

ON BANACH SPACES WITH FEW SPREADING MODELS

BÜNYAMIN SARI

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ABSTRACT. If the set of spreading models of a Banach space X is countable (up to equivalence), then it cannot contain a strictly increasing infinite chain of spreading models generated by normalized weakly null sequences. Moreover, such a space X must have a spreading model which is ‘close’ to c_0 or ℓ_p for some $1 \leq p < \infty$.

1. INTRODUCTION

In this note we address a problem on the structure of the set of spreading models of Banach spaces, and make a few remarks on other related problems.

Recall that, by Ramsey theory, for every normalized basic sequence (y_i) in a Banach space X and for every $(\varepsilon_n) \searrow 0$ there exists a subsequence (x_i) of (y_i) and a normalized basic sequence (\tilde{x}_i) in some Banach space \tilde{X} such that: For all $n \in \mathbb{N}$, $(a_i)_{i=1}^n \in [-1, 1]^n$ and $n \leq k_1 < \dots < k_n$,

$$\left| \left\| \sum_{i=1}^n a_i x_{k_i} \right\| - \left\| \sum_{i=1}^n a_i \tilde{x}_i \right\| \right| < \varepsilon_n.$$

The sequence (\tilde{x}_i) is called *the spreading model* of (x_i) and it is a suppression 1-unconditional basic sequence if (y_i) is weakly null. (See [BL] for further background on spreading models.)

Consider the set $SP(X)$ of all spreading models (\tilde{x}_i) of a Banach space X equipped with the partial order which is defined by domination. $(\tilde{x}_i) \geq (\tilde{y}_i)$ if for some $K < \infty$, $K \left\| \sum_i a_i \tilde{x}_i \right\| \geq \left\| \sum_i a_i \tilde{y}_i \right\|$ for all scalars (a_i) . We identify (\tilde{x}_i) and (\tilde{y}_i) in $SP(X)$ if $(\tilde{x}_i) \geq (\tilde{y}_i)$ and $(\tilde{y}_i) \geq (\tilde{x}_i)$. By $SP_\omega(X)$ denote the subset of $SP(X)$ which consists of spreading models generated by normalized weakly null sequences. A study of the structure of $SP(X)$ was initiated by Androulakis, Odell, Schlumprecht and Tomczak-Jaegermann [AOST]. In particular, they proved that every countable subset of $SP_\omega(X)$ admits an upper bound in $SP_\omega(X)$ (see Theorem 1.1 below).

In this note we consider a problem in this context posed to us by E. Odell. Suppose that $SP_\omega(X)$ contains a strictly increasing sequence $(\tilde{x}_i^1)_i < (\tilde{x}_i^2)_i < \dots < (\tilde{x}_i^n)_i < \dots$ of spreading models. Does it follow that $SP_\omega(X)$ also contains an uncountable increasing chain of spreading models? We show that this problem has a positive answer (Theorem 2.2).

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Examples of Banach spaces X for which $SP_\omega(X)$ is infinite and does not contain an infinite increasing chain exist. For instance, if $2 < p_1 < p_2 < \dots$, then the space $X = (\sum \oplus \ell_{p_i})_{\ell_2}$ is such an example. Clearly (the unit vector bases of), $\ell_2, \ell_{p_1}, \ell_{p_2}, \dots$ form a decreasing infinite chain in $SP_\omega(X)$. Moreover, it is not difficult to show that, up to equivalence, these are the only spreading models of X (cf. p. 20 of [AOST]).

It is an interesting open question whether a Banach space X with few spreading models (e.g., finite or countable) must admit a c_0 or ℓ_p spreading model. This is not known even when X has, up to equivalence, a unique spreading model. This case, a problem due to S. Argyros, is closely related to a problem raised by H. Rosenthal. If a Banach space X has a basis (e_i) with the property that every normalized block sequence has a subsequence which is equivalent to (e_i) , must (e_i) be equivalent to the unit vector basis of ℓ_p or c_0 ? Exploiting this connection and using Theorem 1.1, some partial results to Rosenthal's problem were obtained in [FPR]. In the second part of the paper we obtain some improvements to these results. In particular, we show that if X has countably many spreading models, then one of them must be 'close' to some ℓ_p or c_0 (see Theorem 3.4).

Throughout the paper we adapt the notation from [AOST]. Our main tool is the following *the upper bound theorem* from [AOST], which we state in a slightly different (but equivalent) formulation.

Theorem 1.1 ([AOST]). *Let $(C_n) \subset (0, \infty)$ such that $\sum_n C_n^{-1} < \infty$ and let $(\tilde{x}_i^{(n)})$, $n = 1, 2, \dots$, be in $SP_\omega(X)$. Then there exists (\tilde{y}_i) in $SP_\omega(X)$ and a constant K , where $K \leq (\sum_n C_n^{-1})$, such that (\tilde{y}_i) KC_n -dominates (\tilde{x}_i^n) for all $n \in \mathbb{N}$.*

The value of the constant K above, which is not stated in the original statement in [AOST], can be easily verified by doing the proof carefully to obtain the best possible estimates. Moreover, it also follows from the construction in the proof of the theorem that there exists a constant $D \leq \min\{C_n : n \in \mathbb{N}\}$ such that for all $(a_j)_{j=1}^m$,

$$(1) \quad \left\| \sum_{j=1}^m a_j \tilde{y}_j \right\| \leq D \sum_{i=1}^{\infty} C_i^{-1} \left\| \sum_{j=1}^m a_j \tilde{x}_j^{(i)} \right\|.$$

The construction also yields that if the theorem is applied to finitely many $(\tilde{x}_i^{(n)})$ for $n \leq m$, then (see remark 3.5(b) of [AOST]) (\tilde{y}_i) in the theorem is equivalent to the norm given by

$$(2) \quad \|(a_i)\| = \max_{n \leq m} \left\| \sum_i a_i \tilde{x}_i^{(n)} \right\|.$$

In particular, if $(\tilde{x}_i^{(1)})_i \leq (\tilde{x}_i^{(2)})_i \leq \dots \leq (\tilde{x}_i^{(m)})_i$, then for all $m \in \mathbb{N}$, $(\tilde{y}_i)_i$ above is equivalent to $(\tilde{x}_i^{(m)})_i$.

2. INCREASING CHAIN IMPLIES UNCOUNTABLE

In the proof of the main theorem we will make use of the following transfinite induction principle (cf. Proposition 3.7 of [AOST]).

Lemma 2.1. *Let (M, \leq) be a non-empty partially ordered set satisfying the following two conditions:*

- (i) M does not have a maximal element with respect to \leq .

(ii) For every $(A_i) \subset M$ there exists $A \in M$ such that $A_i \leq A$ for every $i \in \mathbb{N}$.

Then for all ordinals $\alpha < \omega_1$ there exists $A^\alpha \in M$ such that if $\alpha < \beta < \omega_1$, then $A^\alpha < A^\beta$.

Theorem 2.2. *Suppose that a Banach space X admits a strictly increasing infinite sequence of spreading models generated by normalized weakly null sequences. Then X also admits an uncountable strictly increasing chain of such spreading models.*

It was shown in [FPR] (Proposition 9) that for a separable space X , the cardinality of $SP(X)$ is, up to equivalence, either continuum or at most countable. Thus we have

Corollary 2.3. *If a separable space X satisfies the assumption of the above theorem, then X has continuum many non-equivalent spreading models.*

Proof of Theorem 2.2. Let $(\tilde{x}_i^{(1)}) < (\tilde{x}_i^{(2)}) < \dots < (\tilde{x}_i^{(n)}) < \dots$ be a strictly increasing infinite sequence in $SP_\omega(X)$.

Define a function R on c_{00} , the linear space of all finitely supported sequences of reals, by

$$(3) \quad R(a) = \sup_n \left\| \sum_i a_i \tilde{x}_i^{(n)} \right\|,$$

for all $a = (a_i) \in c_{00}$. We will write $(\tilde{y}_i) < R$ if there exists a constant K such that for all $a = (a_i) \in c_{00}$, $\|\sum_i a_i \tilde{y}_i\| \leq KR(a)$, and for all $N \geq 1$ there exists $a = (a_i) \in c_{00}$ so that $R(a) > N\|\sum_i a_i \tilde{y}_i\|$. Then, clearly, we have $(\tilde{x}_i^{(n)}) < R$, for all n .

Let $C = \{(\tilde{y}_i) \in SP_\omega(X) : (\tilde{y}_i) < R\}$. Let \mathcal{P} be the class of all chains (totally ordered subsets) in C which contain $(\tilde{x}_i^{(1)}) < (\tilde{x}_i^{(2)}) < \dots < (\tilde{x}_i^{(n)}) < \dots$. Partially order \mathcal{P} by set inclusion, up to equivalence. That is, for all $A, B \in \mathcal{P}$, $A \subseteq B$ means for every spreading model in A there is an equivalent spreading model in B . Since \mathcal{P} is non-empty, by Zorn's lemma there exists a maximal chain $M \in \mathcal{P}$. We claim that the set M satisfies the assumptions (i) and (ii) of Lemma 2.1 which in turn will yield the theorem.

First, we show that every countable chain $(\tilde{y}_i^{(n)})_i$ in C has an upper bound in C . Indeed, let $(\tilde{y}_i^{(n)})_i$ be a chain in C . For all n , let $A_n \geq 1$ be the best constant satisfying $\|\sum_i a_i \tilde{y}_i^{(n)}\| \leq A_n R(a)$ for all $a = (a_i) \in c_{00}$. Let (\tilde{z}_i) be a spreading model given by Theorem 1.1 applied to $(\tilde{y}_i^{(n)})$ and $C_n = 2^n A_n$, $n = 1, 2, \dots$. Then $(\tilde{z}_i) \geq (\tilde{y}_i^{(n)})$, for all n . That is, (\tilde{z}_i) is an upper bound for $(\tilde{y}_i^{(n)})_i$ in $SP_\omega(X)$. To see that $(\tilde{z}_i) \in C$, let $\varepsilon > 0$ be arbitrary, and let D be as in (1). Let $m \in \mathbb{N}$ such that $\sum_{n=m+1}^\infty 2^{-n} < \varepsilon/2D$. Since $(\tilde{y}_i^{(n)})_i$ is a chain, there is $n_0 \in \{1, \dots, m\}$ such that $(\tilde{y}_i^{(n)}) \leq (\tilde{y}_i^{(n_0)})$, for all $n \leq m$. Let $K_0 = \max\{K_1, \dots, K_m\} < \infty$, where each K_n satisfies $\|\sum_i a_i \tilde{y}_i^{(n)}\| \leq K_n \|\sum_i a_i \tilde{y}_i^{(n_0)}\|$, for all (a_i) . By the fact that $(\tilde{y}_i^{(n_0)})_i \not\leq R$, we can pick $a_\varepsilon = (a_\varepsilon^i) \in c_{00}$ such that

$$\frac{\|\sum_i a_\varepsilon^i \tilde{y}_i^{(n_0)}\|}{R(a_\varepsilon)} < \frac{\varepsilon}{2DK_0}.$$

Then by (1)

$$\begin{aligned} \frac{\|\sum_i a_i^\varepsilon \tilde{z}_i\|}{R(a_\varepsilon)} &\leq D \sum_{n=1}^m 2^{-n} A_n^{-1} \frac{\|\sum_i a_i^\varepsilon \tilde{y}_i^{(n)}\|}{R(a_\varepsilon)} + D \sum_{n=m+1}^\infty 2^{-n} A_n^{-1} \frac{\|\sum_i a_i^\varepsilon \tilde{y}_i^{(n)}\|}{R(a_\varepsilon)} \\ &\leq DK_0 \sum_{n=1}^m 2^{-n} \frac{\|\sum_i a_i^\varepsilon \tilde{y}_i^{(n_0)}\|}{R(a_\varepsilon)} + D \sum_{n=m+1}^\infty 2^{-n} \leq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this means that $(\tilde{z}_i) \not\leq R$. On the other hand, by (1) and (3) we have that $(\tilde{z}_i) \leq R$. Thus $(\tilde{z}_i) < R$, that is, $(\tilde{z}_i) \in C$.

To see that the set M satisfies (ii) of Lemma 2.1, assume to the contrary that there exists a countable chain $(\tilde{y}_i^{(n)}) \subseteq M$ without an upper bound in M . Then by the maximality of M in C , there is no such upper bound in C either. But, since $(\tilde{y}_i^{(n)})$ is a chain (as being a subset of M), by the above argument there is an upper bound (\tilde{z}_i) in C , a contradiction.

To see (i), suppose that M has a maximal element (\tilde{u}_i) . Then $(\tilde{u}_i) < R$. Thus for all $k \in \mathbb{N}$ there exists a sequence of scalars $a^k = (a_i^k)_{i=1}^{m(k)}$ such that

$$(4) \quad \frac{\|\sum_{i=1}^{m(k)} a_i^k \tilde{u}_i\|}{R(a^k)} < \frac{1}{k2^k}.$$

For all n , let $(\tilde{y}_i^{(n)}) \in SP_\omega(X)$ be a spreading model obtained by Theorem 1.1 applied (merely) to two spreading models (\tilde{u}_i) and (\tilde{x}_i^n) , and (coefficients) $C_1^{-1} = C_2^{-1} = 1/2$. Then $(\tilde{y}_i^{(n)})$ 2-dominates (\tilde{x}_i^n) . That is, for all (a_i) , $2\|\sum_i a_i \tilde{y}_i^{(n)}\| \geq \|\sum_i a_i \tilde{x}_i^n\|$. Observe that for all n , $(\tilde{y}_i^{(n)})$ is equivalent to $\max\{(\tilde{u}_i), (\tilde{x}_i^n)\}$, which is, trivially, equivalent to (\tilde{u}_i) . In particular, they are all in C .

Now by (3), for all $k \in \mathbb{N}$ and $a^k = (a_i^k)_{i=1}^{m(k)}$ satisfying (4), there exists $n_k \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{m(k)} a_i^k \tilde{x}_i^{(n_k)} \right\| \geq \frac{1}{2} R(a^k).$$

In particular, for all k , we have

$$\left\| \sum_{i=1}^{m(k)} a_i^k \tilde{y}_i^{(n_k)} \right\| \geq \frac{1}{4} R(a^k).$$

Now let (\tilde{w}_i) be an upper bound obtained by Theorem 1.1 applied to $(\tilde{y}_i^{(n_k)})$ and (2^{-k}) . Then again by the argument in the first part of the proof (since $(\tilde{y}_i^{(n)})$ is a trivial chain, i.e., any two of them are mutually equivalent), we conclude that (\tilde{w}_i) is in C , i.e., $(\tilde{w}_i) < R$. Also, clearly, $(\tilde{w}_i) \geq (\tilde{y}_i^{(n_k)}) \geq (\tilde{u}_i)$, for all k .

Moreover, for all $k \in \mathbb{N}$ we have

$$\left\| \sum_{i=1}^{m(k)} a_i^k \tilde{w}_i \right\| \geq 2^{-k} \left\| \sum_{i=1}^{m(k)} a_i^k \tilde{y}_i^{(n_k)} \right\| \geq \frac{2^{-k}}{4} R(a^k) \geq \frac{k}{4} \left\| \sum_{i=1}^{m(k)} a_i^k \tilde{u}_i \right\|.$$

This shows that $(\tilde{u}_i) \not\leq (\tilde{w}_i)$. Thus $(\tilde{u}_i) < (\tilde{w}_i) < R$. So M is not a maximal chain, a contradiction. This completes the proof. \square

3. THE UPPER ENVELOPE FUNCTION R_X

Define the upper envelope function R_X^ω for $SP_\omega(X)$ on c_{00} as follows. For $a = (a_i) \in c_{00}$, let

$$R_X^\omega(a) = \sup \left\| \sum_i a_i \tilde{x}_i \right\|,$$

where sup is over all $(\tilde{x}_i)_i \in SP_\omega(X)$. It is easy to see that it defines a 1-unconditional and 1-spreading norm on c_{00} . For $a = (a_i) \in c_{00}$, for convenience, we will sometimes write $R_X^\omega(\sum_i a_i e_i)$ for $R_X^\omega(a)$, where (e_i) is the unit vector basis of c_{00} .

Then the following is proved easily using Theorem 1.1 (cf. Remark 3.5(a) of [AOST]). (Observe that Theorem 2.2 extends this proposition.)

Proposition 3.1. *Suppose that R_X^ω is not equivalent to any spreading model in $SP_\omega(X)$. Then $SP_\omega(X)$ contains an uncountable increasing chain.*

Let $(\tilde{x}_i) \in SP_\omega(X)$ and let (\tilde{y}_j) be an identically distributed normalized block basis of (\tilde{x}_i) . That is, there exists $\alpha = (\alpha_i) \in c_{00}$ such that for all j , $\tilde{y}_j = \sum_{i \in \sigma_j} \alpha_i \tilde{x}_i$ with $\|\tilde{y}_j\| = 1$ and $\sigma_1 < \sigma_2 < \dots$, $\sigma_i \subset \mathbb{N}$. Then, clearly $(\tilde{y}_j) \in SP_\omega(X)$ as well. (Note, however, that in general a block basis of a spreading model is not necessarily a spreading model.) This property immediately implies the following sub-multiplicativity property for R_X^ω .

Lemma 3.2 ([Sa]). *Let (e_i) be the unit vector basis of c_{00} . Let (u_i) be an identically distributed block basis of (e_i) such that $R_X^\omega(u_i) = 1$. Then for all $a = (a_i) \in c_{00}$, we have*

$$R_X^\omega\left(\sum_i a_i u_i\right) \leq R_X^\omega\left(\sum_i a_i e_i\right).$$

Let $(\tilde{x}_i) \in SP_\omega(X)$. Then by Krivine's theorem there exists $1 \leq p \leq \infty$ such that for all $k \in \mathbb{N}$ and $\varepsilon > 0$, there is an identically distributed block basis (\tilde{y}_i) of (\tilde{x}_i) so that any k vectors of (\tilde{y}_i) are $(1 + \varepsilon)$ -equivalent to the unit vector basis of ℓ_p^k . The Krivine set of (\tilde{x}_i) is the set of all p 's satisfying the above assertion. Moreover, we define the Krivine set of a Banach space X to be the union of the Krivine sets of all the spreading models of X .

The above lemma in turn implies, again by well-known standard arguments using Krivine's theorem (cf. Proposition 5.2, [Sa]), that

Proposition 3.3. *There exists $1 \leq p \leq \infty$ and for every $\varepsilon > 0$ there exists C_ε such that for all $a \in c_{00}$,*

$$\|a\|_{\ell_p} \leq R_X^\omega(a) \leq C_\varepsilon \|a\|_{\ell_{p-\varepsilon}}.$$

Here it is understood that if $p = 1$, then R_X^ω is equivalent to $\|\cdot\|_{\ell_1}$, and if $p = \infty$, then for all $r < \infty$ there exists C_r such that $R_X^\omega(a) \leq C_r \|a\|_{\ell_r}$.

Observe that if 1 belongs to the Krivine set of a spreading model $(\tilde{x}_i) \in SP_\omega(X)$, then R_X^ω is equivalent to $\|\cdot\|_{\ell_1}$. However, even in this case R_X^ω need not be equivalent to a spreading model in $SP_\omega(X)$ (the space with no ℓ_p spreading models constructed in [OS] is such an example).

From Proposition 3.1 and Proposition 3.3 above we deduce the following, which gives a partial answer to Question 4.7 of [AOST].

Theorem 3.4. *Let X be a Banach space such that $SP_\omega(X)$ is countable. Then there exists $(\tilde{x}_i) \in SP_\omega(X)$ such that the following holds. There exist $1 \leq p \leq \infty$ and $c > 0$, and for every $\varepsilon > 0$ there exists C_ε such that*

$$c\|a\|_{\ell_p} \leq \left\| \sum_i a_i \tilde{x}_i \right\| \leq C_\varepsilon \|a\|_{\ell_{p-\varepsilon}},$$

for all $a = (a_i) \in c_{00}$. If $p = 1$, then (\tilde{x}_i) is equivalent to the unit vector basis of ℓ_1 . Moreover, the Krivine set of (\tilde{x}_i) must be a singleton of $\{p\}$.

Following [FPR] we say that a normalized basis (e_i) has *Rosenthal's property* or is a *Rosenthal basis* if every normalized block basis of (e_i) has a subsequence that is equivalent to (e_i) . It is proved there that a Rosenthal basis is subsymmetric, that is, it is equivalent to its every subsequence and it is unconditional. A question due to H. Rosenthal asks whether such a basis is equivalent to the unit vector basis of c_0 or ℓ_p , for some $1 \leq p < \infty$. If X is non-reflexive, then by James' theorem X contains a copy of either c_0 or ℓ_1 . This quickly implies that (e_i) is equivalent to the unit vector basis of c_0 or ℓ_1 . So the non-trivial case is when X is reflexive. For a reflexive X , clearly, every spreading model of X generated by a block basis is equivalent to (e_i) . Thus, we have the following consequence of Theorem 3.4, which improves on the results in [FPR].

Corollary 3.5. *Let X be a reflexive Banach space with a Rosenthal basis (e_i) . Then there exists $1 < p \leq \infty$ and for every $\varepsilon > 0$ there exists C_ε such that*

$$\|a\|_{\ell_p} \leq \left\| \sum_i a_i e_i \right\| \leq C_\varepsilon \|a\|_{\ell_{p-\varepsilon}},$$

for all $a = (a_i) \in c_{00}$. If $p = \infty$, then for all $r < \infty$ there exists C_r such that $\|\sum_i a_i e_i\| \leq C_r \|a\|_{\ell_r}$. Moreover, the Krivine set I of X must be a singleton, i.e., $I = \{p\}$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SOUTH CAROLINA 29208

E-mail address: bsari@math.sc.edu