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MAPPING SPACES AND HOMOLOGY ISOMORPHISMS

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ABSTRACT. Let $\mathrm{Map}(K, X)$ denote the space of pointed continuous maps from a finite cell complex K to a space X. Let E_* be a generalized homology theory. We use Goodwillie calculus methods to prove that under suitable conditions on K and X, $\mathrm{Map}(K, X)$ will send an E_*–isomorphism in either variable to a map that is monic in E_* homology. Interesting examples arise by letting E_* be K–theory, the finite complex K be a sphere, and the map in the X variable be an exotic unstable Adams map between Moore spaces.

1. INTRODUCTION AND MAIN RESULTS

Let K and X be pointed spaces, with K homotopy equivalent to a finite cell complex, and then let $\mathrm{Map}(K, X)$ denote the space of pointed continuous maps from K to X. Fixing K, this includes many important constructions on X. For example, $\mathrm{Map}(S^n, X) = \Omega^n X$, the n^{th} loopspace of X, and $\mathrm{Map}(S^1_+, X) = \mathcal{L}X$, the free loopspace on X.

Suppose E_* is a generalized homology theory. A fundamental problem is to try to determine to what extent $E_*(\mathrm{Map}(K, X))$ might be determined by $E_*(X)$. This is difficult, and has a long history, even when E_* is ordinary homology with field coefficients and $K = S^1$.

We consider a related problem. A map $f : X \to Y$ will be called an E_*–isomorphism if $E_*(f)$ is an isomorphism. One can ask to what extent does $\mathrm{Map}(K, \ )$ preserve E_*–isomorphisms?

This question is of interest when E_* is a nonconnective theory, as the following simple example illustrates: the constant map $c : K(\mathbb{Z}/p, 2) \to *$ is an isomorphism in complex K–theory K_*, but $\Omega c : K(\mathbb{Z}/p, 1) \to *$ is not. A much more subtle family of examples has been constructed recently by L. Langsetmo and D. Stanley [LS]; see Example 1.4 below.

In this paper we use Goodwillie calculus methods to prove the curious result that under suitable conditions on K and X, $\mathrm{Map}(K, X)$ will send an E_*–isomorphism in either variable to a map that is monic in E_* homology.

1.1. **The main theorem.** We need to define some numerical invariants of spaces.

Let $d(K)$ be the minimal d such that K is homotopy equivalent to a d–dimensional complex.

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Let $e(K)$ be the minimal n such that there exists a parallelizable n–dimensional manifold M (not necessarily compact, and possibly with boundary), together with a closed subcomplex A such that K is homotopy equivalent to M/A. For example, $e(S^n) = n$, as $S^n = D^n/S^{n-1}$.

Let $c(X)$ be the connectivity of X.

Let $s(X)$ be the maximal n such that X is homotopy equivalent to an n–fold suspension.

Armed with these definitions, we can state our main theorem.

Theorem 1.1. *Suppose $e(K) \leq s(X)$ and $d(K) \leq c(X)$.*

(1) *If $f : X \to Y$ is an E_*–isomorphism, then*

$$\mathrm{Map}(K, f) : \mathrm{Map}(K, X) \to \mathrm{Map}(K, Y)$$

is E_*–monic.

(2) *If $g : L \to K$ is an E_*–isomorphism of finite complexes, then*

$$\mathrm{Map}(g, X) : \mathrm{Map}(K, X) \to \mathrm{Map}(L, X)$$

is E_*–monic.

Corollary 1.2. *If Z is connected, and $f : \Sigma^n Z \to Y$ is an E_*–isomorphism, then $\Omega^n f : \Omega^n \Sigma^n Z \to \Omega^n Y$ is E_*–monic.*

Remark 1.3. This corollary seems to be new even when $n = 1$. To the best of the author's knowledge, the only results of this sort in the literature are the author's papers [K1, K2] which contain the $n = \infty$ version of the corollary.

Note that $d(K) \leq e(K)$. Furthermore $s(X) \leq c(X) + 1$ is always true, and very often $s(X) \leq c(X)$. For example, $s(M^n(d)) = c(M^n(d)) = n - 2$, where $M^n(d)$ is the Moore space $D^n \cup_d S^{n-1}$. Thus when the first inequality in the hypotheses of the theorem holds, so usually does the second. In general, $e(K)$ seems hard to compute exactly. The appendix includes some observations of Greg Arone and the author which yield some further explicit calculations, and some general bounds. For example, $e(M^n(d)) = n + 1$, and, $e(K) \leq 2d(k) - 1$ for all K with $d(K) \geq 1$.

The numeric hypotheses of our theorem are easy to explain. The condition $d(K) \leq c(X)$ guarantees the strong convergence of the Goodwillie tower of the functor sending a space X to the suspension spectrum $\Sigma^\infty \mathrm{Map}(K, X)$. The condition $e(K) \leq s(X)$ implies that there is a filtered configuration space approximation to $\mathrm{Map}(K, X)$, as in work of Bödigheimer [Bö], following McDuff [McD] and May [Ma].

When both numeric conditions hold, statement (1) of Theorem 1.1 is proved by using properties of Goodwillie towers to play the two corresponding geometric conditions against each other. Using the existence of Bousfield localization of spaces, statement (2) is then a formal consequence of (1).

The proof of Theorem 1.1 is given in sections 2 and 3. In section 2, we outline how general calculus theory leads to theorems like ours, while in the shorter section 3, we specialize to the case in hand.

1.2. **Examples and applications.**

Example 1.4. Let p be an odd prime. For each (m, n) in an explicit infinite list of pairs, with $m \geq 4$ and both m and $(n - m)$ taking on arbitrarily large values, Langsetmo and Stanley [LS] construct a K_*–isomorphism

$$f : M^n(p) \to M^m(p)$$

such that Ωf is *not* a K_*–isomorphism. For example, with $p = 3$, for all $t \geq 1$, one has such a nondurable K_*–isomorphism $f : M^{4t}(3) \to M^4(3)$.

It is not hard to deduce that then, for all $j \geq 1$, the 3–connected cover of $\Omega^j f$,

$$(\Omega^j f)\langle 3\rangle : \Omega^j M^n(p)\langle 3\rangle \to \Omega^j M^m(p)\langle 3\rangle,$$

is also not a K_*–isomorphism.[1]

In contrast, Corollary 1.2 implies that for all $1 \leq j \leq n-2$,

$$\Omega^j f : \Omega^j M^n(p) \to \Omega^j M^m(p)$$

is K_*–monic.

Combining these results, we conclude that for $1 \leq j \leq n-5$,

$$(\Omega^j f)\langle 3\rangle : \Omega^j M^n(p) \to \Omega^j M^m(p)\langle 3\rangle$$

is K_*–monic but not K_*–epic.

Example 1.5. When the homology theory E_* is $K(r)_*$, the r^{th} Morava K–theory at a prime p, then the corollary has the following computational implication.

Let $f : \Sigma^n Z \to Y$ be a $K(r)_*$–isomorphism, with Z connected and $n \geq 1$, and let F be the fiber of f. The $K(r)_*$ bar spectral sequence associated to the principal fibration

$$\Omega^n \Sigma^n Z \xrightarrow{\Omega^n f} \Omega^n Y \to \Omega^{n-1} F$$

converges to $K(r)_*(\Omega^{n-1}F)$ and has

$$E^2_{*,*} = \operatorname{Tor}^{K(r)_*(\Omega^n\Sigma^n Z)}_{*,*}(K(r)_*(\Omega^n Y), K(r)_*).$$

By the corollary,

$$K(r)_*(\Omega^n\Sigma^n Z) \xrightarrow{(\Omega^n f)_*} K(r)_*(\Omega^n Y)$$

is monic. The map $(\Omega^n f)_*$ is in the category of $\mathcal{K}/p$–Hopf algebras studied by Bousfield in [B2, Appendix]. He shows [B2, Thm.10.8] that objects in this category are flat over subobjects, when viewed as algebras. We conclude that the spectral sequence collapses, giving an isomorphism

$$K(r)_*(\Omega^{n-1}F) \simeq K(r)_*(\Omega^n Y) \otimes_{K(r)_*(\Omega^n\Sigma^n Z)} K(r)_*$$

of $K(r)_*$–coalgebras.[2]

Example 1.6. Suppose $g : L \to K$ is a $K(r)_*$–isomorphism between finite complexes. Let C be the cofiber of g. Applying statement (2) of Theorem 1.1 to Σg, and reasoning as in the last example, we deduce that, for all X such that $e(K) < s(X)$ and $d(K) < c(X)$, one gets an isomorphism of $K(r)_*$–coalgebras

$$K(r)_*(\operatorname{Map}(C, X)) \simeq K(r)_*(\operatorname{Map}(\Sigma L, X)) \otimes_{K(r)_*(\operatorname{Map}(\Sigma K, X))} K(r)_*.$$

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[1] This follows from a theorem of Bousfield [B3, Theorem 11.10], but is easy to prove directly, using the fact that $\tilde{K}_*(K(\mathbb{Z}/p, 2) = 0$.

[2] $K(r)_*$–Hopf algebras if $n > 1$.

2. Goodwillie calculus and E_*–isomorphisms

Let $\mathcal{T}$ denote the category of based spaces, and let $\mathcal{S}$ be a nice model category of spectra, e.g. the category of S–modules of [EKMM]. In this section we find conditions on a functor $F : \mathcal{T} \to \mathcal{S}$ and a space X ensuring that if $f : X \to Y$ is an E_*–isomorphism, then $F(f) : F(X) \to F(Y)$ will be E_*–monic.

2.1. **Review of Goodwillie calculus.** In the series of papers [G1, G2, G3], Tom Goodwillie has developed his theory of polynomial resolutions of homotopy functors. We need to summarize some aspects of Goodwillie's work as they apply to functors $F : \mathcal{T} \to \mathcal{S}$.[3]

As is carefully discussed in [G2, G3], a functor is said to be polynomial of degree r if it takes strongly homotopy cocartesion $(r+1)$–cubical diagrams to homotopy cartesian cubical diagrams. In [G3], given a functor F from one topological model category to another, Goodwillie proves the existence of a tower $\{P_rF\}$ under F so that $F \to P_rF$ is the universal arrow to a polynomial functor of degree r, up to weak equivalence.

The functors F of interest to us in this paper satisfy an additional property: they will be *finitary.* Here, following [G3, Definition 5.10], F is said to be finitary if it commutes with filtered homotopy colimits up to equivalence.

Examination of the construction of P_rF shows that P_r satisfies the following useful properties.

Lemma 2.1 (Compare with [G3, Proposition 1.7])**.**
(1) *If $F(X) \to G(X) \to H(X)$ is a fibration sequence for all X, so is $P_rF(X) \to P_rG(X) \to P_rH(X)$.*
(2) *Given natural transformations $F_1 \to F_2 \to \dots$, the natural map*

$$\operatorname*{hocolim}_s P_rF_s(X) \to P_r(\operatorname*{hocolim}_s F_s)(X)$$

is an equivalence for all r and X.
(3) *If F is finitary, so is P_rF for all r.*

The fact that the suspension of a strongly homotopy cocartesian cube is again strongly homotopy cocartesion implies the next property of Goodwillie towers.

Lemma 2.2 ([G3, Remark 1.1])**.** *There is a natural equivalence*

$$P_r(F \circ \Sigma^d)(X) \simeq (P_rF)(\Sigma^d X).$$

Let $D_rF(X)$ be the homotopy fiber of $P_rF(X) \to P_{r-1}F(X)$. D_rF is homogeneous of degree r: it has degree r, and $P_{r-1}D_rF(X)$ is weakly contractible. (This follows from Lemma 2.1(1); see [G3, Proposition 1.17].) Goodwillie analyzes D_rF. We need his description when F is also finitary and takes values in a stable model category like $\mathcal{S}$.

Proposition 2.3 ([G3, Theorems 3.5, 6.1])**.** *If $F : \mathcal{T} \to \mathcal{S}$ is finitary, then, for each r, there is a spectrum $t_r(F)$ with an action of the r^{th} symmetric group Σ_r, and a natural weak equivalence*

$$D_rF(X) \simeq (t_r(F) \wedge X^{\wedge r})_{h\Sigma_r}.$$

[3] As surveyed in [K3], Goodwillie's general theory applies to functors $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are topological or simplicial model categories. As is apparent from the proofs, many of the results in this section would apply to functors with domain $\mathcal{T}$ replaced by any such $\mathcal{C}$, and range $\mathcal{S}$ replaced by a model category that is also stable.

Corollary 2.4. *If $F : \mathcal{T} \to \mathcal{S}$ is finitary, and $f : X \to Y$ is an E_*–isomorphism, then $P_rF(f) : P_rF(X) \to P_rF(Y)$ is also an E_*–isomorphism.*

Proof. Standard spectral sequences show that any construction of the form $(C \wedge X^{\wedge r})_{h\Sigma_r}$ preserves E_*–isomorphisms in the X variable. The proposition thus implies that the maps on fibers, $D_rF(f) : D_rF(X) \to D_rF(Y)$, are E_*–isomorphisms. The proposition then follows by induction on r. □

Remark 2.5. The corollary is false without the finitary hypothesis. Examples can easily be constructed using homological localization functors, which are homogeneous and linear, but not, in general, finitary.

2.2. **Strongly split towers of spectra.** Suppose we are given a tower of spectra under another spectrum:

$$\begin{tikzcd}
& \vdots \arrow[d] \\
& C_2 \arrow[d, "p_1"] \\
& C_1 \arrow[d, "p_0"] \\
C \arrow[r, "e_0"] \arrow[ur, "e_1"] \arrow[uur, "e_2"] & C_0.
\end{tikzcd}$$

We will say that the tower is *strongly convergent* if the connectivity of the maps e_r goes to infinity as r goes to infinity.

We will say that the tower is *strongly split* if there exists a homotopy commutative diagram

$$\begin{tikzcd}
& \vdots \\
& C_2 \arrow[u] \arrow[ddl, "i_2"'] \\
& C_1 \arrow[u, "j_1"'] \arrow[dl, "i_1"'] \\
C & C_0. \arrow[l, "i_0"'] \arrow[u, "j_0"']
\end{tikzcd}$$

such that, for all r, the composite

$$C_r \xrightarrow{i_r} C \xrightarrow{e_r} C_r$$

is an equivalence.

The following lemma is evident.

Lemma 2.6. *If a tower as above is both strongly convergent and strongly split, then the induced map*

$$\operatorname*{hocolim}_r i_r : \operatorname*{hocolim}_r C_r \to C$$

is an equivalence. Thus, if E_* is a homology theory, then

$$\operatorname*{colim}_r E_*(C_r) \to E_*(C)$$

is an isomorphism.

Remark 2.7. This lemma says all that we will need to know about strongly split towers for our purposes. However, it is illuminating to note the following. If a tower is strongly split, one can, if needed, modify the splitting data so that $e_r \circ i_r : C_r \to C_r$ is homotopic to the identity. In this case, it will also be true that the composite

$$C_r \xrightarrow{j_r} C_{r+1} \xrightarrow{p_r} C_r$$

will be homotopic to the identity for all r. If the tower is also strongly convergent, then there will be a wedge decomposition

$$C \simeq \bigvee_{r=1}^{\infty} \text{hofiber}\{p_r : C_r \to C_{r-1}\}.$$

2.3. **A useful proposition.**

Proposition 2.8. *Let $F : \mathcal{T} \to \mathcal{S}$ be finitary. Suppose the Goodwillie tower of F is both strongly convergent and strongly split when evaluated at a space X. Then, if $f : X \to Y$ is an E_*–isomorphism, then $F(f) : F(X) \to F(Y)$ is E_*–monic.*

Proof. Let $i_r : P_rF(X) \to F(X)$ and $j_r : P_rF(X) \to P_{r+1}F(X)$ denote the maps splitting the tower $\{P_rF(X)\}$. Suppose $f : X \to Y$ is an E_*–isomorphism, and consider the diagram

$$\begin{tikzcd}
P_rF(X) \arrow[r, "i_r"] & F(X) \arrow[r, "e_r"] \arrow[d, "F(f)"] & P_rF(X) \arrow[d, "P_rF(f)"] \\
& F(Y) \arrow[r, "e_r"] & P_rF(Y).
\end{tikzcd}$$

By assumption, the top composite is an equivalence, and thus is an E_*–isomorphism. Since f is an E_*–isomorphism, so is the right vertical map, by Corollary 2.4. We conclude that $E_*(F(f))$ is monic when restricted to the image of $E_*(i_r)$. But $E_*(F(X))$ is the colimit over r of these images, by Lemma 2.6, and thus $E_*(F(f))$ is also monic. □

To apply this proposition, we need criteria ensuring that a Goodwillie tower $\{P_rF(X)\}$ strongly splits. This is the topic of our next two subsections.

2.4. **Goodwillie towers of functors with polynomial filtration.** Say that a functor $C : \mathcal{T} \to \mathcal{S}$ has a *polynomial filtration* if it is filtered by functors $F_0C \to F_1C \to \dots$ such that

$$\operatorname*{hocolim}_r F_rC(X) \to C(X)$$

is an equivalence, and the homotopy cofiber functor

$$F_rC/F_{r-1}C$$

is homogeneous of degree r for all r.

The following lemma is well–known folk knowledge.

Lemma 2.9. *In this situation, the composite*

$$F_rC(X) \to C(X) \to P_rC(X)$$

is an equivalence for all r and X. It follows that the Goodwillie tower $\{P_rC(X)\}$ will be strongly split.

Proof. We have a homotopy commutative diagram with rows that are cofibration sequences

$$\begin{tikzcd}
F_rC(X) \arrow[r] \arrow[d] & C(X) \arrow[r] \arrow[d] & C/F_rC(X) \arrow[d] \\
P_rF_rC(X) \arrow[r] & P_rC(X) \arrow[r] & P_r(C/F_rC)(X).
\end{tikzcd}$$

As F_rC has degree r, the left vertical map is an equivalence. If we check that $P_r(C/F_rC)(X) \simeq *$, then the bottom left map will be an equivalence, and we will be done. To check this we have

$$P_r(C/F_rC)(X) \simeq \operatorname*{hocolim}_s P_r(F_sC/F_rC)(X) \simeq *,$$

as $P_r(F_sC/F_rC)(X) \simeq *$ for $s \geq r$. □

2.5. **Stable natural equivalences.** Call a natural transformation $\Theta(X) : C(X) \to G(X)$ a *stable equivalence* if it is an equivalence for all suitably connected spaces X.

Goodwillie notes in [G3, Remark 1.1] that his construction of P_rF depends only on the behavior of F on highly connected spaces. To say this more precisely, we first recall that P_rF is constructed as the hocolimit over i of functors T_r^iF. In turn, each $T_r^iF(X)$ is constructed as the homotopy limit of a finite diagram of spectra obtained by applying F to a finite diagram of spaces determined by X, and these spaces are all at least $(i-1)$–connected. The constructions are natural in both the variables X and F, and so the next lemma follows.

Lemma 2.10. *If $\Theta : C \to G$ is a stable equivalence, then*

$$P_r\Theta(X) : P_rC(X) \to P_rG(X)$$

is an equivalence for all X.

Corollary 2.11. *Suppose $\Theta : C \to G$ is a stable equivalence. If, for a particular X, the Goodwillie tower $\{P_rC(X)\}$ is strongly split, then so also is the Goodwillie tower $\{P_rG(X)\}$.*

3. Proof of Theorem 1.1

The Goodwillie tower of the functor from spaces to spectra sending X to $\Sigma^\infty \operatorname{Map}(K, X)$ consists of a diagram of functors

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_3^K(X) \arrow[d] \\
 & P_2^K(X) \arrow[d] \\
\Sigma^\infty \operatorname{Map}(K,X) \arrow[r, "e_1"] \arrow[ur, "e_2"] \arrow[uur, "e_3"] & P_1^K(X).
\end{tikzcd}$$

In [G2, Example 4.5], it is shown that this tower is strongly convergent if $d(K) \leq c(X)$. Thus Theorem 1.1(1) will follow from Proposition 2.8 once we show that the tower is strongly split whenever $e(K) \leq s(X)$. Otherwise said, we wish to show that if $n \geq e(K)$, then the tower $\{P_r^K(\Sigma^n Z)\}$ is strongly split for all spaces Z. By Lemma 2.2, this tower agrees with the tower associated to the functor sending a space Z to $\Sigma^\infty \operatorname{Map}(K, \Sigma^n Z)$.

In the terminology of the last section, the main constructions and theorems of [Bö] states that if $n \geq e(K)$, then there is a filtered configuration space $C(K, Z)$ such that $\Sigma^\infty C(K, Z)$ is a functor with a polynomial filtration, and a natural map of spaces

$$C(K, Z) \to \operatorname{Map}(K, \Sigma^n Z)$$

such that

$$\Sigma^\infty C(K, Z) \to \Sigma^\infty \operatorname{Map}(K, \Sigma^n Z)$$

is a stable equivalence. Then Lemma 2.9 and Corollary 2.11 combine to say that the tower associated to $\Sigma^\infty \operatorname{Map}(K, \Sigma^n Z)$ is strongly split.

Statement (2) of Theorem 1.1 turns out to follow easily from statement (1). The following argument was observed by Pete Bousfield.

Suppose $g : L \to K$ is an E_*–isomorphism between finite complexes. Let $X \to X_E$ be Bousfield localization of the space X with respect to E_* [B1].

Consider the diagram

$$\begin{tikzcd}
\operatorname{Map}(K,X) \arrow[r] \arrow[d, "\operatorname{Map}(g,X)"] & \operatorname{Map}(K,X_E) \arrow[d, "\operatorname{Map}(g,X_E)"] \\
\operatorname{Map}(L,X) \arrow[r] & \operatorname{Map}(L,X_E).
\end{tikzcd}$$

As $X \to X_E$ is an E_*–isomorphism, statement (1) of Theorem 1.1 applies to say that the top map is E_*–monic. The right vertical map is a homotopy equivalence as X_E is E_*–local, and is thus an E_*–isomorphism. Thus the left vertical map is E_*–monic.

Remark 3.1. Though we have not needed this here, there is an explicit model for the tower $\{P^K_r(X)\}$ for $\Sigma^\infty \operatorname{Map}(K,X)$; see [Ar, AK]. From this model, it follows that a version of Corollary 2.4 holds for the K–variable: if E_* is a ring theory, and $g : L \to K$ is an E_*–isomorphism, then so is $P^f_r(X) : P^K_r(X) \to P^L_r(X)$. This leads to an alternative proof of Theorem 1.1(2), under the ring theory hypothesis.

Appendix A. Computations of $e(K)$

By Greg Arone and Nicholas J. Kuhn

Recall that $e(K)$ is the minimal n such that there exists a parallelizable n–dimensional manifold M, together with a closed subcomplex A such that $K \simeq M/A$.

In this Appendix we make some observations allowing for some general estimates and explicit computations of $e(K)$.

A.1. **Upper bounds.** If $K \simeq M/A$, with M parallelizable, we will say that the pair (M,A) *represents* K.

Lemma A.1. $e(K \wedge L) \leq e(K) + e(L)$.

Proof. If (M,A) represents K and (N,B) represents L, then $(M\times N, A\times N \cup M\times B)$ represents $K \wedge L$. □

Corollary A.2. $e(\Sigma^r K) \leq e(K) + r$.

Lemma A.3. *Suppose that L is a finite complex equivalent to a stably parallelizable m–manifold. If $n > m$, and K is obtained from L by attaching n–cells, then $e(K) \leq m+n$.*

Proof. Suppose $g : L \to M$ is an equivalence, with M a stably parallelizable m–manifold. Let

$$\coprod_{i=1}^{c} f_i : \coprod_{i=1}^{c} S^{n-1} \to L$$

be the attaching maps for constructing K from L. Then $\coprod_{i=1}^{c} S^{n-1}$ embeds in the parallelizable manifold $D^n \times M$ so that $(D^n \times M, \coprod_{i=1}^{c} S^{n-1})$ represents K. □

Corollary A.4. *If K is the mapping cone of a map $f : S^n \to S^m$, then $e(K) \leq m+n+1$.*

Lemma A.5. *If $d(K) \geq 1$, then $e(K) \leq 2d(K) - 1$.*

Proof. We can assume K is a $d = d(K)$–dimensional C.W. complex. Let L be its $d-1$ skeleton, and let

$$\coprod_{i=1}^{c} f_i : \coprod_{i=1}^{c} S^{d-1} \to L$$

denote the attaching maps of the d—cells of K. The complex L can be embedded in $\mathbb{R}^{2d-1}$, and we let U be a regular neighborhood. Thus $L \hookrightarrow U$ is an equivalence, and U is a $2d-1$–dimensional parallelizable manifold. The composite

$$\coprod_{i=1}^{c} S^{d-1} \xrightarrow{\coprod_{i=1}^{c} f_i} L \hookrightarrow U$$

is homotopic to an embedding, and then $(U, \coprod_{i=1}^{c} S^{d-1})$ will represent K. □

Slava Krushkal has told us of an unpublished result of Stallings [S] that says that any d–dimensional and c–connected finite complex is (simple) homotopy equivalent to a subcomplex of $\mathbb{R}^{2d-c}$. This implies our final upper bound.

Lemma A.6. $e(K) \leq 2d(K) - c(K)$.

A.2. **Lower bounds.** The obvious lower bound for $E(K)$ comes from dimension

$$d(K) \leq e(K).$$

Stronger lower bounds arise from the contrapositive forms of the following proposition and corollary.

Proposition A.7. *If $n \geq e(K)$, then $\operatorname{Map}_{\mathcal{S}}(K, S^n)$ is equivalent to a suspension spectrum.*

Here $\operatorname{Map}_{\mathcal{S}}(K, X)$ denotes the function spectrum of stable maps between K and X, so that $\operatorname{Map}_{\mathcal{S}}(K, S^n)$ is the n–dual of K.

Proof. It is easy to check (see e.g. [G1]) that the degree 1 approximation to $\Sigma^\infty \operatorname{Map}_{\mathcal{T}}(K, X)$ is $\operatorname{Map}_{\mathcal{S}}(K, X)$. Then, as in §3, we can conclude that, if $n \geq e(K)$, then the composite

$$\Sigma^\infty F_1 C(K, S^0) \to \Sigma^\infty C(K, S^0) \to \Sigma^\infty \operatorname{Map}_{\mathcal{T}}(K, S^n) \to \operatorname{Map}_{\mathcal{S}}(K, S^n)$$

is an equivalence, where $F_1 C(K, S^0)$ is the first filtration of the configuration space $C(K, S^0)$. □

The implications of the proposition for homology are the following.

Corollary A.8. *Suppose $n \geq e(K)$.*

(1) *The reduced integral cohomology groups of K satisfy*

$$\tilde{H}^m(K; \mathbb{Z}) \text{ is } \begin{cases} 0 & \text{if } m > n, \\ \text{free abelian} & \text{if } m = n. \end{cases}$$

(2) *For all primes p, $\mathbb{F}_p \oplus \Sigma^n \tilde{H}_{-*}(K; \mathbb{F}_p)$ admits the structure of an unstable algebra over the mod p Steenrod algebra.*

For certain two cell complexes, the lower bound of the proposition matches our upper bounds. Call a map $f : S^{m+k} \to S^m$ *stably minimal* if whenever $f' : S^{m'+k} \to S^{m'}$ is a map so that f and f' represent the same element in the stable homotopy group π_k^S, one has $m \leq m'$.

Theorem A.9. *If $f : S^n \to S^m$ is stably minimal, and K is the mapping cone of f, then $e(\Sigma^r K) = m + n + 1 + r$.*

Proof. As S–duality induces the identity on π_*^S, one deduces that

$$\operatorname{Map}_{\mathcal{S}}(\Sigma^r K, S^{r+m+n+1}) \simeq \operatorname{Map}_{\mathcal{S}}(K, S^{m+n+1}) \simeq \Sigma^\infty K.$$

Since f is stably minimal, $\Sigma^{-i}\Sigma^\infty K$ is not a suspension spectrum for any $i > 0$. The proposition thus implies that $e(\Sigma^r K) \geq m+n+1+r$. The theorem follows, as Corollary A.4 and Corollary A.2 combine to show that $e(\Sigma^r K) \leq m+n+1+r$. □

A.3. **Examples.** Theorem A.9 applies to all the classic 2–cell complexes. Explicitly, we have

(1) $e(M^{r+2}(d)) = e(\Sigma^r M^2(d)) = 3 + r$,
(2) $e(\Sigma^r \mathbb{C}P^2) = 6 + r$,
(3) $e(\Sigma^r \mathbb{H}P^2) = 12 + r$,
(4) $e(\Sigma^r(\text{Cayley plane})) = 24 + r$, and
(5) $e(\Sigma^r(D^{2p+1} \cup_\alpha S^3)) = 2p + 4 + r$ if p is an odd prime and $\alpha \in \pi_{2p}(S^3)$ is an element of order p.

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