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ON THE COMMUTANT AND ORBITS OF CONJUGATION

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Abstract. In this note, we analyse the relationship between the commutant of a bounded linear operator A and the algebra of similarity B_A that was introduced in the late 70s as a characterization of nest algebras. Necessary and sufficient conditions are also obtained for an operator to commute with real scalar generalized operators in the sense of Colojoară-Foiaş in Banach spaces. In the second part, we analyse the relationship between the generalized inverse, the generalized commutant and the orbits of conjugation.

0. Introduction

Let $\mathcal{B}(X)$, $\mathcal{B}(H)$ and $\mathcal{A}$ denote respectively the algebra of all bounded linear operators on a complex Banach space X, the algebra of all bounded linear operators on the complex separable infinite-dimensional Hilbert space H, and a complex Banach algebra. The symbols $\sigma(S)$ and $r(S)$ denote respectively the spectrum and the spectral radius of the operator $S \in \mathcal{B}(X)$, and as usual S is called quasi-nilpotent if $\sigma(S) = \{0\}$. Given an invertible operator A, the study of operators T whose conjugation orbit $\{A^n T A^{-n}\}$ is bounded, that is, for $T \in \mathcal{B}(X)$ and A invertible,

$$\sup_{n\geq 0} \|A^n T A^{-n}\| < \infty, \tag{1}$$

was initiated by J. A. Deddens in the 1970s when he gave a characterization of nest algebras in terms of the algebra B_A, where B_A is the set of operators T in $\mathcal{B}(X)$ satisfying (1). It is clear that B_A contains the commutant $\{A\}'$ of A, but in general this inclusion may be proper. J. A. Deddens and T. K. Wong [3] showed that, if $A \in \mathcal{B}(H)$ is of the form $A = \alpha I + N$ where $0 \neq \alpha \in \mathbb{C}$ and N is nilpotent, then $B_A = \{A\}'$. Furthermore, if H is finite dimensional, then the converse holds; see [3] and [4]. In [4], Deddens raised the question: does the converse result still hold in infinite-dimensional Hilbert spaces? A negative answer to Deddens' question was provided by P. Roth [15]. The algebra B_A was further studied by J. P. Williams [17] and J. Stampfli [16].

Recently, a quantitative version of these results was given in [6] (see also [7]). It provides us with a bound on $\|e^S T e^{-S} - T\|$ in terms of the spectral radius $r(\Delta_S)$ of the commutator $\Delta_S(T) = ST - TS$. In the first part of this paper, we improve Deddens-Wong's result and extend it in the more general situation of Banach spaces

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(Theorem 1.2). An extension of Deddens-Williams's result is given in Theorem 1.1 and Theorem 1.4, where the spectrum can be more than one point; more precisely, the spectrum can be a compact subset of any half-line with origin at the point O. In the second part, we analyse the relationship between the generalized inverse and the orbits of conjugation in C^*-algebras.

We start by recalling some necessary tools on local spectral theory and growth of entire functions.

Let f be an entire function and let $M_f(r) = \max_{|z|=r} |f(z)|$. We say that f is of *finite order* if there exists $k \geq 0$ such that

$$M_f(r) \leq e^{r^k} \text{ for } r \text{ large.}$$

The infimum of all k satisfying this inequality is called the *order of* f and is denoted by $\tau(f)$. It is easy to verify that

$$\tau(f) = \lim_{r\to\infty} \frac{\log\log M_f(r)}{\log r}.$$

Now suppose that f is an entire function of finite order $\tau(f)$. We define the *type of* f, denoted by $\sigma(f)$, to be the infimum of all nonnegative numbers a such that

$$M_f(r) \leq e^{ar^{\tau(f)}}.$$

We then have

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\tau(f)}}.$$

When $\sigma(f) = 0$, we say that f is of *minimal type*. If the entire function f is of order at most one, then by Levin's result (see [1] and the references therein), the type of f is given by

$$\sigma(f) = \limsup_{n\to\infty} |f^{(n)}(0)|^{\frac{1}{n}}.$$

Let $T \in \mathcal{B}(X)$ and $x \in X$. We define $\rho_T(x)$ to be the set of $\alpha \in \mathbb{C}$ for which there exists a neighbourhood V_α of α with u analytic on V_α having values in X such that $(zI - T)u(z) = x$ on V_α. This set is open and contains the complement of the spectrum $\sigma(T)$ of T. The function u is called a local resolvent of T on V_α. By definition the *local spectrum* of T at x, denoted by $\sigma_x(T)$, is the complement of $\rho_T(x)$, so it is then a compact subset of $\sigma(T)$.

In general, the local resolvent $u(\alpha)$ is not unique. We say that T has the single-valued extension propery (in abbreviation SVEP) if

$$(zI - T)v(z) = 0 \quad \text{implies } v = 0$$

for any analytic function v defined on any domain D of $\mathbb{C}$ with values in a Banach space X. It is easy to see that an operator T having spectrum without interior points has the SVEP (for more details see [1], [2] and [5]). Further, in this case the *local spectral radius*

$$r_T(x) = \max\{|z| : z \in \sigma_T(x)\} = \limsup_{k\to\infty} \|T^k x\|^{\frac{1}{k}}.$$

1. Relationship between the commutant and the orbits of conjugation

Let $S \in \mathcal{B}(X)$ be a real scalar generalized operator in the sense of Colojoară-Foiaş [2], i.e. there exists $N > 0$ such that $\|e^{itS}\| = O(t^N)$, for every real number t. We have the following result.

Theorem 1.1. *Let S be a real scalar generalized operator. If T is in $B_{e^S} \cap B_{e^{-S}}$, then $ST - TS = 0$.*

Proof. Consider $f(z) = u(e^{zS}Te^{-zS})$, where u is a functional of norm one on a Banach space $\mathcal{B}(X)$. Since T is in $B_{e^S} \cap B_{e^{-S}}$, we have

$$\sup_{t\in\mathbb{R}} |f(t)| < \infty. \tag{2}$$

On the other hand, f is an entire function such that for $z = r + it$, we obtain $|f(z)| \le \|e^{itS}e^{rS}Te^{-rS}e^{-itS}\| \le K|z|^N$. Hence, f is a polynomial of degree N. But then

$$\lim_{|z|\to\infty} \inf |z^{-N}f(z)| > 0.$$

In particular, this holds for a real number z, so that $N = 0$ follows from (2). In other words, $f'(0) = 0$, which implies $u(ST - TS) = 0$. The result follows by applying the Hahn-Banach theorem.

For the particular case of Hermitian-equivalent operators ($N = 0$), we obtain that T is in the commutant $\{S\}'$. Related results on positive operators in Hilbert spaces can be found in [4] and [17].

Remark 1. **1.** We note that the above example of a real scalar generalized operator shows that there exists an operator A for which $B_A \cap B_{A^{-1}} = \{A\}'$ but $\sigma(A) \ne \{1\}$ (see also [4, Lemma 4(c)] and [15, Corollary 3]).

2. Theorem 1.1 cannot be extended to the case of normal operators even in Hilbert spaces. In fact, take A as a unitary operator ($A \ne \lambda I$). Then by Williams-Stampfli's characterization we have $B_A = B(H)$, which is obviously different from the commutant of A. However, for $A = U|A|$, with $|A|^2 = AA^*$, applying Fuglede-Putnam's theorem, we can obtain easily that $B_A = B_{|A|} = \{A^2\}'$.

On the other hand, in the case of Hilbert space, J. A. Deddens and T. K. Wong showed that, if $A \in \mathcal{B}(H)$ is of the form $A = \alpha I + N$ where $0 \ne \alpha \in \mathbb{C}$ and N is nilpotent, then $B_A = \{A\}'$. We present a short and elementary proof of an extension of this result in the general situation of Banach spaces. We should remark that the proof of the special case of this theorem for Hilbert space operators given in [3] depends on the special structure of Hilbert spaces and therefore cannot be extended to Banach spaces.

Theorem 1.2. *Let A be a bounded linear operator in a Banach space X. Suppose that $A = \lambda I + N$ where N is a nilpotent operator and $\lambda \in \mathbb{C}$. Then $B_A^l = \{A\}'$, where $B_A^l = \{T \in \mathcal{L}(X) : \|A^nTA^{-n}\| = o(n)\}$.*

Proof. Let $A = e^S$. Without loss of generality we may suppose that S is nilpotent of degree 2, that is, $S^2 = 0$. We then have

$$\|e^{nS}Te^{-nS}\| = \|(1 + nS)T(1 - nS)\| = \|T + n(ST - TS) - n^2STS\|. \tag{3}$$

Dividing both sides by n^2, we get

$$\frac{\|e^{nS}Te^{-nS}\|}{n^2} = \|\frac{T}{n^2} + \frac{(ST-TS)}{n} - STS\|.$$

Since T is in $B^l_{e^S}$, we obtain $STS = 0$. Therefore, the equality (2) becomes

$$\|e^{nS}Te^{-nS}\| = \|(1+nS)T(1-nS)\| = \|T + n(ST-TS)\|. \tag{4}$$

Dividing again both sides of the equality by n and using the fact that T is in $B^l_{e^S}$, we obtain $ST - TS = 0$. As we can see the proof works for any degree of nilpotency of S.

Remark 2. **1.** The reader may notice that the above proof works as well in the general situation of Banach algebras.

2. It would be interesting to know whether Theorem 1.2 can be extended to the case of algebraic operators.

Corollary 1.3. *Let $A \in \mathcal{B}(X)$ be in the form $A = \alpha I + N$ where $0 \neq \alpha \in \mathbb{C}$ and N is nilpotent. Then $B_A = \{A\}'$.*

Suppose that $T \in B_A \cap B_{A^{-1}}$. Then the local spectrum of D_A at T is included in the unit circle Γ, where $D_A(T) = ATA^{-1}$. In fact, T in B_A implies that $\sup_{n\geq 0} \|D^n_A(T)\| < \infty$. Similarly, if T is in $B_{A^{-1}}$, then $\sup_{n\geq 0} \|D^{-n}_A(T)\| < \infty$. But, the local spectral radius $r_x(T)$ is given by

$$r_T(x) = \max\{|z| : z \in Sp_T(x)\} \leq \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}}.$$

Hence, the condition $T \in B_A \cap B_{A^{-1}}$ implies that $r_{D_A}(T) \leq 1$ and $r_{D_A^{-1}}(T) \leq 1$. Therefore, by the local spectral mapping theorem (see [5] and the references therein), we get $\sigma_{D_A}(T) \subset \Gamma$.

Theorem 1.4. *Let A be an invertible operator with real spectrum $\sigma(A)$ of A. Then $T \in B_{e^A} \cap B_{e^{-A}}$ implies $TA = AT$.*

Proof. By the spectral mapping theorem, we have

$$\sigma(D_A) = \sigma(A)\sigma(A^{-1}).$$

In fact, this comes easily from the Lumer-Rosenblum theorem (see [10]) and the observation

$$e^{\Delta_A}(T) = D_{e^A}(T).$$

By the observation below Corollary 1.3, the local spectrum of D_{e^A} at T is in the unit circle. So, by hypothesis we obtain $\sigma_{D_{e^A}}(T) \subset \{1\}$. On the other hand, let $f(z) = u(e^{z\Delta_A}(T))$, where u is a functional of norm 1. Then $T \in B_{e^A} \cap B_{e^{-A}}$ implies that f is an entire function which is bounded on the real axis. By the local spectral mapping theorem, we have $\sigma_{\Delta_A}(T) \subset \{0\}$. Since A has a real spectrum, it follows from [1, Theorem 2.5 or Corollary 2.6, p. 576] that $\Delta_A = L_A - R_A$ has the single valued extension property (SVEP) and therefore, by the Dunford-Schwartz lemma, $\sigma_{\Delta_A}(T) \neq \emptyset$ for $T \neq 0$. Hence, $\sigma_{\Delta_A}(T) = \{0\}$. Consequently, the function f is of minimal type. Hence, by Bernstein's theorem, f is constant. So, $T \in \{A\}'$.

Remark 3. As the reader may notice the above proof shows more. It shows that $B_{e^A} \cap B_{e^{-A}} = \{A\}'$ when the spectrum of A has no interior points.

2. Generalized inverse and orbits of conjugation

Let $\mathcal{A}$ be a C^*-algebra. Let a be in $\mathcal{A}$. An element b of $\mathcal{A}$ is said to be a *generalized inverse* of a if

$$aba = a \quad \text{and} \quad bab = b. \tag{5}$$

We notice that the first equality in (5) is a necessary and sufficient condition for a to have a generalized inverse in $\mathcal{A}$. In this case a will be called *regular*. In general, the generalized inverse is not unique. But, if (ab) and (ba) are self-adjoint, then the generalized inverse is unique and it is called the *Moore-Penrose inverse* of a and denoted by a^+. We recall here an important property of regular elements:

$$a \in \mathcal{A} \text{ is regular if and only if } a\mathcal{A} \text{ is closed.} \tag{6}$$

For $a \in \mathcal{A}$, we define

$$\gamma(a) = \inf \sigma(|a|) \setminus \{0\}, \quad \text{with } |a|^2 = aa^*.$$

$\gamma(a)$ is the so-called reduced minimum modulus of a. This definition is equivalent to the one introduced by Kato for operators in Banach spaces (see [9], Chapter IV). We recall here two important properties of $\gamma(a)$ (see [8] and [12] for more details):

$$\gamma(a) > 0 \quad \text{if and only if} \quad a\mathcal{A} \text{ is closed;} \tag{7}$$

$$\gamma(a) = \frac{1}{\|a^+\|}. \tag{8}$$

We denote

$$\mathcal{A}^{-1} = \{x \in \mathcal{A} : x^{-1} \in \mathcal{A}\},$$
$$\mathcal{S}(a) = \{w^{-1}aw : w \in \mathcal{A}^{-1}\},$$
$$\Gamma_\delta = \{x \in \mathcal{A} : \gamma(x) \geq \delta\}.$$

Let $\mathcal{S}_M(a) = \{w^{-1}aw : w \in \mathcal{A}^{-1} \text{ and } M(w) \leq M\}$, where $M(w) = \|w\|\|w^{-1}\|$.

Lemma 2.1. *The set of regular elements is stable by similarity.*

Proof. Straightforward.

Lemma 2.2. *Assume that $a' \in \mathcal{S}_M(a)$. Then*

$$\frac{1}{M}\gamma(a) \leq \gamma(a') \leq M\gamma(a). \tag{9}$$

Proof. It follows from [8, Theorem 2] that $\gamma(a) > 0$ if and only if a is regular. From Lemma 2.1, we have $\gamma(a) = 0$ if and only if $\gamma(a') = 0$. Therefore, we may suppose that $\gamma(a) > 0$. In this case there exists $a^+ \in \mathcal{A}$ such that $w^{-1}a^+w$ is a generalized inverse of a', by Lemma 2.1. Using again [8, Theorem 2], we obtain

$$\begin{aligned} \frac{1}{\gamma(a')} &= \|(w^{-1}aw)^+\| \leq \|w^{-1}a^+w\| \\ &\leq M(w)\|a^+\| = M(w).\frac{1}{\gamma(a)}. \end{aligned}$$

So, $\frac{1}{M(w)}\gamma(a) \leq \gamma(a')$. In a similar way by taking $a = wa'w^{-1}$, we obtain $\frac{1}{M(w)}\gamma(a')$ $\leq \gamma(a)$, which implies $\gamma(a') \leq M(w)\gamma(a)$.

For $a \in \mathcal{A}$, a generalization of the spectrum $\sigma(a)$ is given by replacing the notion of invertibility which appears in the classical definition of the spectrum by the existence of an analytic generalized inverse. More precisely, we denote by $reg(a)$ the *regular set* of a, defined by

$$\begin{aligned} reg(a) \quad := \quad & \{\lambda \in \mathbb{C} : \text{ there is a neighborhood } U_\lambda \text{ of } \lambda \text{ and an analytic} \\ & \text{function } b : U_\lambda \longrightarrow \mathcal{A}, \text{ such that } b(\mu) \text{ is a generalized} \\ & \text{inverse of } a - \mu I \text{ for any } \mu \in U_\lambda\}. \end{aligned}$$

The complement $\sigma_{\mathrm{g}}(a) = \mathbb{C} \setminus reg(a)$ of $reg(a)$ in $\mathbb{C}$ is called the *generalized spectrum* of a (see [11]).

Remark 4. **1.** $\partial\sigma(a) \subseteq \sigma_{\mathrm{g}}(a) \subseteq \sigma(a)$;

2. $\sigma_{\mathrm{g}}(a^*) = \overline{\sigma_{\mathrm{g}}(a)}$;

3. if f is an analytic function on some neighbourhood of the spectrum of a, then

$$\sigma_{\mathrm{g}}(f(a)) = f(\sigma_{\mathrm{g}}(a)).$$

Our main result in this section is as follows.

Theorem 2.3. *Assume that $a' \in \overline{\mathcal{S}_M(a)}$. Then*

(i) $\frac{1}{M}\gamma(a) \leq \gamma(a') \leq M\gamma(a)$;

(ii) *a is regular if and only if a' is regular;*

(iii) $\sigma_g(a') = \sigma_g(a)$.

Proof. Let $a' \in \overline{\mathcal{S}_M(a)}$. Then there exist a sequence $\{w_n\}$ in $\mathcal{A}^{-1}$ such that for every $n \geq 0$, we have $\|w_n\|\|w_n^{-1}\| \leq M$ and $w_n^{-1}aw_n \to a'$. Applying Lemma 2.2 to $a' = w_n^{-1}aw_n$, we obtain for every $n \geq 0$,

$$\frac{1}{M}\gamma(a) \leq \gamma(w_n^{-1}aw_n) \leq M\gamma(a).$$

So, if $\gamma(a) > 0$, then $(w_n^{-1}aw_n)_{n\geq 0} \subset \Gamma_{\frac{\gamma(a)}{M}}$, or by [8, (7.2)], $\Gamma_{\frac{\gamma(a)}{M}}$ is a closed set and therefore $\gamma(a') \geq \frac{\gamma(a)}{M}$.

Similarly, we obtain $w_n a' w_n^{-1} \to a$. So, $\gamma(a) \geq \frac{\gamma(a')}{M}$. Hence, the inequality (i) is shown in the case of $\gamma(a) > 0$.

Suppose that $\gamma(a) = 0$ and $\gamma(a') > 0$. Then $w_n a' w_n^{-1} \to a$ and as above $\gamma(a) \geq \frac{\gamma(a')}{M} > 0$, which is a contradiction.

(ii) This is a consequence of (i) since

$$\gamma(a) > 0 \text{ if and only if } \gamma(a') > 0$$

and

$$\gamma(x) > 0 \text{ if and only if } x \text{ is regular.}$$

(iii) Note that $\sigma_g(a) = \sigma_g(L_a)$, where $L_a : \mathcal{A} \to \mathcal{A}$ is the left-multiplication operator defined by $L_a(x) = ax$ for all $x \in \mathcal{A}$. Then $\gamma(a) = \gamma(L_a)$ (see [8] and [12]). Applying [13, Theorem 4.1], we obtain

$$\sigma_g(L_a) = \{z : \lim_{w \to z} \gamma(L_a - w) > 0\}.$$

Now we apply (i) to get the conclusion.

Let $\{a\}^c = \{x \in \mathcal{A} : \text{ there exists } y \in \mathcal{A}, \Delta_{x,y}(a) = 0\}$, where $\Delta_{x,y}(a) = ax - ya$ is the generalized derivation. Let $B_a^l = \{x \in \mathcal{A} : \|a^n x a^{-n}\| = o(n), \text{ as } n \to \infty\}$. Let $b \in \mathcal{A}$ be such that $aba = a$. Then it is easy to see is that $(ab)^2 = ab$ and $(ba)^2 = ba$. Also easy to see is that $\{a\}' \subset \{a\}^c$. Our main concern here is to

investigate the relationship between the algebras B_{ba+1}, B_{ab+1}, and $\{a\}^c$, given that b is the generalized inverse of a.

Theorem 2.4. *For $a \in \mathcal{A}, B_{ba+1} = \{a\}^c$.*

Proof. For the inclusion $B_{ba+1} \subset \{a\}^c$.

Step 1: From $(ba)^2 = ba$, we obtain

$$(1+ba)^n = 1 + ba(\sum_{k=1}^{n} alpha_k), \quad \text{with } \sum_{k=1}^{n} \alpha_k = 2^n - 1,$$

and

$$(1+ba)^{-n} = 1 + ba((\sum_{k=1}^{n} \beta_k), \quad \text{with } \sum_{k=1}^{n} \beta_k = 2^{-n} - 1.$$

Therefore,

$$(1+ba)^n x(1+ba)^{-n} = (1+(2^n-1)ba)x(1+(2^{-n}-1)ba).$$

So, if $x \in B_{1+ba}$, then we obtain

$$\sup_{n\geq 0} 2^n \|bax - baxba\| < \infty.$$

Hence, $bax = baxba$.

Step 2: Since $aba = a$,we obtain from the above relationship that $ax(1-ba) = 0$. So, there exists $y = axb$ such that $ax - ya = 0$, i.e. $x \in \{a\}^c$.

Let $x \in \{a\}^c$. Then there exists $y \in \mathcal{A}$ such that $ax - ya = 0$. Hence $axba = ax$, which implies $baxba = bax$. In a similar way as in step 1, we obtain

$$(1+ba)^n x(1+ba)^{-n} = (1+(2^n-1)ba)x(1+(2^{-n}-1)ba).$$

It is easy to see that the condition $baxba - bax = 0$ implies

$$\sup_{n\geq 0} \|(1+ba)^n x(1+ba)^{-n}\| < \infty.$$

Remark 5. As the reader may notice the above proof shows more. It shows that for any complex number λ, for which $ba + \lambda$ is invertible, we have $B_{ba+\lambda} = \{a\}^c$ if $aba = a$.

In [6, Theorem 4], it was shown that if T and S are two bounded linear operators in a Banach space X, then $T \in B_{e^S} \cap B_{e^{-S}}$ implies

$$\|e^S T e^{-S} - T\| \leq 2\tan(\frac{r(\Delta_S(T))}{2})C,$$

where $C = \sup_{n\geq 0} \|e^{nS} T e^{-nS}\| < \infty$. Using the so-called Kleineke-Shirokov Theorem on commutators, we obtain that $B_{e^S} \cap B_{e^{-S}} \cap \{S\}'' = \{S\}'$, for any $S \in \mathcal{L}(X)$, with $r(\Delta_S(T)) < 2\pi$, where $\{S\}'' = \{T \in \mathcal{L}(X) : S\Delta_S(T) = \Delta_S(T)S\}$.

In this paper we obtain the following extension.

Lemma 2.5. *Let a be in $\mathcal{A}$. Then $B^l_{a-\lambda} \cap \{a\}'' = \{a\}'$, for any $\lambda \in \mathbb{C} \setminus \sigma(a)$.*

Proof. It is easy to see that for $x \in \mathcal{A}$,

$$(*) \qquad (a-\lambda)x(a-\lambda)^{-1} = x - \Delta_a(x)(a-\lambda)^{-1}.$$

So, if $x \in \{a\}''$, then using $(*)$,

$$\begin{aligned}(a-\lambda)^2 x(a-\lambda)^{-2} &= (a-\lambda)[x - \Delta_a(x)(a-\lambda)^{-1}](a-\lambda)^{-1} \\ &= x - 2\Delta_a(x)(a-\lambda)^{-1}.\end{aligned}$$

By induction, we get

$$(a-\lambda)^n x(a-\lambda)^{-n} = x - n\Delta_a(x)(a-\lambda)^{-1}.$$

Therefore, if $x \in B^l_{a-\lambda}$, then the above equality implies that $\Delta_a(x) = 0$, i.e. $x \in \{a\}'$. □

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