\( \ell_p \ (p > 2) \) DOES NOT COARSELY EMBED INTO A HILBERT SPACE

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Abstract. We show that a Banach space with a normalized symmetric basis
behaving like that of \( \ell_p \ (p > 2) \) cannot coarsely embed into a Hilbert space.

A (not necessarily continuous) map \( f \) between two metric spaces \((X, d)\) and
\((Y, \delta)\) is called a coarse embedding (see [G, 7.G]) if there exist two non-decreasing
functions \( \varphi_1 : [0, \infty) \to [0, \infty) \) and \( \varphi_2 : [0, \infty) \to [0, \infty) \) such that
1. \( \varphi_1(d(x,y)) \leq \delta(f(x), f(y)) \leq \varphi_2(d(x,y)) \),
2. \( \varphi_1(t) \to \infty \) as \( t \to \infty \).

Nowak [N], improving a theorem due to A. N. Dranishnikov, G. Gong, V. Laf-forgue, and G. Yu [DGLY], gave a characterization of coarse embeddability of
general metric spaces into a Hilbert space using a result of Schoenberg on neg-
ative definite kernels. He used this characterization to show that the spaces
\( L_p(\mu) \) coarsely embed into a Hilbert space for \( p < 2 \). In this article, we show that \( \ell_p \)
does not coarsely embed into a Hilbert space when \( p > 2 \). It was already proved
in [DGLY] that the Lipschitz universal space \( c_0 \) (see [A]) does not coarsely embed
into a Hilbert space.

In its full generality, the statement of our result is as follows:

Theorem 1. Suppose that a Banach space \( X \) has a normalized symmetric basis
\((e_n) \) and that \( \lim \inf_{n \to \infty} n^{-\frac{1}{2}} \left\| \sum_{i=1}^{n} e_i \right\| = 0 \). Then \( X \) does not coarsely embed into a
Hilbert space.

In [Y], Yu proved that a discrete metric space with bounded geometry must
satisfy the coarse geometric Novikov conjecture if it coarsely embeds into a Hilbert
space, and in [KY] G. Kasparov and Yu proved that to get the same conclusion it
is sufficient that the metric space coarsely embeds into a uniformly convex Banach
space. Our theorem suggests that the result of [KY] cannot be deduced from the
earlier theorem in [Y], but as yet there is no example of a discrete metric space
with bounded geometry which coarsely embeds into \( \ell_p \) for some \( 2 < p < \infty \) but not

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into $\ell_2$. (The reader should be warned that what we called a “coarse embedding” is called a “uniform embedding” in many places, including [DGLY], [KY], and [Y]. Following [N], we use the term coarse embedding to avoid confusion with the closely related notion of uniform embedding as it is used in non-linear Banach space theory [BL]: i.e., a bi-uniformly continuous mapping.)

Besides Schoenberg’s classical work [S] on positive definite functions, an important tool for proving the theorem is Theorem 5.2 in [AMM], which asserts that the hypothesis on $X$ in the theorem implies that every symmetric continuous positive definite function on $X$ is constant. We present the proof in five steps.

**Step 0: Reducing to the $\alpha$-Hölder case**

Let $f : X \to H$ be a coarse embedding satisfying

1. $\varphi_1(||x - y||) \leq ||f(x) - f(y)|| \leq \varphi_2(||x - y||)$,
2. $\varphi_1(t) \to \infty$ as $t \to \infty$.

Our first claim is that we do not lose generality by assuming that $\varphi_2(t) = t^\alpha$ with $0 < \alpha < \frac{1}{2}$.

To prove this claim, note first that $(x, y) \mapsto ||f(x) - f(y)||^2$ is a negative definite kernel on $X$. This can be seen by direct computations (see [N, Proposition 3.1]). We refer the reader to [BL, Chapter 8] or [N, Section 2] for the definitions of negative definite kernels and negative definite functions.

So ([N, Lemma 4.2]), for any $0 < \alpha < 1$, the kernel $N(x, y) = ||f(x) - f(y)||^{2\alpha}$ is also negative definite and satisfies $N(x, x) = 0$ (such a negative definite kernel is called normalized).

As a result, a theorem of Schoenberg ([S] and [BL, Chapter 8]) allows us to find a Hilbert space $H_\alpha$ and a function $f_\alpha : X \to H_\alpha$ such that $N(x, y) = ||f_\alpha(x) - f_\alpha(y)||^2$.

On the other hand, since $X$, being a normed space, is (metrically) convex, the original function $f : X \to H$ is Lipschitz for large distances. In fact, using the upper bound $\varphi_2(\cdot)$, we can use the same proof as in [BL] proof of Proposition 1.11]. Namely, assume $||x - y|| \geq 1$ and let $n$ be the smallest integer greater than or equal to 1; breaking the segment $[x, y]$ into $n$ pieces of length less than or equal to 1, we get $||f(x) - f(y)|| \leq n\varphi_2(1) \leq 2\varphi_2(1)||x - y||$. Consequently, without loss of generality, we can assume by rescaling that we have the following for $||x - y|| \geq 1$:

$$||f(x) - f(y)||_H \leq ||x - y||$$

and

$$(\varphi_1(||x - y||))^\alpha \leq ||f_\alpha(x) - f_\alpha(y)||_{H_\alpha} \leq ||x - y||^\alpha.$$

Now, let $N$ be a 1-net in $X$ (i.e. $N$ is a maximal 1-separated subset of $X$). The restriction of $f_\alpha$ to $N$ is $\alpha$-Hölder, so if $0 < \alpha < \frac{1}{2}$, then we can extend $f_\alpha$ to an $\alpha$-Hölder map $\tilde{f_\alpha}$ defined on the whole of $X$ (see [WW] last statement of Theorem 19.1):

$$\tilde{f_\alpha} : X \to H_\alpha,$$

$$\forall x \in N, \tilde{f_\alpha}(x) = f_\alpha(x),$$

and

$$\forall x, y \in X, ||f_\alpha(x) - f_\alpha(y)||_{H_\alpha} \leq ||x - y||^\alpha.$$
This finishes the proof of our reduction to the case where $f$ is $\alpha$-Hölder and thus uniformly continuous. So from now on we will assume that our coarse embedding is a map $f : X \to H$ satisfying the following for all $x, y \in X$:

$$\varphi_1(\|x - y\|) \leq \|f(x) - f(y)\| \leq \|x - y\|^\alpha$$

where $\varphi_1(t) \to \infty$ as $t \to \infty$.

**Step 1**

Set $N(x, y) = \|f(x) - f(y)\|^2$. Then $N$ is a normalized (i.e. $N(x, x) = 0$) negative definite kernel on $X$ (see [N, Proposition 3.1]). Now if we write $\phi_1(t) = (\varphi_1(t))^2$ and $\phi_2(t) = t^{2\alpha}$, then $N$ satisfies:

$$\begin{align*}
\phi_1(\|x - y\|) &\leq N(x, y) \leq \phi_2(\|x - y\|), \\
\phi_1(t) &\to \infty \text{ as } t \to \infty.
\end{align*}$$

**Step 2**

The argument in this step comes from [AMM, Lemma 3.5].

Let $\mu$ be an invariant mean on the bounded functions on $X$ (see e.g. [BL] for the definition of invariant means). Define:

$$g(x) = \int_X N(y + x, y) \, d\mu(y).$$

Then we have the following for $g$:

- $g$ is well defined because the map $y \mapsto N(y + x, y)$ is bounded for each $x \in X$.
- $g(0) = \int_X N(y, y) \, d\mu(y) = 0$.
- For scalars $(c_i)_{1 \leq i \leq n}$ satisfying $\sum_{i=1}^n c_i = 0$, we have:

$$\sum_{i,j=1}^n c_ic_j g(x_i - x_j) = \sum_{i,j} c_ic_j \int_X N(y + x_i - x_j, y) \, d\mu(y)$$

$$= \sum_{i,j=1}^n c_ic_j \int_X N(y + x_i, y + x_j) \, d\mu(y)$$

$$= \int_X \left( \sum_{i,j=1}^n c_ic_j N(y + x_i, y + x_j) \right) \, d\mu(y)$$

$$= \int_X (\leq 0) \, d\mu(y)$$

$$\leq 0.$$ 

This is because $\mu$ is translation invariant, and $N$ is negative definite. This shows that $g$ is a negative definite function on $X$. 

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Finally, since \( \int_X d\mu(y) = 1 \), we have:

\[
\phi_1(\|x\|) \leq g(x) \leq \phi_2(\|x\|).
\]

In summary, we have found a negative definite function \( g \) on \( X \) which satisfies \( g(0) = 0 \) and \( \phi_1(\|x\|) \leq g(x) \leq \phi_2(\|x\|) \), where \( \phi_1(t) \to \infty \) as \( t \to \infty \).

**Step 3**

Let \( (e_n)_n \) be the normalized symmetric basis for \( X \). This means that for any choice of signs \( (\theta_n)_n \in \{-1,+1\} \) and any choice of permutation \( \sigma : \mathbb{N} \to \mathbb{N} \),

\[
\| \sum_n \theta_n a_n e_{\sigma(n)} \|_X = \| \sum_n a_n e_n \|_X.
\]

The purpose of this step is to show that the negative definite function \( g \) we found in the previous step can be chosen to be symmetric, i.e. to satisfy for any choice of signs \( (\theta_n)_n \in \{-1,+1\} \) and any choice of permutation \( \sigma : \mathbb{N} \to \mathbb{N} \) the equality:

\[
g \left( \sum_n \theta_n a_n e_{\sigma(n)} \right) = g \left( \sum_n a_n e_n \right).
\]

For \( x = \sum_{n=1}^{\infty} x_n e_n \in X \), define \( g_m(x) \) to be the average of \( g \left( \sum_{n=1}^{\infty} \theta_n x_n e_{\sigma(n)} \right) \) over all choices of signs \( \theta \) and permutations \( \sigma \) with the restrictions that \( \theta_n = 1 \) for \( n > m \) and \( \sigma(n) = n \) for \( n > m \).

It follows that for all such \( \theta, \sigma \), and for all \( x = \sum_{n=1}^{\infty} x_n e_n \in X \),

\[
g_m \left( \sum_{n=1}^{\infty} \theta_n x_n e_{\sigma(n)} \right) = g_m \left( \sum_{n=1}^{\infty} x_n e_n \right).
\]

Moreover, we also have

\[
\phi_1(\|x\|) \leq g_m(x) \leq \phi_2(\|x\|).
\]

Next we show that the sequence \( (g_m)_m \) is equicontinuous. To check this, let us first check the continuity of \( g \):

\[
|g(a) - g(b)| \leq \int_X |N(y + a, y) - N(y + b, y)| \, d\mu(y)
\]

\[
= \int_X \|f(y + a) - f(y)\|^2 - \|f(y + b) - f(y)\|^2 \, d\mu(y)
\]

\[
= \int_X (\|f(y + a) - f(y)\| + \|f(y + b) - f(y)\|) \cdot \|f(y + a) - f(y)\| - \|f(y + b) - f(y)\| \, d\mu(y)
\]

\[
\leq \int_X (\|f(y + a) - f(y)\| + \|f(y + b) - f(y)\|) \cdot \|f(y + a) - f(y)\| \, d\mu(y)
\]

\[
\leq \int_X (\|a\|^\alpha + \|b\|^\alpha) \|a - b\|^\alpha \, d\mu(y).
\]

So \( |g(a) - g(b)| \leq \|a - b\|^\alpha (\|a\|^\alpha + \|b\|^\alpha) \) and \( g \) is continuous.
Now for the equicontinuity of \((g_m)_m\):
\[
|g_m(a) - g_m(b)| = \| \text{ave} \left( g \left( \sum \theta_n a_n e_{\sigma(n)} \right) - g \left( \sum \theta_n b_n e_{\sigma(n)} \right) \right) \| \\
\leq \| \text{ave} \left( g \left( \sum \theta_n a_n e_{\sigma(n)} \right) - g \left( \sum \theta_n b_n e_{\sigma(n)} \right) \right) \| \\
\leq \text{ave} \left( \| \sum \theta_n a_n e_{\sigma(n)} - \sum \theta_n b_n e_{\sigma(n)} \|^\alpha \right) \\
\cdot \left( \| \sum \theta_n a_n e_{\sigma(n)} \|^\alpha + \| \sum \theta_n b_n e_{\sigma(n)} \|^\alpha \right) \\
= \text{ave} (\| a - b \|\alpha (\| a \|^\alpha + \| b \|^\alpha)) \\
= \| a - b \|\alpha (\| a \|^\alpha + \| b \|^\alpha) .
\]

So by Ascoli’s theorem [R, Chapter 7, Section 10], there is a subsequence \((g_{m_k})_k\) of \((g_m)_m\) which converges pointwise to a continuous function \(\tilde{g}\). The property of the \(g_m\)’s implies that \(\tilde{g}\) must necessarily be symmetric. We have that \(\tilde{g}(0) = 0\), and that \(\phi_1(\|x\|) \leq \tilde{g}(x) \leq \phi_2(\|x\|)\). Finally, as it is easily checked that the \(g_m\)’s are negative definite functions, it also follows easily that \(\tilde{g}\) is a negative definite function.

**Step 4**

There is a relation between negative and positive definite kernels as given by a result of Schoenberg [S]; see also [BL, Chapter 8]. This result states that a kernel \(K\) on \(X\) is negative definite if and only if \(e^{-tK}\) is positive definite for every \(t > 0\).

Since \(\lim_{n \to \infty} \left\| \frac{e_1 + e_2 + \cdots + e_n}{\sqrt{n}} \right\| = 0\), and \(\tilde{f} = e^{-\tilde{g}}\) is a symmetric continuous positive definite function on \(X\), we conclude by a theorem of Aharoni, Maurey and Mityagin (see [AMM, Theorem 5.2]), that \(\tilde{f}\) is constant.

On the other hand, \(\tilde{f}(0) = e^{-\tilde{g}(0)} = 1\), while \(0 \leq \tilde{f}(x) \leq e^{-\phi_1(\|x\|)} \to 0\) as \(\|x\| \to \infty\). This gives a contradiction and finishes the proof. 

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