MAULDIN-WILLIAMS GRAPHS, MORITA EQUIVALENCE AND ISOMORPHISMS

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Abstract. We describe a method for associating some non-self-adjoint algebras to Mauldin-Williams graphs and we study the Morita equivalence and isomorphism of these algebras.

We also investigate the relationship between the Morita equivalence and isomorphism class of the $C^*$-correspondences associated with Mauldin-Williams graphs and the dynamical properties of the Mauldin-Williams graphs.

1. Introduction

In this note we follow the notation from [8]. By a Mauldin-Williams graph (see [14]), we mean a system $G = (G, \{T_v, \rho_v\}_{v \in V}, \{\phi_e\}_{e \in E})$, where $G = (V, E, r, s)$ is a graph with a finite set of vertices $V$, a finite set of edges $E$, a range map $r$ and a source map $s$, and where $\{T_v, \rho_v\}_{v \in V}$ and $\{\phi_e\}_{e \in E}$ are families such that:

1. Each $T_v$ is a compact metric space with a prescribed metric $\rho_v$, $v \in V$.

2. For $e \in E$, $\phi_e$ is a continuous map from $T_{r(e)}$ to $T_{s(e)}$ such that

$$c_1 \rho_{r(e)}(x, y) \leq \rho_{s(e)}(\phi_e(x), \phi_e(y)) \leq c \rho_{r(e)}(x, y)$$

for some constants $c_1, c$ satisfying $0 < c_1 \leq c < 1$ (independent of $e$) and all $x, y \in T_{r(e)}$.

We shall assume, too, that the source map $s$ and the range map $r$ are surjective. Thus, we assume that there are no sinks and no sources in the graph $G$.

In [8] we associated to a Mauldin-Williams graph $G = (G, \{T_v, \rho_v\}_{v \in V}, \{\phi_e\}_{e \in E})$ a so-called $C^*$-correspondence $X$ over the $C^*$-algebra $A = C(T)$, where $T = \bigsqcup_{v \in V} T_v$ is the disjoint union of the spaces $T_v, v \in V$, as follows. Let $E \times_G T = \{(e, x) | x \in T_{r(e)}\}$. Then, by our finiteness assumptions, $E \times_G T$ is a compact space. We set $X = C(E \times_G T)$ and view $X$ as a $C^*$-correspondence over $C(T)$ via the formulae:

$$\xi \cdot a(e, x) := \xi(e, x)a(x),$$

$$a \cdot \xi(e, x) := a \circ \phi_e(x)\xi(e, x)$$

and

$$\langle \xi, \eta \rangle_A(x) := \sum_{e \in E} \overline{\xi(e, x)}\eta(e, x),$$

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where $a \in C(T)$ and $\xi, \eta \in C(E \times G T)$. With these data we can form the tensor algebra $T_+(\mathcal{X})$ as prescribed in [15] and [16]. Our main result is:

**Theorem 1.1.** For $i = 1, 2$, let $G_i = (G_i, (K^v_i)_{v \in V}, (\phi_e^i)_{e \in E_i})$ be two Mauldin-Williams graphs. Let $A_i = C(K^v_i)$ and let $\mathcal{X}_i$ be the associated $C^*$-algebras and $C^*$-correspondences. Then the following are equivalent:

1. $T_+(\mathcal{X}_1)$ is strongly Morita equivalent to $T_+(\mathcal{X}_2)$ in the sense of [2].
2. $\mathcal{X}_1$ and $\mathcal{X}_2$ are strongly Morita equivalent in the sense of [16].
3. $\mathcal{X}_1$ and $\mathcal{X}_2$ are isomorphic as $C^*$-correspondences.
4. $T_+(\mathcal{X}_1)$ is completely isometrically isomorphic to $T_+(\mathcal{X}_2)$.

We find this result especially remarkable in light of Theorem 1.1 from [8] (see also Section 4.2 from [18]), which states that the Cuntz-Pimsner algebra, $\mathcal{O}(\mathcal{X})$, which is the $C^*$-envelope of the tensor algebra $T_+(\mathcal{X})$, depends only of the structure of the underlying graph. In particular, our results lead to examples of different non-self-adjoint algebras which are not completely isometrically isomorphic, but have the same $C^*$-envelope, namely $\mathcal{O}_n$.

To understand further the relationship between the tensor algebra and the Mauldin-Williams graph, we study the isomorphism class of our $C^*$-algebras and tensor algebras in terms of the dynamics of the Mauldin-Williams graph. Roughly, we find that two $C^*$-correspondences associated to two Mauldin-Williams graphs, $(G_i, (K^v_i)_{v \in V}, (\phi_e^i)_{e \in E_i}), i = 1, 2$, are isomorphic if the maps $\phi_e^1$ and $\phi_e^2$ are locally conjugate in a sense that will be made precise later.

2. **Non-self-adjoint algebras associated with Mauldin-Williams graphs**

**Definition 2.1.** An invariant list associated with a Mauldin-Williams graph $G = (G, \{T_v\}_{v \in V}, \{\phi_e\}_{e \in E})$ is a family $(K_v)_{v \in V}$ of compact sets, such that $K_v \subset T_v$, for all $v \in V$ and such that

$$K_v = \bigcup_{(e, v, e) \in E, (e) = v} \phi_e(K_{r(e)}).$$

Since each $\phi_e$ is a proper contraction, $G$ has a unique invariant list (see [13] Theorem 1). We set $T := \bigcup_{v \in V} T_v$ and $K := \bigcup_{v \in V} K_v$, and we call $K$ the invariant set of the Mauldin-Williams graph.

In the particular case when we have one vertex $v$ and $n$ edges, i.e. in the setting of an iterated function system, the invariant set is the unique compact subset $K := K_v$ of $T = T_v$, such that

$$K = \phi_1(K) \cup \cdots \cup \phi_n(K).$$

Note that the $*$-homomorphism $\Phi : A \rightarrow \mathcal{L}(\mathcal{X})$, $(\Phi(a)\xi)(e, x) = a \circ \phi_e(x)\xi(e, x)$, which gives the left action of the $C^*$-correspondence associated to a Mauldin-Williams graph, is faithful if and only if $K = T$. In this note we assume that $T$ equals the invariant set $K$.

Kajiwara and Watatani have proved in [10] Lemma 2.3 that, if the contractions are proper, the invariant set of an iterated function system has no isolated points. Their proof can be easily generalized to the invariant set of a Mauldin-Williams graph. Hence $K$ has no isolated points.

For a $C^*$-correspondence $\mathcal{X}$ over a $C^*$-algebra $A$, the (full) Fock space over $\mathcal{X}$ is

$$\mathcal{F}(\mathcal{X}) = A \oplus \mathcal{X} \oplus \mathcal{X}^\otimes 2 \oplus \cdots.$$
We write $\Phi_\infty$ for the left action of $A$ on $\mathcal{F}(\mathcal{X})$, $\Phi_\infty(a) = \text{diag}(a, \Phi^{(1)}(a), \Phi^{(2)}(a), \cdots)$, where $\Phi^{(n)}$ is the left action of $A$ on $\mathcal{X}^\otimes n$ $(\Phi^{(1)} = \Phi$, the left action of $A$ on $\mathcal{X})$. For $\xi \in \mathcal{X}$, the creation operator determined by $\xi$ is defined by the formula $T_\xi(\eta) = \xi \otimes \eta$, for all $\eta \in \mathcal{F}(\mathcal{X})$.

**Definition 2.2.** The tensor algebra of $\mathcal{X}$, denoted by $\mathcal{T}_+(\mathcal{X})$, is the norm closed subalgebra of $\mathcal{L}(\mathcal{F}(\mathcal{X}))$ generated by $\Phi_\infty(A)$ and the creation operators $T_\xi$, for $\xi \in \mathcal{X}$ (see [15] and [16]). The $C^\ast$-algebra generated by $\mathcal{T}_+(\mathcal{X})$ is denoted by $\mathcal{T}(\mathcal{X})$ and it is called the Toeplitz algebra of the $C^\ast$-correspondence $\mathcal{X}$.

We may regard each finite sum $\sum_{n=0}^{N} \mathcal{X}^\otimes n$ as a subspace of $\mathcal{F}(\mathcal{X})$ and we may regard $\mathcal{L}(\sum_{n=0}^{N} \mathcal{X}^\otimes n)$ as a subalgebra of $\mathcal{L}(\mathcal{F}(\mathcal{X}))$ in the obvious way. Let $B$ be the $C^\ast$-subalgebra of $\mathcal{L}(\mathcal{F}(\mathcal{X}))$ generated by all the $\mathcal{L}(\sum_{n=0}^{N} \mathcal{X}^\otimes n)$ as $N$ ranges over the non-negative integers. Then $\mathcal{T}(\mathcal{X}) \subset M(B)$, the multiplier algebra of $B$. The Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{X})$ is defined to be the image of $\mathcal{T}(\mathcal{X})$ in the corona algebra $M(B)/B$ (see [15] and [17]).

By a homomorphism from an $A_1 - B_1$ $C^\ast$-correspondence $\mathcal{X}_1$, to an $A_2 - B_2$ $C^\ast$-correspondence $\mathcal{X}_2$ we mean a triple $(\alpha, V, \beta)$, where $\alpha : A_1 \to A_2$, $\beta : B_1 \to B_2$ are $C^\ast$-homomorphisms and $V : \mathcal{X}_1 \to \mathcal{X}_2$ is a linear map such that $V(a'b) = \alpha(a)V(\xi)\beta(b)$ and such that $V(\langle \xi, \eta \rangle)_{B_2} = \beta(\langle \xi, \eta \rangle)_{B_1}$ (see [16] Section 1). When $A_1 = A_2$ and $B_1 = B_2$, we will consider $\alpha \in \text{Aut}(A_1)$ and $\beta \in \text{Aut}(B_1)$. This then forces $V$ to be isometric. If $V$ is also surjective, we shall say that $V$ is a correspondence isomorphism over $(\alpha, \beta)$. If, moreover, $A_1 = B_1$ and $\alpha = \beta$, we say that $V$ is a correspondence isomorphism over $\alpha$.

A central concept for our work in this note is the strong Morita equivalence for $C^\ast$-correspondences defined in [16] Definition 2.1, which we review here.

**Definition 2.3.** If $\mathcal{X}$ is a $C^\ast$-correspondence over a $C^\ast$-algebra $A$, and \( \mathcal{Y} \) is a $C^\ast$-correspondence over a $C^\ast$-algebra $B$, we say that $\mathcal{X}$ and $\mathcal{Y}$ are strongly $\text{Morita equivalent}$ if $A$ and $B$ are strongly Morita equivalent via an $A-B$ equivalence bimodule $\mathcal{Z}$ (in which case we write $A \simeq_{\text{SM}} B$), for which there is an $A-B$ correspondence isomorphism $(id, W, id)$ from $\mathcal{Z} \otimes_B \mathcal{Y}$ onto $\mathcal{X} \otimes_A \mathcal{Z}$. This means, in particular, that $W(a\xi b) = aW(\xi)b$ for all $a \in A, b \in B$ and $\xi \in \mathcal{Z} \otimes_B \mathcal{Y}$ and that $\langle W(\xi), W(\eta) \rangle_B = \langle \xi, \eta \rangle_B$.

We say that a $C^\ast$-correspondence $\mathcal{X}$ over a $C^\ast$-algebra $A$ is aperiodic if: for all $n \geq 1$, for all $\xi \in \mathcal{X}^\otimes n$ and for all hereditary subalgebras $B \subseteq A$, we have
\[
\inf \left\{ \| (\Phi^{(n)}(a)\xi a) \| \mid a \geq 0, a \in B, \|a\| = 1 \right\} = 0.
\]

It was proved in [16] Theorem 3.2, Theorem 3.5 that if $\mathcal{X}$ and $\mathcal{Y}$ are strongly Morita equivalent, then $\mathcal{T}_+(\mathcal{X})$ and $\mathcal{T}_+(\mathcal{Y})$ (respectively $\mathcal{T}(\mathcal{X})$ and $\mathcal{T}(\mathcal{Y})$, $\mathcal{O}(\mathcal{X})$ and $\mathcal{O}(\mathcal{Y})$) are strongly Morita equivalent. Also, if $\mathcal{X}$ and $\mathcal{Y}$ are aperiodic $C^\ast$-correspondences over the $C^\ast$-algebras $A$ and $B$, respectively, and if $\mathcal{T}_+(\mathcal{X})$ and $\mathcal{T}_+(\mathcal{Y})$ are strongly Morita equivalent in the sense of [2], then $\mathcal{X}$ and $\mathcal{Y}$ are strongly Morita equivalent (see [16] Theorem 7.2).

To study the aperiodicity and strong Morita equivalence of $C^\ast$-correspondences associated to Mauldin-Williams graphs, we need the following lemma which gives an equivalent description of when a $C^\ast$-correspondence is aperiodic.

**Lemma 2.4 (16 Lemma 5.2).** The $C^\ast$-correspondence $\mathcal{X}$ is aperiodic if and only if given $a_0 \in A$, $a_0 \geq 0$, $\xi^k \in \mathcal{X}^\otimes k$, $1 \leq k \leq n$, and $\varepsilon > 0$, there is an $x$ in the
hereditary subalgebra \( \overline{a_0Aa_0} \), with \( x \geq 0 \) and \( \|x\| = 1 \), such that

\[
\|xa_0x\| > \|a_0\| - \varepsilon
\]

and

\[
\|\Phi^{(k)}(x)x^k\| < \varepsilon \text{ for } 1 \leq k \leq n.
\]

For a directed graph \( G = (V,E,r,s) \) and for \( k \geq 2 \), we define

\[
E^k := \{ \alpha = (\alpha_1,\cdots,\alpha_k) : \alpha_i \in E \text{ and } r(\alpha_i) = s(\alpha_{i+1}), i = 1,\cdots,k-1 \}
\]

to be the set of paths of length \( k \) in the graph \( G \). We define also the infinite path space to be

\[
E^\infty := \{(\alpha_i)_{i \in \mathbb{N}} : \alpha_i \in E \text{ and } r(\alpha_i) = s(\alpha_{i+1}) \text{ for all } i \in \mathbb{N}\}.
\]

For \( \alpha \in E^k \), we write \( \phi_\alpha = \phi_{\alpha_1} \circ \cdots \circ \phi_{\alpha_k} \).

**Proposition 2.5.** Let \( G = (G,(K_r)_{r \in V},(\phi_e)_{e \in E}) \) be a Mauldin-Williams graph with the invariant set \( K \). Let \( A = C(K) \) be the associated \( C^* \)-algebra and let \( \mathcal{X} \) be the associated \( C^* \)-correspondence. Then the \( C^* \)-correspondence \( \mathcal{X} \) is aperiodic.

**Proof.** Note that \( \phi_\alpha : K_{r(\alpha)} \to K_{s(\alpha)} \), with \( \alpha \in E^k \) and \( k \in \mathbb{N} \), has a fixed point if and only if \( r(\alpha) = s(\alpha) \), i.e. \( \alpha \) is a cycle in the graph \( G \).

Fix \( n_0 \in \mathbb{N} \), choose \( k \in \mathbb{N}, 1 \leq k \leq n_0 \); let \( a_0 \in A \) with \( a_0 \geq 0 \); let \( \xi^k \in \mathcal{X}^{\otimes k} \) and let \( \varepsilon > 0 \). We verify the criterion in Lemma 2.4 first when \( n_0 = k = 1 \).

Without loss of generality, we assume that \( \|a_0\| = 1 \). Then we can find \( t_0 \in K \) such that \( |a_0(t_0)| \geq 1 - \varepsilon \) and \( t_0 \) is not a fixed point for any \( \phi_e, e \in E \). Let \( v_0 \in V \) be such that \( t_0 \in K_{v_0} \). Choose \( \delta_1 > 0 \) such that \( B(t_0,\delta_1) \subset K_{v_0} \) and \( B(\phi_e(t_0),\delta_1) \cap B(t_0,\delta_1) = \emptyset \) for all \( e \in E \) for which \( r(e) = v_0 \). Let

\[
\delta_2 := \begin{cases} 
\min\{\rho_{v_0}(t_0,t) \mid a_0(t) = 0\}, & \text{if } \{t \in K_{v_0} : a_0(t) = 0\} \neq \emptyset, \\
\delta_1, & \text{otherwise}.
\end{cases}
\]

Set \( \delta = \min\{\delta_1,\delta_2\} \) and let \( x \in A, x \geq 0 \) be such that

\[
x(t) = \begin{cases} 
1, & \text{if } t = t_0, \\
0, & \text{if } t \in K \setminus B(t_0,\delta).
\end{cases}
\]

Since \( x(t) > 0 \) only when \( a_0(t) > 0 \), it follows that \( x \in \overline{a_0Aa_0} \). Moreover, \( x(t_0)a_0(t_0)x(t_0) > 1 - \varepsilon \), hence \( \|xa_0x\| > 1 - \varepsilon \).

Fix \( t \in K \). If \( t \in B(t_0,\delta) \), then \( \phi_e(t) \notin B(t_0,\delta) \), by our choice of \( \delta_1 \) and the fact that each map \( \phi_e \) is a contraction, for all \( e \in E \) such that \( r(e) = v_0 \); so \( x \circ \phi_e(t)x(t) = 0 \). If \( t \notin B(t_0,\delta) \), then \( x(t) = 0 \), hence \( x \circ \phi_e(t)x(t) = 0 \), for all \( e \in E \) such that \( t \in K_{r(e)} \). Therefore, \( \langle \Phi(x)x, \Phi(x)x \rangle_A(t) = \sum_{e \in E} (x \circ \phi_e(t))^2 |\xi(e,t)|^2 x(t)^2 \), we see that \( \|\Phi(x)x\| = 0 \).

For \( n_0 = 2 \), we choose \( t_0 \in K \) such that \( a_0(t_0) > 1 - \varepsilon \) and \( t_0 \) is not a fixed point for any \( \phi_\alpha \) with \( \alpha \in E^2 \). Let \( v_0 \in V \) be such that \( t_0 \in K_{v_0} \). Let \( \delta_1 > 0 \) be such
that \( B(\phi_\alpha(t_0), \delta_1) \cap B(t_0, \delta_1) = \emptyset \), for all \( \alpha \in E^2 \) for which \( r(\alpha) = \nu_0 \), and such that \( B(t_0, \delta_1) \subset K_{\nu_1} \). Choosing \( \delta_2, \delta \) and \( x \) as before, we conclude that \( x \in \overline{a_0Aa_0} \) and \( \|x\| > 1 - \varepsilon \). Moreover, we have \( x \circ \phi_\alpha(t) = 0 \) for all \( t \in K, \alpha \in E \cup E^2 \) (since \( \phi_\alpha \) is a contraction, for all \( \alpha \in E \cup E^2 \)); and since

\[
\left\langle \Phi^{(2)}(x)x, \Phi^{(2)}(x)x \right\rangle_A(t) = \sum_{\alpha \in K^2} (x \circ \phi_\alpha(t))^2 \left| \xi_2^2(\alpha_1, \phi_\alpha(t)) \right|^2 x(t)^2 = 0,
\]

it follows that \( \|\Phi^{(k)}(x)x\| = 0 \) for \( k = 1, 2 \). Applying the same argument inductively, we see that \( \mathcal{X} \) is an aperiodic \( C^* \)-correspondence.

Let \( K^1 \) and \( K^2 \) be two compact metric spaces. Let \( A_1 = C(K^1) \) and \( A_2 = C(K^2) \). If \( A_1 \sim_{\text{SMIE}} A_2 \), then the Rieffel correspondence determines a unique homeomorphism \( f : K^1 \to K^2 \) and a unique Hermitian line bundle \( \mathcal{L} \) over \( \text{Graph}(f) = \{(x, f(x)) : x \in K^1\} \), such that \( \mathcal{Z} \) is isomorphic to \( \Gamma(\mathcal{L}) \) (see \cite{Ma99}, Section 3.3 and Example 4.55), \cite{Ma06} Appendix (A)), where \( \Gamma(\mathcal{L}) \) is the imprimitivity bimodule of the cross sections of \( \mathcal{L} \) endowed with the following structure:

\[
(a \cdot s \cdot b)(x, f(x)) = a(x)s(x, f(x))b(f(x)),
\]

\[
\langle s_1, s_2 \rangle_{A_2}(y) = s_1(f^{-1}(y), y)s_2(f^{-1}(y), y),
\]

\[
\alpha, \langle s_1, s_2 \rangle(x) = s_1(x, f(x))s_2(x, f(x)),
\]

for all \( a \in A_1, b \in A_2, s, s_1, s_2 \in \Gamma(\mathcal{L}) \). We write \( \mathcal{Z}(f, \mathcal{L}) \) for \( \Gamma(\mathcal{L}) \).

We are ready to prove the main theorem.

**Proof of Theorem.** By Proposition \ref{prop:1}, \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) are aperiodic \( C^* \)-correspondences. Using \cite{Ma06} Theorem 7.2, we obtain that \ref{prop:1} implies \ref{prop:2}.

Now we show that \ref{prop:2} implies \ref{prop:3}. Suppose that \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) are strongly Morita equivalent. This implies that \( A_1 \) and \( A_2 \) are strongly Morita equivalent via an imprimitivity bimodule \( \mathcal{Z} \) such that \( \mathcal{Z} \otimes \mathcal{X}_2 \) is isomorphic to \( \mathcal{X}_1 \otimes \mathcal{Z} \). Let \( f : K^1 \to K^2 \) and \( \mathcal{L} \) be the homeomorphism and the line bundle determined by the Rieffel correspondence. We have that \( \mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2 \) is isomorphic to \( \mathcal{X}_1 \otimes \mathcal{Z}(f, \mathcal{L}) \). Hence \( \mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2 \otimes \mathcal{Z}(f, \mathcal{L}) \) is isomorphic to \( \mathcal{X}_1 \), where \( \mathcal{Z}(f, \mathcal{L}) \) is the dual imprimitivity bimodule (see \cite{Ma99} Proposition 3.18). We prove that \( \mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2 \otimes \mathcal{Z}(f, \mathcal{L}) \) is isomorphic to \( \mathcal{X}_2 \) over an isomorphism \( \alpha \) of \( A_1 \) and \( A_2 \).

Let \( \alpha : A_1 \to A_2 \) be defined by the formula \( \alpha(a) = a \circ f^{-1} \) and let \( V : \mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2 \otimes \mathcal{Z}(f, \mathcal{L}) \to \mathcal{X}_2 \) be defined by the formula

\[
V(s_1 \otimes \xi \otimes s_2)(e, y) = s_1(f^{-1}(\phi_\xi^2(y)), \phi_\xi^2(y))\xi(e, x)s_2(f^{-1}(y), y).
\]

Then \( \alpha \) is an isomorphism and

\[
V(a \cdot s_1 \otimes \xi \otimes s_2 \cdot b) = a \cdot V(s_1 \otimes \xi \otimes s_2) \cdot b,
\]
for all \(a, b \in A, s_1, s_2 \in \mathcal{Z}(f, L), \xi \in \mathcal{X}_2\). Moreover, we have that
\[
\langle V(s_1 \otimes \xi \otimes s_2), V(t_1 \otimes \eta \otimes t_2) \rangle_{A_2}(y) = \sum_{e \in E, y \in K_E^{(e)}} \overline{V(s_1 \otimes \xi \otimes s_2)(e, y)V(t_1 \otimes \eta \otimes t_2)(e, y)}
\]
\[
= \sum_{e \in E, y \in K_E^{(e)}} \left( s_1(f^{-1}T_2^2(y)), T_2^2(y))\xi(e, x)s_2(f^{-1}T_2(y), y) \\
+ t_1(f^{-1}T_2^2(y)), T_2^2(y))\eta(e, x)t_2(f^{-1}T_2(y), y) \right)
\]
\[
= \langle s_1 \otimes \xi \otimes s_2, t_1 \otimes \eta \otimes t_2 \rangle_{A_2},
\]
for all \(s_1, s_2, t_1, t_2 \in \mathcal{Z}(f, L)\) and \(\xi, \eta \in \mathcal{X}_2\). Also, for \(\xi \in \mathcal{X}_2, V(1 \otimes \xi \otimes 1) = \xi\). Hence \(V\) is a correspondence isomorphism. Thus \(A_1\) is isomorphic to \(A_2\).

The rest is clear.

It was shown in [8, Theorem 2.3] that the Cuntz-Pimsner algebra of the \(C^*\)-correspondence built from a Mauldin-Williams graph is isomorphic to the Cuntz-Krieger algebra of the underlying graph \(G = (V, E, r, s)\) (as defined in [12]). Hence, for \(C^*\)-correspondences associated to Mauldin-Williams graphs with the same underlying graph which are not isomorphic, we obtain tensor algebras which are not Morita equivalent, but have the same \(C^*\)-envelope, namely the Cuntz-Krieger algebra of the graph \(G\).

3. THE ISOMORPHISM CLASS OF THE \(C^*\)-CORRESPONDENCES ASSOCIATED WITH MAULDIN-WILLIAMS GRAPHS

In the following we analyze the relation between the isomorphism class of the \(C^*\)-correspondences associated with two Mauldin-Williams graphs, \(\mathcal{G}_i = (G_i, (K_i^e)_{e \in V}, (\phi_i^e)_{e \in E}), i = 1, 2\), and the topological and dynamical properties of the Mauldin-Williams graphs.

Since, by [18, Section 4.] and [8, Theorem 2.3], the Cuntz-Pimsner algebra associated to a Mauldin-Williams graph depends only on the structure of the underlying graph \(G\), we will consider only Mauldin-Williams graphs having the same underlying graph \(G = (V, E, r, s)\).

Next we determine necessary and sufficient conditions for the isomorphism of the \(C^*\)-correspondences associated to two Mauldin-Williams graphs.

Proposition 3.1. For \(i = 1, 2\), let \(\mathcal{G}_i = (G_i, (K_i^e)_{e \in V}, (\phi_i^e)_{e \in E})\) be two Mauldin-Williams graphs over the same underlying graph \(G\). Let \(A_i = C(K_i), i = 1, 2\), be the associated \(C^*\)-algebras and let \(\mathcal{X}_i, i = 1, 2\), be the associated \(C^*\)-correspondences. If there is a homeomorphism \(f : K_1 \to K_2\), a partition of open subsets \(\{U_1, \ldots, U_m\}\) for \(K_1\), for some \(m \in \mathbb{N}\), and if for each \(U_j\) there is a permutation \(\sigma_j \in S_n\), where \(n = |E|\), such that \(f^{-1} \circ \phi_j^e \circ f|_{U_j} = \phi_i^e|_{U_j}\) and \(f(K_1^e) = K_2^e|_{\sigma_j(e)}\) for all \(e \in E, j \in \{1, \ldots, m\}\), then \(\mathcal{X}_1\) and \(\mathcal{X}_2\) are isomorphic.

Proof. Since \(f\) is a homeomorphism, the map \(\beta : A_2 \to A_1\), defined by the equation \(\beta(b) = b \circ f\) for all \(b \in A_2\), is a \(C^*\)-isomorphism. Define \(V : \mathcal{X}_2 \to \mathcal{X}_1\) by the formula
\[
V(\xi)(e, x) = \sum_{k=1}^m \xi_{\sigma_k(e)}(f(x)) \cdot 1_{U_k}(x),
\]
for all \((e, x) \in E \times_G K\), where \(\xi_{\sigma_k(e)}(f(x)) := \xi(\sigma_k(e), f(x))\). We show that \(V\) is a \(C^*\)-correspondence isomorphism over \(\beta\). Let \(b_1, b_2 \in A_2\) and \(\xi \in X_2\). We have

\[
V(b_1 \cdot \xi \cdot b_2)(e, x) = \sum_{k=1}^{m} b_1 \circ \phi^2_{\sigma_k(e)}(f(x))\xi_{\sigma_k(e)}(f(x))b_2(f(x))1_{U_k}(x)
\]

\[
= \sum_{k=1}^{m} b_1 \circ \phi^1_{\sigma_k(e)}(f(x))\xi_{\sigma_k(e)}(f(x))1_{U_k}(x) \cdot \beta(b_2)(x)
\]

\[
= \beta(b_1) \cdot V(\xi) \cdot \beta(b_2)(e, x).
\]

Also

\[
\langle V(\xi), V(\eta) \rangle_{A_1}(x) = \sum_{f \in E, \sigma(e) \in K_{\sigma(e)}} \sum_{k=1}^{m} \xi_{\sigma(e)}(f(x))\eta_{\sigma_k(e)}(f(x))1_{U_k}(x)
\]

hence \(\langle V(\xi), V(\eta) \rangle_{A_1} = \beta(\langle \xi, \eta \rangle_{A_2})\). Finally, one can see that \(V\) is onto, hence \(V\) is a \(C^*\)-correspondence isomorphism.

Recall that, for \(k \geq 2\), \(E^k := \{\alpha = (\alpha_1, \cdots, \alpha_k) : \alpha_i \in E\) and \(r(\alpha_i) = s(\alpha_{i+1}), i = 1, \cdots, k - 1\},\) is the set of paths of length \(k\) in the graph \(G\). Let \(E^* = \bigcup_{k \in \mathbb{N}} E^k\) be the space of finite paths in the graph \(G\). Also the infinite path space, \(E^\infty\), is defined to be

\[
E^\infty := \{(\alpha_i)_{i \in \mathbb{N}} : \alpha_i \in E\ \text{and} \ r(\alpha_i) = s(\alpha_{i+1})\ \text{for all} \ i \in \mathbb{N}\}.
\]

For \(v \in V\), we also define \(E^k(v) := \{\alpha \in E^k : s(\alpha) = v\}\), and \(E^*(v)\) and \(E^\infty(v)\) are defined similarly. We consider \(E^\infty(v)\) to be endowed with the metric:

\[
\delta_\varepsilon(\alpha, \beta) = \varepsilon^{\alpha \wedge \beta}\ 
\]

if \(\alpha \neq \beta\) and 0 otherwise, where \(\alpha \wedge \beta\) is the longest common prefix of \(\alpha\) and \(\beta\), and \(|w|\) is the length of the word \(w \in E^*\) (see \([5, \text{Page 116}]\)). Then \(E^\infty(v)\) is a compact metric space, and, since \(E^\infty\) equals the disjoint union of the spaces \(E^\infty(v)\), \(E^\infty\) becomes a compact metric space in a natural way. Define the maps \(\theta_v : E^\infty(v) \to E^\infty(s(v))\) by the formula \(\theta_v(\alpha) = e \alpha\), for all \(\alpha \in E^\infty\) and for all \(e \in E\). Then \((G, (E^\infty(v))_{v \in V}, (\theta_v)_{e \in E})\) is a Mauldin-Williams graph. We set \(A_E := C(E^\infty)\) and we set \(E\) be the \(C^*\)-correspondence associated to this Mauldin-Williams graph. Let \(M = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})\) be a Mauldin-Williams graph. For \((\alpha_1, \cdots, \alpha_n) \in E^n\) let \(K_{(\alpha_1, \cdots, \alpha_n)} := \phi_{\alpha_1} \circ \cdots \phi_{\alpha_n}(K_{r(\alpha_n)})\). Then, for any infinite path \(\alpha = (\alpha_n)_{n \in \mathbb{N}} \in E^\infty, \bigcap_{n \geq 1} K_{(\alpha_1, \cdots, \alpha_n)}\) contains only one point. Therefore, we can define a map \(\pi : E^\infty \to K\) by \(\{\pi(x)\} = \bigcap_{n \geq 1} K_{(\alpha_1, \cdots, \alpha_n)}\). Since \(\pi(E^\infty)\) is also an invariant set, \(\pi\) is a continuous, onto map and \(\pi(E^\infty(v)) = K_v\). Moreover, \(\pi \circ \theta_v = \phi_e \circ \pi\).

We say that a Mauldin-Williams graph \(M = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})\) is totally disconnected if \(\phi_{\varepsilon}(K_{r(e)}) \cap \phi_{f}(K_{r(f)}) = \emptyset\) if \(s(e) = s(f)\) and \(e \neq f\).

**Corollary 3.2.** Let \(M = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})\) be a totally disconnected Mauldin-Williams graph. Let \(A\) be the \(C^*\)-algebra and let \(\mathcal{X}\) be the \(C^*\)-correspondence associated to this Mauldin-Williams graph. Then \(\mathcal{X}\) is isomorphic with \(E\), as \(C^*\)-correspondences. In particular, one obtains that for any two totally disconnected Mauldin-Williams graphs having the same underlying graph \(G\), the \(C^*\)-correspondences and tensor algebras associated are isomorphic.
Proof. If the Mauldin-Williams graph is totally disconnected, then the map \( \pi : E^\infty \to K \) defined above is a homeomorphism. Moreover, \( \pi \circ \theta_e \circ \pi^{-1} = \phi_e \) for all \( e \in E \), therefore the associated \( C^* \)-correspondences are isomorphic. \( \square \)

The next theorem is a converse of the Proposition 3.1. We note, however, that the family of open sets \( \{U_i\} \) here is not required to be a partition of the compact set \( K^1 \), but only a finite open cover of it.

**Theorem 3.3.** For \( i = 1, 2 \), let \( G_i = (G, (K^i_e)_{e \in V}, (\delta^i_e)_{e \in E}) \) be two Mauldin-Williams graphs over the same underlying graph \( G \). Let \( A_i = C(K^i) \), \( i = 1, 2 \), be the associated \( C^* \)-algebras and let \( \chi_i, i = 1, 2 \), be the associated \( C^* \)-correspondences. If \( \chi_1 \) and \( \chi_2 \) are isomorphic, then there is a homeomorphism \( f : K^1 \to K^2 \), a finite open cover of \( K^1 \), \( \{U_1, \ldots, U_m\} \), and for each \( U_j \) there is a permutation \( \sigma_j \in S_n \) (\( n = |E| \)) such that

\[
\tag{3.1}
f^{-1} \circ \phi^2_e \circ f|_{U_j} = \phi^1_{\sigma(j)}|_{U_j} \text{ for all } e \in E, i \in \{1, \ldots, m\}.
\]

Proof. Since \( \chi_1 \) and \( \chi_2 \) are isomorphic, there is a \( C^* \)-isomorphism \( \beta : A_2 \to A_1 \) and a \( C^* \)-correspondence isomorphism \( W : X_2 \to X_1 \) such that \( W(b_1 \cdot \xi \cdot b_2) = \beta(b_1)W(\xi)\beta(b_2) \) and \( W(\xi, W(\eta))_{A_1} = \beta(\xi, \eta)_{A_2} \), for all \( b_1, b_2 \in A_2 \), \( \xi, \eta \in X_2 \). Let \( f : K^1 \to K^2 \) be the homeomorphism which implements \( \beta \), that is, \( \beta(b) = b \circ f \), for all \( b \in A_2 \).

Let \( \delta_e \in X_2 \), defined by

\[
\delta_e(g, y) = \begin{cases} 
1, & \text{if } e = g, \\
0, & \text{otherwise},
\end{cases}
\]

for \( e \in E \), be the natural basis in \( X_2 \) and let \( (\delta_e')_{e \in E} \subset X_1 \) be the natural basis in \( X_1 \), which is defined similarly.

For \( \xi \in X_2 \), \( \xi = \sum_{g \in E} \delta_g \cdot \xi_g \), where \( \xi_g(y) = \xi(g, y) \) for all \( y \in K^2_{r(g)} \) and is 0 otherwise. With respect to the bases, we can write

\[
W(\xi) = W\left( \sum_{g \in E} \delta_g \cdot \xi_g \right) = \sum_{g \in E} W(\delta_g) \cdot \xi_g \circ f \tag{3.2}
\]

where

\[
W(\delta_g) = \sum_{e \in E} \delta'_e \cdot w_{eg} \delta_g \circ f, \tag{3.3}
\]

and \( w_{eg} \) are given by the formula \( w_{eg} = \langle \delta'_e, W(\delta_g) \rangle_{A_1} \), for all \( e, g \in E \). We call \( (w_{eg})_{e, g \in E} \) the matrix of \( W \) with respect to the basis \( (\delta'_e)_{e \in E} \) and \( (\delta_g)_{g \in E} \) (it is an \( n \times n \) matrix, where \( n = |E| \)). Since \( W \) preserves the inner product, we see that

\[
\langle W(\delta_g), W(\delta_e) \rangle = \delta_{ge}, \tag{3.4}
\]

where \( \delta_{ge}(x) = 1 \) if \( e = g \) and \( x \in K_{r(e)} \) and is 0 otherwise. Also,

\[
\langle W(\delta_g), W(\delta_e) \rangle = \sum_{f \in E} w_{fg} w_{fe}. \tag{3.5}
\]
Equations (3.4) and (3.5) imply that for every $x \in K^1$ the matrix $(w_{xf}(x))_{e, f \in E}$ is invertible. Hence there is $\sigma_x \in S_n$ such that $w_{\sigma_x(e)}(x) \neq 0$ for all $e \in E$. Therefore, there is a neighborhood $U_x$ of $x$ such that
\begin{equation}
(3.6) \quad w_{\sigma_x(e)}(x) \neq 0 \text{ for all } e \in E, y \in U_x \text{ and } x \in K^1.
\end{equation}

Let $b \in A_2$. Then, for $h \in E$ we have that
\[
W(b \cdot \delta_h) = \sum_{e \in E} \delta'_c w_{eh} b \circ \phi^2_h \circ f
\]
and
\[
\beta(b) \cdot W(\delta_h) = \sum_{e \in E} \delta'_c b \circ f \circ \phi^1_c w_{eh}.
\]

Fix $x \in K^1$ and let $\sigma_x \in S_n$ and $U_x$ be defined as in Equation (3.6). Then
\[
W(b \cdot \delta_h)(\sigma_x(h), y) = w_{\sigma_x(h)}(y) b \circ \phi^2_h \circ f(y)
\]
and
\[
(\beta(b) \cdot W(\delta_h))(\sigma_x(h), y) = b \circ f \circ \phi^1_{\sigma_x(h)}(y) w_{\sigma_x(h)}(y)
\]
for all $y \in U_x$ and for all $h \in E$. Since $W$ is a $C^*$-correspondence isomorphism and $w_{\sigma_x(h)}(y) \neq 0$ for all $y \in U_x$, for any $x \in K^1$, there is a neighborhood $U_x$ of $x$ in $K^1$ and there is a permutation $\sigma_x \in S_n$ such that
\[
f^{-1} \circ \phi^2_h \circ f \mid_{U_x} = \phi^1_{\sigma_x(h)} \mid_{U_x} \quad \text{for all } h \in E.
\]
Hence we can find a finite cover $\{U_1, \ldots, U_m\}$ of $K^1$ and for each $U_i$ we can find a permutation $\sigma_i \in S_n$ such that the Equation (3.1) holds.

In the special case when the two Mauldin-Williams graphs are totally disconnected, more can be said about the choice of the permutations $\sigma_i$.

**Corollary 3.4.** Let $G_i = (G_i((K^1_i))_{e \in V_i}, (\phi^e_i)_{e \in E})$ be two Mauldin-Williams graphs. Let $A_i = C(K^1_i)$ and let $X_i$, $i = 1, 2$, be the associated $C^*$-algebras and $C^*$-correspondences. If $G_1$ is totally disconnected and if $X_1$ is isomorphic with $X_2$ there is a continuous map $h : K^1 \to S_n$ such that $f^{-1} \circ \phi^2_e \circ f(x) = \phi_{h(x)(e)}(x)$, for all $x \in K^1$.

**Proof.** Recall that if $G_1$ is totally disconnected, then $\phi^e_1(K_r(e)) \cap \phi^f_1(K_r(f)) = \emptyset$ if $e \neq f$. From the Theorem 3.3 we know that there are open sets $\{U_1, \ldots, U_m\}$, for some $m \in \mathbb{N}$, and permutations $\sigma_1, \ldots, \sigma_m \in S_n$ such that
\begin{equation}
(3.7) \quad f^{-1} \circ \phi^2_e \circ f \mid_{U_i} = \phi^1_{\sigma_i(e)} \mid_{U_i} \quad \text{for all } e \in E, i \in \{1, \ldots, m\}.
\end{equation}
If $U_i \cap U_j \neq \emptyset$ for some $i \neq j$, then it follows that $\phi^1_{\sigma_i(e) \mid U_i \cap U_j} = \phi^1_{\sigma_j(e) \mid U_i \cap U_j}$ for all $e \in E$, hence $\sigma_i(e) = \sigma_j(e)$ for all $e \in E$, so $\sigma_i = \sigma_j$. Therefore, we can choose the cover $U_1, \ldots, U_m$ such that $U_i \cap U_j = \emptyset$ if $i \neq j$.

Let $x \in K^1$. Then there is a unique $i \in \{1, \ldots, n\}$ such that $x \in U_i$. We define $h(x) = \sigma_i$. Then $h : K^1 \to S_n$ is a well-defined map. Moreover, $h$ is continuous (considering $S_n$ endowed with the discrete topology), since for every $\sigma \in S_n$, $h^{-1}(\sigma) = \emptyset$ or $h^{-1}(\sigma) = U_i$, for some $i \in \{1, \ldots, n\}$. Finally, from the Equation (3.7) we obtain that
\[
f^{-1} \circ \phi^2_e \circ f(x) = \phi^1_{h(x)(e)}(x) \quad \text{for all } x \in K^1 \text{ and } e \in E.
\]
\[\square\]
Suppose that $G_2 = (\{ G, (K_i)_{i \in V}, (\phi_e^i)_{e \in E} \})$ are two Mauldin-Williams graphs that satisfy the hypothesis of the Corollary 3.4. We claim that $G_2$ must also be totally disconnected. Suppose that there are $e, f \in E$, $e \neq f$, such that $\phi_e^2(K_i^2(e)) \cap \phi_f^2(K_i^2(f)) \neq \emptyset$. Then there is an $x \in K^1$ such that $y := f(x) \in \phi_e^2(K_i^2(e)) \cap \phi_f^2(K_i^2(f))$. Then $\phi_e^1(K_i^2(e))(x) = \phi_f^1(K_i^2(f))(x)$, which is a contradiction since $G_1$ is totally disconnected. So $G_2$ is totally disconnected.

**Example 3.5.** Let $K$ be the Cantor set, let $\phi_i : K \to K$, $i = 1, 2$, be the maps defined by the formulae $\phi_1(x) = \frac{1}{3}x$ and $\phi_2(x) = \frac{1}{3}x + \frac{2}{3}$. Then $K$ is the invariant set of $(\phi_1, \phi_2)$. Let $T = [0, 1]$ and let $\psi_i : T \to T$, $i = 1, 2$, be the maps defined by the formulae $\psi_1(x) = \frac{1}{2}x$ and $\psi_2(x) = -\frac{1}{2}x + 1$. Then $T$ is the invariant set of $(\psi_1, \psi_2)$. Since $(\psi_1, \psi_2)$ is not totally disconnected, we see that the associated $C^*$-correspondences are not strongly Morita equivalent. Hence the tensor algebras fail to be strongly Morita equivalent in the sense of [2], yet their $C^*$-envelopes coincide with $\Omega_2$.

**Example 3.6.** Let $T$ be the regular triangle in $\mathbb{R}^2$ with vertices $A = (0, 0)$, $B = (1, 0)$ and $C = (1/2, \sqrt{3}/2)$. Let $\phi_1(x, y) = \left( \frac{x}{2} + \frac{1}{2}, \frac{y}{2} + \frac{\sqrt{3}}{2} \right)$, $\phi_2(x, y) = \left( \frac{x}{2}, \frac{y}{2} \right)$ and $\phi_3(x, y) = \left( \frac{x}{2}, \frac{y}{2} \right)$. Then the invariant set $K$ of this iterated function system is the Sierpinski gasket. Let $\psi_1 = \sigma_1 \circ \phi_1$, $\psi_2 = \phi_2$ and $\psi_3 = \sigma_2 \circ \phi_3$, where $\sigma_i$ is the symmetry about the median from the vertex $\phi_i(C)$ of the triangle $\phi_i(T)$, for $i = 1, 3$. Then the invariant set of this iterated function system is also the Sierpinski gasket, but the $C^*$-correspondences associated to $(\phi_1, \phi_2, \phi_3)$ and $(\psi_1, \psi_2, \psi_3)$ are not isomorphic.

**References**


