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MULTIPLICATION AND DIVISION BY INNER FUNCTIONS IN THE SPACE OF BLOCH FUNCTIONS

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ABSTRACT. A subspace X of the Hardy space H^1 is said to have the f-property if $h/I \in X$ whenever $h \in X$ and I is an inner function with $h/I \in H^1$. We let $\mathcal{B}$ denote the space of Bloch functions and $\mathcal{B}_0$ the little Bloch space. Anderson proved in 1979 that the space $\mathcal{B}_0 \cap H^\infty$ does not have the f-property. However, the question of whether or not $\mathcal{B} \cap H^p$ $(1 \le p < \infty)$ has the f-property was open. We prove that for every $p \in [1, \infty)$ the space $\mathcal{B} \cap H^p$ does not have the f-property.

We also prove that if B is any infinite Blaschke product with positive zeros and G is a Bloch function with $|G(z)| \to \infty$, as $z \to 1$, then the product BG is not a Bloch function.

1. Introduction and statement of results

We denote by Δ the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and by H^p $(0 < p \le \infty)$ the classical Hardy spaces of analytic functions in Δ (see [6] and [9]). A function I, analytic in Δ, is said to be an inner function if $I \in H^\infty$ and I has a radial limit $I(e^{i\theta})$ of modulus one for almost every $e^{i\theta} \in \partial\Delta$.

Given a function $v \in L^\infty(\partial\Delta)$, the associated *Toeplitz operator* T_v is defined by

$$(T_v f)(z) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{v(\zeta) f(\zeta)}{\zeta - z}\, d\zeta \quad (f \in H^1,\ z \in \Delta).$$

Definition 1.1. A subspace X of H^1 is said to have the K-property if $T_{\overline{\psi}}(X) \subset X$ for any $\psi \in H^\infty$.

Definition 1.2. A subspace X of H^1 is said to have the f-property if $h/I \in X$ whenever $h \in X$ and I is an inner function with $h/I \in H^1$.

These notions were introduced by Havin [12] and Korenblum [14]. The K-property implies the f-property: indeed, if $h \in H^1$, I is inner and $h/I \in H^1$, then $h/I = T_{\overline{I}} h$.

In addition to the Hardy spaces H^p $(1 < p < \infty)$ many other spaces such as the Dirichlet space $\mathcal{D}$ ([12], [14]) and several spaces of Dirichlet type (see [7], [15] and

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[19]), the spaces $BMOA$ and $VMOA$ [13] and the spaces Q_p $(0 < p < 1)$ [8], have the K-property.

Clearly, H^1 has the f-property but an argument of duality shows that it does not possess the K-property. Hedenmalm proved in [13] that $VMOA \cap H^\infty$ has the f-property but does not have the K-property. More generally, it is proved in [13] that no subspace of H^∞ containing the disc algebra has the K-property.

The first example of a space not possessing the f-property was given by Gurarii [11] who proved that the space of analytic functions in Δ with an absolutely convergent power series does not have the f-property.

Recall that if f is an analytic function in Δ, then f is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\Delta}(1-|z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions is denoted by $\mathcal{B}$. The little Bloch space $\mathcal{B}_0$ consists of those $f \in \mathcal{B}$ such that $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. Alternatively, $\mathcal{B}_0$ is the closure of the polynomials in the Bloch norm. We mention [2] for the theory of Bloch functions. Let us recall that $H^\infty \subsetneq BMOA \subsetneq \mathcal{B}$ and $VMOA \subsetneq \mathcal{B}_0$.

Anderson proved in [1] that $\mathcal{B}_0 \cap H^\infty$ does not have the f-property. Consequently, the same is true for $\mathcal{B}_0 \cap H^p$ for every $p \in [1, \infty)$. It is natural to ask the following question: Does $\mathcal{B} \cap H^p$ $(1 \le p \le \infty)$ have the f- or K-property?

Since $H^\infty \subset \mathcal{B}$, we see that $\mathcal{B} \cap H^\infty = H^\infty$ which has the f-property but does not have the K-property. It is easy to prove the following result.

Proposition 1.3. *If $1 \le p < \infty$, then $\mathcal{B} \cap H^p$ does not have the K-property.*

In fact, we can prove the following stronger result.

Theorem 1.4. *If $1 \le p < \infty$, then $\mathcal{B} \cap H^p$ does not have the f-property.*

Next we shall consider products of the form $B \cdot f$ with $B \in H^\infty$ and $f \in \mathcal{B}$ but before doing so it is convenient to recall some definitions and results. If a sequence of points $\{a_n\}$ in Δ satisfies the *Blaschke condition* $\sum_{n=1}^\infty (1-|a_n|) < \infty$, the corresponding Blaschke product B is defined as

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}.$$

Such a product is analytic in Δ, and, in fact, it is an inner function. If there exists a positive constant δ such that $\prod_{m\neq n}\left|\frac{a_n - a_m}{1-\overline{a}_n a_m}\right| \ge \delta$, for all n, we say that the sequence $\{a_n\}$ is *uniformly separated* and that B is an *interpolating Blaschke product.* Equivalently,

$$(1.1) \qquad B \text{ is an interpolating Blaschke product} \quad \Leftrightarrow \quad \inf_{n\ge 1}(1-|a_n|^2)|B'(a_n)| > 0.$$

Thus no interpolating Blascke product belongs to $\mathcal{B}_0$. Sarason [20] proved that $\mathcal{B}_0$ contains infinite Blaschke products. Other constructions of such products were given by Stephenson in [21] and Bishop in [3], where a description of all H^∞-functions in $\mathcal{B}_0$ is also given.

We recall that a function f which is meromorphic in Δ is a normal function in the sense of Lehto and Virtanen if

$$\sup_{z\in\Delta}(1-|z|^2)\frac{|f'(z)|}{1+|f(z)|^2} < \infty.$$

We refer to [2] and [17] for the theory of normal functions. Certainly, any Bloch function is normal. Using (1.1) we can deduce the following result.

Proposition 1.5. *If B is an interpolating Blaschke product whose sequence of zeros is $\{a_n\}_{n=1}^{\infty}$ and G is an analytic function in Δ with $G(a_n) \to \infty$, as $n \to \infty$, then the function $f = B \cdot G$ is not normal (and, hence, it is not a Bloch function).*

Proposition 1.5 has been used by several authors (see e.g. [16], [5], [22], [23], [10] and [4]) to construct distinct classes of non-normal functions. We can prove a result of this kind dealing with Blaschke products with zeros in a radius but not necessarily interpolating.

Theorem 1.6. *Let B be an infinite Blaschke product whose sequence of zeros $\{a_n\}$ is contained in the radius $(0, 1)$ and let G be a Bloch function such that $G(z) \to \infty$, as $z \to 1$. Then the function $f = B \cdot G$ is not a Bloch function.*

2. Division by inner functions

Even though Proposition 1.3 follows from Theorem 1.4, we shall give a direct proof of it. We shall use the following easy lemma.

Lemma 2.1. *The space $\mathcal{B}_0 \cap H^p$ $(1 \leq p < \infty)$ is the closure of the polynomials in $\mathcal{B} \cap H^p$, that is, for any $f \in \mathcal{B}_0 \cap H^p$ there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ which converges to f both in the Bloch norm and in the H^p-norm.*

Proof. Take $f \in \mathcal{B}_0 \cap H^p$ with $1 \leq p < \infty$. For $0 < r < 1$, set $f_r(z) = f(rz)$ $(z \in \Delta)$. Also set $r_n = 1 - \frac{1}{n}$, $n = 1, 2, \dots$. Using Theorem 2.1 of [2] and Theorem 2.6 of [6], we see that

$$\max\left(\|f - f_{r_n}\|_{\mathcal{B}}, \|f - f_{r_n}\|_{H^p}\right) \longrightarrow 0, \quad \text{as } n \to \infty. \tag{2.1}$$

For each n, f_{r_n} is analytic in $\{|z| \leq 1\}$, hence, we can find a polynomial $P_n(z)$ such that $|f_{r_n}(z) - P_n(z)| < \frac{1}{n}$ for $|z| \leq 1$. Note that if $g \in H^{\infty}$, then $\|g\|_{\mathcal{B}} \leq 2\|g\|_{H^{\infty}}$ (see p. 13 of [2]) and $\|g\|_{H^p} \leq \|g\|_{H^{\infty}}$. This and (2.1) give

$$\begin{aligned} &\max(\|f - P_n\|_{\mathcal{B}}, \|f - P_n\|_{H^p}) \\ &\qquad \leq \max(\|f - f_{r_n}\|_{\mathcal{B}}, \|f - f_{r_n}\|_{H^p}) + \frac{2}{n} \xrightarrow[n \to \infty]{} 0. \end{aligned}$$

□

Proof of Proposition 1.3. Take $p \in [1, \infty)$ and suppose that $\mathcal{B} \cap H^p$ has the K-property. Then for any inner function I, we would have that $T_{\overline{I}}(\mathcal{B} \cap H^p) \subset \mathcal{B} \cap H^p$. Using the closed graph theorem, it would follow that $T_{\overline{I}}$ is continuous from $\mathcal{B} \cap H^p$ into itself. Now, bearing in mind that $T_{\overline{I}}$ maps polynomials into polynomials and using Lemma 2.1, we would deduce that $T_{\overline{I}}(\mathcal{B}_0 \cap H^p) \subset \mathcal{B}_0 \cap H^p$ for any inner function I. Hence, $\mathcal{B}_0 \cap H^p$ would possess the f-property. But, as mentioned above, this is not true. □

Before embarking into the proof of Theorem 1.4, let us note that if f is an analytic function in Δ we shall set

$$M_{\infty}(r, f) = \sup_{|z| \leq r} |f(z)|, \quad 0 < r < 1.$$

Proof of Theorem 1.4. Let B be an infinite Blaschke product in $\mathcal{B}_0$ whose sequence of zeros $\{a_n\}_{n=1}^{\infty}$ contains a subsequence which tends to 1. Set

$$\varphi(r) = \sup_{r\le|z|<1}(1-|z|^2)|B'(z)|, \quad \phi(r) = \frac{1}{\varphi(r)}, \quad 0 \le r < 1. \tag{2.2}$$

Since $B \in \mathcal{B}_0$, it follows that $\phi(r) \uparrow \infty$, as $r \uparrow 1$.

Let $\{\gamma_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers with $0 < \gamma_n < 1$ for all n such that if

$$D = \Delta \cup \left(\bigcup_{n=1}^{\infty} \{z \in \mathbb{C} : |\operatorname{Im} z| < \gamma_n, \operatorname{Re} z > 0, \, n \le |z| < n+1\} \right) \tag{2.3}$$

and F is the conformal mapping from Δ onto D with $F(0) = 0$ and $F'(0) > 0$, then

$$M_\infty(r, F) = \mathrm{O}\left(\phi(r)\right), \quad \text{as } r \to 1. \tag{2.4}$$

Such a sequence $\{\gamma_n\}$ can be constructed using the Carathèodory kernel theorem (see Theorem 3.2 of [4] and its proof). Since $\operatorname{Im} F$ is bounded, it follows that

$$F \in BMOA \subset \mathcal{B}. \tag{2.5}$$

Also, it is clear that

$$F(z) \to \infty, \quad \text{as } z \to 1. \tag{2.6}$$

Set

$$f(z) = B(z)F(z), \quad z \in \Delta. \tag{2.7}$$

Using (2.2), (2.4), (2.5) and bearing in mind that $|B(z)| \le 1$, we deduce that there exist two positive constants C_1 and C_2 such that

$$\begin{aligned}(1-|z|^2)|f'(z)| &\le (1-|z|^2)\,|B'(z)|\,|F(z)| + (1-|z|^2)\,|F'(z)|\,|B(z)| \\ &\le C_1\varphi(|z|)\phi(|z|) + C_2 = C_1 + C_2,\end{aligned}$$

for all $z \in \Delta$. Hence, $f \in \mathcal{B}$. Note that (2.5) implies that $F \in H^p$ for all $p < \infty$, which, since B is bounded, implies that the same is true for f. Consequently, we have proved that

$$f \in \mathcal{B} \cap H^p, \quad 1 \le p < \infty. \tag{2.8}$$

Now let $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence of the sequence $\{a_n\}$ such that

$$a_{n_k} \to 1, \quad \text{as } k \to \infty, \tag{2.9}$$

and

$$\{a_{n_k}\}_{k=1}^{\infty} \quad \text{is uniformly separated.} \tag{2.10}$$

Let B_1 be the Blaschke product whose sequence of zeros is $\{a_{n_k}\}_{k=1}^{\infty}$ and set

$$B_2 = \frac{B}{B_1}, \quad g = \frac{f}{B_2} = B_1 F. \tag{2.11}$$

It is clear that B_2 is a Blaschke product and that $g \in H^p$ for all $p < \infty$. Next we are going to prove that $g \notin \mathcal{B}$. Clearly, this implies that $\mathcal{B} \cap H^p$ $(1 \le p < \infty)$ does not possess the f-property.

Since $\{a_{n_k}\}$ is uniformly separated, there exists $A > 0$ such that

$$(1-|a_{n_k}|^2)|B_1'(a_{n_k})| \ge A, \quad \text{for all } k.$$

Then, using (2.9), (2.6), (2.11) and Proposition 1.5, we deduce that $g = f/B_2$ is not a Bloch function. □

3. Multiplication by Blaschke products with zeros in a radius

Proof of Theorem 1.6. Let $\{a_n\}$, B and G be as in the statement of Theorem 1.6 and set $g = BG$. Take $\alpha \in \Delta \setminus (-1,1)$ such that α is not in the set $\{B(a) : a \in \Delta,\ B'(a) = 0\}$. Clearly, $B\left((0,1)\right) \subset (-1,1)$ and, hence, α is not a cluster point of $B|_{(0,1)}$. Then, using a result of Marshall and Sarason (see [18]), we deduce that the Frostman shift B_α defined by

$$B_\alpha(z) = \frac{B(z) - \alpha}{1 - \overline{\alpha} B(z)}, \quad z \in \Delta, \tag{3.1}$$

is an interpolating Blaschke product. Thus, if $\{b_n\}_{n=1}^{\infty}$ is the sequence of zeros of B_α, there exists a positive constant δ such that

$$(1 - |b_n|^2)|B'_\alpha(b_n)| \geq \delta, \quad \text{for all } n. \tag{3.2}$$

Also, using Theorem 6.1 on p. 75 of [9], we easily see that $b_n \to 1$, as $n \to \infty$. Note that there exists two positive constants A_1 and A_2 such that

$$A_1|B'_\alpha(z)| \leq |B'(z)| \leq A_2|B'_\alpha(z)|, \quad z \in \Delta.$$

This and (3.2) give

$$\begin{aligned}(1 - |b_n|^2)|g'(b_n)| &\geq (1 - |b_n|^2)|B'(b_n)||G(b_n)| - (1 - |b_n|^2)|G'(b_n)||B(b_n)| \\ &\geq A_1(1 - |b_n|^2)|B'_\alpha(b_n)||G(b_n)| - \|G\|_{\mathcal{B}} \\ &\geq A_1\delta|G(b_n)| - \|G\|_{\mathcal{B}} \to \infty, \quad \text{as } n \to \infty.\end{aligned}$$

Consequently, g is not a Bloch function. □

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