ with $O_{\alpha,n} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$. Let us put $\mathcal{U}_\alpha = \{U_{\alpha,\xi} | \xi \in \Xi\}$ for each $\alpha \in \Lambda$. Then \mathcal{U}_α is an open cover of U_α . Since U_α is ww - $\delta\theta$ -refinable, by Lemma 3, there is a sequence $\langle \mathcal{L}_{\alpha,m} \rangle_{m \in \mathbf{N}}$ of open covers of U_α satisfying:

- (1) $_{\alpha}$. $\mathcal{L}_{\alpha,m} \prec \mathcal{U}_\alpha$.
- (2) $_{\alpha}$. For each $y \in U_\alpha$, there are an $m(y) \in \mathbf{N}$ and a countable subset $A(y)$ of Ξ such that $y \in \bigcap_{\xi \in A(y)} U_{\alpha,\xi}$ and $\text{st}(y, \mathcal{L}_{\alpha,m}) \subset \bigcup_{\xi \in A(y)} U_{\alpha,\xi}$ for each $m \geq m(y)$.

For each $n, m \in \mathbf{N}$, let us put $\mathcal{V}_m^n = \{\pi_\alpha^{-1}(L) \cap O_{\alpha,n} | L \in \mathcal{L}_{\alpha,m}, \alpha \in \Lambda\}$. Then

- (3) \mathcal{V}_m^n is an open cover of G for each $n, m \in \mathbf{N}$.
- (4) $\langle \mathcal{V}_m^n \rangle_{n,m \in \mathbf{N}}$ is a point-star \dot{C} -refining sequence of \mathcal{G} . \square

Proof. Let $x \in G$. Then there exists an n_x such that $\text{ord}(x, \mathcal{O}_{n_x}) < \omega$. Let $\{\alpha_i | i = 1, 2, \dots, k\}$ be a finite subset of Ξ such that $x \in O_{\alpha,n_x} \iff \alpha = \alpha_i$ for some $i = 1, 2, \dots, k$. Since for each i , $x_{\alpha_i} \in U_{\alpha_i}$, by (2) $_{\alpha_i}$, there are an $m_i \in \mathbf{N}$ and a countable subset A_i of Ξ such that $x_{\alpha_i} \in \bigcap_{\xi \in A_i} U_{\alpha_i,\xi}$ and $\text{st}(x_{\alpha_i}, \mathcal{L}_{\alpha_i,m}) \subset \bigcup_{\xi \in A_i} U_{\alpha_i,\xi}$ if $m \geq m_i$. Let $m = \max\{m_i | i = 1, 2, \dots, k\}$ and $A = \bigcup_{i=1}^k A_i$. Put $n = n_x$. Then $x \in \bigcap_{\xi \in A} G_\xi$ and $\text{st}(x, \mathcal{V}_m^n) \subset \bigcup_{\xi \in A} G_\xi$. \square

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, SHIZUOKA UNIVERSITY, OHYA,
SURUGAKU, SHIZUOKA, 422-8529 JAPAN

E-mail address: smktiba@ipc.shizuoka.ac.jp