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MODULAR DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH REGULAR SINGULARITIES AT ELLIPTIC POINTS FOR $SL_2(\mathbb{Z})$

HIROYUKI TSUTSUMI

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Abstract. We give a definition of the modular differential equations of weight k for a discrete subgroup for $\Gamma \subset SL_2(\mathbb{R})$; in this paper we set $\Gamma = SL_2(\mathbb{Z})$. We solve such equations admitting regular singularities at elliptic points for $SL_2(\mathbb{Z})$ in terms of the Eisenstein series and the Gauss hypergeometric series. Furthermore, we give a series of such modular differential equations parametrized by an even integer k, and discuss some properties of solution spaces. We find several equations which are solved by a modular form of weight k.

1. Introduction

Let $\mathfrak{H}$ be the complex upper half-plane and τ a variable in $\mathfrak{H}$. Let $A(\tau)$ and $B(\tau)$ be meromorphic functions on $\mathfrak{H}$ which are bounded when $\Im(\tau) \to \infty$. The symbol f' denotes the derivative $(2\pi\sqrt{-1})^{-1} df/d\tau = q \cdot df/dq$ ($q = e^{2\pi\sqrt{-1}\tau}$), and f'' the second derivative $(2\pi\sqrt{-1})^{-2} d^2f/d\tau^2 = q^2 \cdot d^2f/dq^2 + q \cdot df/dq$. We consider the differential equation

$$f''(\tau) + A(\tau)f'(\tau) + B(\tau)f(\tau) = 0 \tag{1}$$

with regular singularities on $\mathfrak{H}$. For a discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ and a fixed rational number k, we call equation (1) the modular differential equation of weight k for Γ if, for any solution $f(\tau)$, the transformed function $(c\tau + d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$ is again a solution of (1) for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

If a differential equation (1) has two linearly independent (over $\mathbb{C}$) modular form solutions of weight k for Γ, this equation is a modular differential equation of weight k for Γ. But the converse is not true in general. It is a fundamental problem to find all modular form solutions of a given modular differential equation. For $\Gamma = SL_2(\mathbb{Z})$ with the coefficients $A(\tau)$ and $B(\tau)$ being holomorphic, the problem was extensively studied in M. Kaneko and M. Koike [2]. In particular, they showed that the modular differential equation of weight k for $SL_2(\mathbb{Z})$ with holomorphic coefficients is essentially unique (see Remark 2 for a precise form).

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In this paper, we study modular differential equations which have regular singularities only at the elliptic points for $SL_2(\mathbb{Z})$. This amounts to allowing the coefficients $A(\tau)$ and $B(\tau)$ to have poles at elliptic points. In §2, we determine the general form of such equations. We give explicit solutions in §3 in terms of hypergeometric functions. In §4, we give a series of the modular differential equations which is parametrized by an even integer k. The solution spaces of the equations in this series contain a modular form of weight k for $SL_2(\mathbb{Z})$; in particular, for $k = 12, 16, 18, 20, 22$ and 26, those spaces coincide with the spaces of modular forms of weight k for $SL_2(\mathbb{Z})$. Moreover the series (parametrized by k) of those solution spaces has some symmetries centered at $k = 6, 10, 12, 14, 16$ and 20.

2. Normal form of the modular differential equation

Consider the modular differential equations of weight k for $SL_2(\mathbb{Z})$ which have regular singularities only at the elliptic points for $SL_2(\mathbb{Z})$. The main purpose of this section is to give an expression of its coefficients by the Eisenstein series of weight 2, 4 and 6. Certain normalizations will also be made.

2.1. **Expression of the coefficients of the modular differential equation by the Eisenstein series.** Recall the Eisenstein series $E_2(\tau)$, $E_4(\tau)$, and $E_6(\tau)$ of weights 2, 4, and 6:

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}(\sum_{d|n} d)q^n = 1 - 24q - 72q^2 - 96q^3 - \cdots,$$

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}(\sum_{d|n} d^3)q^n = 1 + 240q + 2160q^2 + 6720q^3 + \cdots,$$

and

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}(\sum_{d|n} d^5)q^n = 1 - 504q - 16632q^2 - 122976q^3 - \cdots.$$

The form $E_2(\tau)$ is not quite modular but "quasimodular", and the transformation property is given by

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2E_2(\tau) + \frac{6}{\pi\sqrt{-1}}c(c\tau+d),$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. We have

Theorem A. *Any modular differential equation of weight k for $SL_2(\mathbb{Z})$ which has regular singularities only at elliptic points for $SL_2(\mathbb{Z})$ is given by*

$$f''(\tau) + A(\tau)f'(\tau) + B(\tau)f(\tau) = 0$$

where

$$A(\tau) = \frac{\alpha E_4(\tau)^3 + \beta E_6(\tau)^2}{E_4(\tau)E_6(\tau)} - \frac{k+1}{6}\cdot E_2(\tau) \tag{2}$$

and

$$B(\tau)=\frac{k(k+1)}{144}E_2(\tau)^2-\frac{k}{12}\frac{\alpha E_4(\tau)^3+\beta E_6(\tau)^2}{E_4(\tau)E_6(\tau)}E_2(\tau)+\frac{\gamma E_4(\tau)^6+\delta E_4(\tau)^3E_6(\tau)^2+\epsilon E_6(\tau)^4}{E_4(\tau)^2E_6(\tau)^2}\tag{3}$$

for some $\alpha,\beta,\gamma,\delta,\epsilon\in\mathbb{C}$.

Proof. The function $E_4(\tau)$ (resp. $E_6(\tau)$) has simple zeros exactly at the elliptic points equivalent to $\rho=e^{2\pi\sqrt{-1}\pi/3}$ (resp. $i=\sqrt{-1}$). Because any elliptic point of $SL_2(\mathbb{Z})$ is equivalent to ρ or i, by the definition of regular singular points, the functions

$$E_4(\tau)E_6(\tau)\cdot A(\tau)$$

and

$$E_4(\tau)^2E_6(\tau)^2\cdot B(\tau)$$

are holomorphic on $\mathfrak{H}$.

Let f and g be linearly independent solutions. By the assumption of modularity of the equation, we see by an elementary calculation that f satisfies

$$f''\left(\frac{a\tau+b}{c\tau+d}\right)+\left((c\tau+d)^2A(\tau)-\frac{k+1}{\pi\sqrt{-1}}c(c\tau+d)\right)f'\left(\frac{a\tau+b}{c\tau+d}\right)+\left((c\tau+d)^4B(\tau)-\frac{k}{2\pi\sqrt{-1}}c(c\tau+d)^3A(\tau)+\frac{k(k+1)}{(2\pi\sqrt{-1})^2}c^2(c\tau+d)^2\right)f\left(\frac{a\tau+b}{c\tau+d}\right)=0$$

for any $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$. Comparing this with

$$f''\left(\frac{a\tau+b}{c\tau+d}\right)+A\left(\frac{a\tau+b}{c\tau+d}\right)f'\left(\frac{a\tau+b}{c\tau+d}\right)+B\left(\frac{a\tau+b}{c\tau+d}\right)f\left(\frac{a\tau+b}{c\tau+d}\right)=0,$$

we have

$$\begin{aligned}f'\left(\frac{a\tau+b}{c\tau+d}\right)&\left[A\left(\frac{a\tau+b}{c\tau+d}\right)-(c\tau+d)^2A(\tau)+\frac{k+1}{\pi\sqrt{-1}}c(c\tau+d)\right]\\&=f\left(\frac{a\tau+b}{c\tau+d}\right)\left[B\left(\frac{a\tau+b}{c\tau+d}\right)-(c\tau+d)^4B(\tau)\right.\\&\qquad\left.+\frac{k}{2\pi\sqrt{-1}}c(c\tau+d)^3A(\tau)-\frac{k(k+1)}{(2\pi\sqrt{-1})^2}c^2(c\tau+d)^2\right].\end{aligned}\tag{4}$$

Assume that there exist an element $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$ and a point $\tau_0\in\mathfrak{H}$ which satisfy

$$A\left(\frac{a\tau_0+b}{c\tau_0+d}\right)\neq(c\tau_0+d)^2A(\tau_0)-\frac{k+1}{\pi\sqrt{-1}}c(c\tau_0+d).$$

Since g also satisfies the same relation as (4), we have

$$\frac{f(\tau)}{f'(\tau)}=\frac{g(\tau)}{g'(\tau)}$$

in a small neighborhood of τ_0. This contradicts the linear independence of f and g. Thus we have

$$A\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2A(\tau)-\frac{k+1}{\pi\sqrt{-1}}c(c\tau+d) \tag{5}$$

and

$$B\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^4B(\tau)-\frac{k}{2\pi\sqrt{-1}}c(c\tau+d)^3A(\tau)+\frac{k(k+1)}{(2\pi\sqrt{-1})^2}c^2(c\tau+d)^2.$$

From the quasimodular property of $E_2(\tau)$ as well as the holomorphy of $E_4(\tau)E_6(\tau)\cdot A(\tau)$, we conclude from equation (5) that the function

$$E_4(\tau)E_6(\tau)\left((c\tau+d)^2A(\tau)+\frac{k+1}{6}E_2(\tau)\right)$$

is a holomorphic modular form of weight 12 for $SL_2(\mathbb{Z})$. Because the space of modular forms of weight 12 for $SL_2(\mathbb{Z})$ is spanned by $E_4(\tau)^3$ and $E_6(\tau)^2$, we have equation (2) for some $\alpha,\beta\in\mathbb{C}$. Similarly, the function

$$E_4(\tau)^2E_6(\tau)^2\left(B(\tau)-\frac{k(k+1)}{144}E_2(\tau)^2\right) + \frac{k}{12}E_2(\tau)E_4(\tau)E_6(\tau)\left(\alpha E_4(\tau)^3+\beta E_6(\tau)^2\right)$$

is a modular form of weight 24 for $SL_2(\mathbb{Z})$, the space of such forms being three dimensional and spanned by $E_4(\tau)^6, E_4(\tau)^3E_6(\tau)^2$ and $E_6(\tau)^4$. Thus we have equation (3) for some $\gamma,\delta,\epsilon\in\mathbb{C}$. □

2.2. **Normalization of the modular differential equations.** By Theorem A, we transform the modular differential equation in $\tau\in\mathfrak{H}$ into the equation with variable $q=e^{2\pi\sqrt{-1}\tau}$. Without loss of generality (see Remark 1 below), we assume that the equation has a power series solution of the form $1+c_1q+c_2q^2+\cdots$ ($c_i\in\mathbb{C}$). Under this assumption, we reduce the number of parameters by one and obtain the following normalized form of the differential equation.

Theorem B. *If a modular differential equation* (1) *of weight k for $SL_2(\mathbb{Z})$ has regular singularities only at elliptic points for $SL_2(\mathbb{Z})$ and has a power series solution of the form $1+c_1q+c_2q^2+\cdots$, then it can be expressed as*

(6) $\mathcal{D}_k(\alpha,\beta,\gamma,\epsilon): f''(\tau)+A(\tau)f'(\tau)+B(\tau)f(\tau)=0,$

$$A(\tau)=\frac{\alpha E_4(\tau)^3+\beta E_6(\tau)^2}{E_4(\tau)E_6(\tau)}-\frac{k+1}{6}\cdot E_2(\tau),$$

$$B(\tau)=\frac{k(k+1)}{12}E_2'(\tau)-\frac{k}{12}\frac{3\alpha E_4'(\tau)E_4(\tau)^2+2\beta E_6'(\tau)E_6(\tau)}{E_4(\tau)E_6(\tau)} + \frac{(E_4(\tau)^3-E_6(\tau)^2)(\gamma E_4(\tau)^3-\epsilon E_6(\tau)^2)}{E_4(\tau)^2E_6(\tau)^2},$$

for some $\alpha,\beta,\gamma,\epsilon\in\mathbb{C}$.

Proof. The characteristic polynomial of the equation in Theorem A at $q=0$ is

$$X^2+\left(\alpha+\beta-\frac{k+1}{6}\right)X+\frac{k(k+1)}{144}-\frac{k}{12}(\alpha+\beta)+\gamma+\delta+\epsilon.$$

By the assumption that the equation has a power series solution $1+c_1q+c_2q^2+\cdots$, this polynomial should have $X=0$ as a solution. Thus we have

$$\delta = -\frac{k(k+1)}{144} + \frac{k}{12}(\alpha+\beta) - \gamma - \epsilon.$$

Substituting this into (3) and using

$$\begin{aligned} &3E_4'(\tau) = E_2(\tau)E_4(\tau) - E_6(\tau), \quad 12E_2'(\tau) = E_2(\tau)^2 - E_4(\tau), \\ &2E_6'(\tau) = E_2(\tau)E_6(\tau) - E_4(\tau)^2, \end{aligned} \tag{7}$$

we obtain the theorem. □

Remark 1. The function $\Delta(\tau)$ is the "discriminant" form of weight 12:

$$\Delta(\tau) = \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2) = q - 24q^2 + 252q^3 - 1472q^4 + \cdots.$$

If $f(\tau)$ is a solution of a modular differential equation in Theorem A for some $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}$, then it is easy to see that the function $f(\tau)\Delta(\tau)^l$ is a solution to the modular differential equation of the same type with parameters $\alpha, \beta, \gamma, \delta + l/12, \epsilon$. Thus we may shift solutions by a power of $\Delta(\tau)$ to achieve the requirement of Theorem B.

Remark 2. The equation $\mathcal{D}_k(0,0,0,0)$ was extensively studied by M. Kaneko and M. Koike in [2].

3. Hypergeometric expressions of solutions to the modular differential equations

In this section, we express solutions of the differential equation $\mathcal{D}_k(\alpha, \beta, \gamma, \epsilon)$ explicitly in terms of $E_4(\tau)$, $E_6(\tau)$, the elliptic modular invariant $j(\tau)$, and the Gauss hypergeometric series. Here the elliptic modular invariant $j(\tau)$ is given by

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots.$$

The Gauss hypergeometric differential equation is given as

$$x(1-x)\frac{d^2u}{dx^2} + \{c - (a+b+1)x\}\frac{du}{dx} - abu = 0, \tag{8}$$

where a, b and c are complex parameters. If $c, a-b$ and $c-a-b$ are non-integers, the two functions

$$F(a,b,c;x) \quad \text{and} \quad x^{1-c}F(a-c+1, b-c+1, 2-c; x) \tag{9}$$

are linearly independent solutions, where $F(a,b,c;x)$ represents the Gauss hypergeometric series

$$F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(1)_n(c)_n}x^n, \tag{10}$$

which is convergent for $|x|<1$. The series $F(a,b,c;x)$ is a solution to (8) when c is not a non-positive integer. Moreover, when a and c (resp. b and c) are negative integers with $a > c$ (resp. $b > c$), the polynomial (also denoted by $F(a,b,c;x)$) which is obtained by truncating the sum of (10) at the point where the numerator vanishes is a solution to (8).

Theorem C. *For given $\alpha, \beta, \gamma, \epsilon \in \mathbb{C}$, put*

$$s = \frac{1}{2}\left(2\alpha + 1 - \sqrt{(2\alpha+1)^2 - 16\gamma}\right), \tag{11}$$

$$r = \frac{1}{4}(k - 6s), \tag{12}$$

$$c = \alpha + \beta - \frac{k-5}{6}, \tag{13}$$

and let a and b be the solutions of the equation

$$X^2 - (\beta - \frac{2r}{3} + \frac{1}{3})X + \epsilon - \frac{r}{3}(\beta - \frac{r}{3} + \frac{1}{3}) = 0. \tag{14}$$

Suppose either (i) *c is not an integer, or* (ii) *a and c are negative integers with $a > c$, or* (iii) *b and c are negative integers with $b > c$. Then the differential equation $\mathcal{D}_k(\alpha, \beta, \gamma, \epsilon)$ has two linearly independent solutions*

$$E_4(\tau)^r E_6(\tau)^s F(a, b, c; \frac{1728}{j(\tau)})$$

and

$$E_4(\tau)^r E_6(\tau)^s F(a-c+1, b-c+1, 2-c; \frac{1728}{j(\tau)}) \left(\frac{1728}{j(\tau)}\right)^{1-c}$$

near $i\infty$.

Proof. We transform the Gauss hypergeometric equation (8) by a change of variables into the equation $\mathcal{D}_k(\alpha, \beta, \gamma, \epsilon)$. First put $x = 1728/j(\tau)$. From (7), we have

$$\frac{d}{dx} = \left(\frac{1}{2\pi\sqrt{-1}}\right) \frac{E_4(\tau)^4}{2^6 3^3 E_6(\tau)\Delta(\tau)} \cdot \frac{d}{d\tau}$$

and

$$\begin{aligned} \frac{d^2}{dx^2} = &\left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \frac{E_4(\tau)^8}{2^{12} 3^6 E_6(\tau)^2 \Delta(\tau)^2} \cdot \frac{d^2}{d\tau^2} \\ &+ \left(\frac{1}{2\pi\sqrt{-1}}\right) \frac{(3E_4(\tau)^3 - 8E_6(\tau)^2 - E_2(\tau)E_4(\tau)E_6(\tau))E_4(\tau)^7}{2^{13} 3^7 E_6(\tau)^3 \Delta(\tau)^2} \cdot \frac{d}{d\tau}. \end{aligned}$$

Thus equation (8) is transformed into the equation

$$u''(\tau) + \left(\frac{pE_4(\tau)^3 + qE_6(\tau)^2}{E_4(\tau)E_6(\tau)} - \frac{1}{6}E_2(\tau)\right) u'(\tau) - \frac{1728ab\Delta(\tau)}{E_4(\tau)^2} u(\tau) = 0,$$

where $p = c - a - b - 1/2$ and $q = a + b - 1/3$. Secondly, by changing the unknown $u(\tau) = E_4(\tau)^{-r} E_6(\tau)^{-s} F(\tau)$, the above differential equation transforms into F:

$$F''(\tau) + \left(\frac{p^* E_4(\tau)^3 + q^* E_6(\tau)^2}{E_4(\tau) E_6(\tau)} - \frac{l+1}{6} E_2(\tau) \right) F'(\tau) \\ \left[\frac{l(l+1)}{144} E_2(\tau)^2 - \frac{l}{12} \frac{E_2(\tau) \left(p^* E_4(\tau)^3 + q^* E_6(\tau)^2 \right)}{E_4(\tau) E_6(\tau)} \right. \\ \frac{s}{2} \left(c - a - b + \frac{s}{2} \right) \frac{E_4(\tau)^4}{E_6(\tau)^2} + \left(ab + \frac{r}{3} (a + b + \frac{r}{3}) \right) \frac{E_6(\tau)^2}{E_4(\tau)^2} \\ \left. \left(\frac{r}{3} (c - a - b + s) + \frac{s}{2} (a + b) - ab - \frac{11 l}{144} \right) E_4(\tau) \right] F(\tau) = 0, \tag{15}$$

where $p^* = c - a - b + s - 1/2, q^* = a + b + 2r/3 - 1/3$ and $l = 4r + 6s$. Equating the coefficients of (6) and those of (15), we have

$$k = 4r + 6s, \tag{16}$$

$$\alpha = c - a - b + s - \frac{1}{2}, \tag{17}$$

$$\beta = a + b + \frac{2r}{3} - \frac{1}{3}, \tag{18}$$

$$\gamma = \frac{s}{2} (c - a - b + \frac{s}{2}), \tag{19}$$

$$\epsilon = ab + \frac{r}{3} (a + b + \frac{r}{3}), \tag{20}$$

$$\frac{k}{12} (\alpha + \beta) - \frac{k(k+1)}{144} - \gamma - \epsilon = \frac{r}{3} (c - a - b + s) + \frac{s}{2} (a + b) - ab - \frac{11}{144} (4r + 6s). \tag{21}$$

The last equation (21) is automatic if the first five (16)−(20) are satisfied. Now equations (17) and (19) give

$$s^2 - (2\alpha + 1) s + 4\gamma = 0. \tag{22}$$

Choose a solution s which is given by equation (11). (See Remark 4 below for the effect of choosing another solution as s.) Determine r by (12) so that (16) is satisfied. Then (18) and (20) give a and b as solutions of (14). By (17) and (18) we obtain the value c as in (13). Thus the parameters a, b, c, r, s given as (11)-(14) solve the equations (16)-(21). Since equation (15) comes from the hypergeometric differential equation, this admits two typical solutions (9). This completes the proof. □

Remark 3. For the reader's convenience, we give expressions of the parameters a, b, c, r and s in terms of α, β, γ and δ:

$$r = \frac{1}{4} \left(-3 + k - 6\alpha + 3\sqrt{(1 + 2\alpha)^2 - 16\gamma} \right),$$

$$s = \frac{1}{2} \left(2\alpha + 1 - \sqrt{(2\alpha + 1)^2 - 16\gamma} \right),$$

$$a = \frac{1}{12} \left(5 - k + 6\alpha + 6\beta - 3\sqrt{(1 + 2\alpha)^2 - 16\gamma} + 2\sqrt{(1 + 3\beta)^2 - 36\epsilon} \right),$$

$$b = \frac{1}{12} \left(5 - k + 6\alpha + 6\beta - 3\sqrt{(1 + 2\alpha)^2 - 16\gamma} - 2\sqrt{(1 + 3\beta)^2 - 36\epsilon} \right)$$

and

$$c = \alpha + \beta - \frac{k-5}{6},$$

where the square roots $\sqrt{(1+2\alpha)^2 - 16\gamma}$ and $\sqrt{(1+3\beta)^2 - 36\epsilon}$ are fixed once and for all.

Remark 4. If we choose

$$s = \frac{1}{2}\left(2\alpha + 1 + \sqrt{(2\alpha+1)^2 - 16\gamma}\right)$$

as a solution of (22), we are led to a different hypergeometric equation and hence different expressions of solutions. However, the two expressions are related with each other by the identity

$$F(a, b, c; x) = (1-x)^{c-a-b} F(c-a, c-b, c; x).$$

For example, when $\alpha = \beta = \gamma = \delta = \epsilon = 0$, the two expressions (originally due to [2, Theorem 1])

$$E_4(\tau)^{\frac{k}{4}} F\left(-\frac{k}{12}, -\frac{k-4}{12}, -\frac{k-5}{6}; \frac{1728}{j(\tau)}\right)$$

and

$$E_4(\tau)^{\frac{k-6}{4}} E_6(\tau) F\left(-\frac{k-6}{12}, -\frac{k-10}{12}, -\frac{k-5}{6}; \frac{1728}{j(\tau)}\right)$$

give the same solution.

By using Theorem C, we give a sufficient condition for the differential equation $\mathcal{D}_k(\alpha, \beta, \gamma, \epsilon)$ to have a modular form solution.

Corollary. *Assume that the coefficients of the equation $\mathcal{D}_k(\alpha, \beta, \gamma, \epsilon)$ have at most simple poles at elliptic points (this is equivalent to $\gamma = \epsilon = 0$). Suppose k is a positive integer, and if $-(k-5)/6 + \alpha + \beta$ is a negative integer we moreover suppose $k > 2(5 + 6\alpha + 6\beta)$.*

(i) *When $k \equiv 0 \mod 12$, the equation $\mathcal{D}_k(\alpha, \beta, 0, 0)$ has the modular form*

$$E_4(\tau)^{\frac{k}{4}} F\left(-\frac{k}{12}, -\frac{k-4}{12} + \beta, -\frac{k-5}{6} + \alpha + \beta; \frac{1728}{j(\tau)}\right)$$

of weight k on $SL_2(\mathbb{Z})$ as a solution.

(ii) *If the poles of the coefficients are only at points equivalent to $i = \sqrt{-1}$ (this is equivalent to $\beta = 0$), the equation $\mathcal{D}_k(\alpha, 0, 0, 0)$ has the modular form*

$$E_4(\tau)^{\frac{k}{4}} F\left(-\frac{k}{12}, -\frac{k-4}{12}, -\frac{k-5}{6} + \alpha; \frac{1728}{j(\tau)}\right)$$

of weight k on $SL_2(\mathbb{Z})$ as a solution when $k \equiv 0 \mod 12$ or $k \equiv 4 \mod 12$.

(iii) *If the poles of the coefficients are only at points equivalent to $\rho = e^{2\pi\sqrt{-1}\pi/3}$ (this is equivalent to $\alpha = 0$), the equation $\mathcal{D}_k(0, \beta, 0, 0)$ has the modular* *form*

$$E_4(\tau)^{\frac{k}{4}}F\left(-\frac{k}{12},-\frac{k-4}{12}+\beta,\beta-\frac{k-5}{6};\frac{1728}{j(\tau)}\right)$$
$$=E_4(\tau)^{\frac{k-6}{4}}E_6(\tau)F\left(-\frac{k-6}{12},-\frac{k-10}{12}+\beta,-\frac{k-5}{6}+\beta;\frac{1728}{j(\tau)}\right)$$

of weight k on $SL_2(\mathbb{Z})$ as a solution when $k\equiv 0 \mod 6$.

Proof. We only need to check that the hypergeometric series in each expression becomes a polynomial if the assumption is satisfied, and that the expression is indeed a holomorpihc modular form, which is easily seen. □

4. Equations whose solution spaces have some special properties

In this section, we give a series of the modular differential equations which is parametrized by an even integer k and state some properties of the solution spaces of the equations.

For an even integer k, put

$$i=\begin{cases}0 & \text{for } \frac{k}{2}\equiv 0 \mod 2,\\ 1 & \text{for } \frac{k}{2}\equiv 1 \mod 2,\end{cases}\quad \text{and}\quad j=\begin{cases}0 & \text{for } k\equiv 0 \mod 3,\\ 1 & \text{for } k\equiv 1 \mod 3,\\ 2 & \text{for } k\equiv 2 \mod 3\end{cases}$$

and abbreviate the differential equation $\mathcal{D}_k\left(\frac{1}{2}+i,\frac{2}{3}(1+j),\frac{3}{4}i,\frac{j^2+3j}{9}\right)$ as $\mathcal{D}_k$. Then we have

Proposition 1. *Let k be an even integer. If $k=0$ or $k>2$, the equation $\mathcal{D}_k$ has a modular form of weight k for $SL_2(\mathbb{Z})$ as a solution. If $k=2$ or $k<0$, the solutions of the equation $\mathcal{D}_k$ are obtained by multiplying solutions of $\mathcal{D}_{12+12i+8j-k}(i,j)$ by $\Delta(\tau)^{\frac{1}{6}(6+6i+4j-k)}$. In particular, we have a meromorphic modular form of weight k which is holomorphic on $\mathfrak{H}$ as a solution of $\mathcal{D}_k$ if $k=2$ or $k<0$.*

Proof. From Theorem C, $\mathcal{D}_k$ has two linearly independent solutions

$$E_4(\tau)^{\frac{1}{4}(-6i+k)}E_6(\tau)^i \cdot F\left(\frac{1}{12}(12+6i+4j-k),\frac{1}{12}(6i+4j-k),\frac{1}{6}(12+6i+4j-k);\frac{1728}{j(\tau)}\right) \tag{23}$$

and

$$E_4(\tau)^{\frac{1}{4}(12+6i+8j-k)}E_6(\tau)^i\Delta(\tau)^{\frac{1}{6}(-6-6i-4j+k)} \cdot F\left(\frac{1}{12}(-6i-4j+k),\frac{1}{12}(-12-6i-4j+k),\frac{1}{6}(-6i-4j+k);\frac{1728}{j(\tau)}\right) \tag{24}$$

near $\sqrt{-1}\infty$. Noting the congruence $6i+4j\equiv 0 \mod 12$, we see that the function (23) gives a modular solution when $k=0$ or $k>2$.

If we put $k'=12+12i+8j-k$, functions (23) and (24) become

$$E_4(\tau)^{\frac{1}{4}(12+6i+8j-k')}E_6(\tau)^i \cdot F\left(\frac{1}{12}(-6i-4j+k'),\frac{1}{12}(-12-6i-4j+k'),\frac{1}{6}(-6i-4j+k');\frac{1728}{j(\tau)}\right)$$

and

$$E_4(\tau)^{\frac{1}{4}(-6i+k')}E_6(\tau)^i\Delta(\tau)^{\frac{1}{6}(6+6i+4j-k')}$$
$$\cdot F\left(\frac{1}{12}(12+6i+4j-k'), \frac{1}{12}(6i+4j-k'), \frac{1}{6}(12+6i+4j-k'); \frac{1728}{j(\tau)}\right),$$

respectively. Comparing these with (23) and (24), we conclude the proposition. □

From the proof of Proposition 1, we find that the series of the solution spaces of $\{\mathcal{D}_{12l+\mu}\}_{l\in\mathbb{Z}}$ are symmetrical with centers $k = 6, 10, 12, 14, 16$ and 20 for $\mu = 0, 4, 6, 8, 10$ and 14 respectively. Moreover, when $k = 12, 16, 18, 20, 22$, the equations $\mathcal{D}_k$ have the following properties.

Proposition 2. *When $k = 12, 16, 18, 20, 22$ and 26, the space of solutions of the differential equation $\mathcal{D}_k$ agrees with the space of modular forms of weight k for $SL_2(\mathbb{Z})$.*

Proof. The equation $\mathcal{D}_{12}$ is given by

$$f''(\tau) + \left(\frac{1}{2}\frac{E_4(\tau)^2}{E_6(\tau)} + \frac{2}{3}\frac{E_6(\tau)}{E_4(\tau)} - \frac{13}{6}E_2(\tau)\right)f'(\tau) + \left(13E_2'(\tau) - \frac{3}{2}\frac{E_4'(\tau)E_4(\tau)}{E_6(\tau)} - \frac{4}{3}\frac{E_6'(\tau)}{E_4(\tau)}\right)f(\tau) = 0. \tag{25}$$

By a direct computation using (7), we can check that $E_4(\tau)^3$ and $E_6(\tau)^2$ are the solutions of (25). Because $E_4(\tau)^3$ and $E_6(\tau)^2$ constitute a basis of the space of modular forms of weight 12 for $SL_2(\mathbb{Z})$, we proved the case of $k = 12$. For other values of k, we proceed just in the same way. □

Remark 5. The space of modular forms of weight k for $SL_2(\mathbb{Z})$ is two dimensional only when $k = 12, 18, 20, 22$ and 26.

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Department of Mathematics, Shimane University, Matsue 690-8504 Japan
E-mail address: tsutsumi@math.shimane-u.ac.jp
Current address: Osaka University of Health and Sports Science, 1-1 Asashirodai, Kumatori-cho, Sennan-gun, Osaka 590-0496, Japan
E-mail address: tsutsumi@ouhs.ac.jp