

HANKEL-TYPE OPERATORS ON THE SPACE OF BOUNDED HARMONIC FUNCTIONS

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Dedicated to Professor Kazuyuki Tsurumi on the occasion of his 70th birthday

ABSTRACT. We study Hankel-type operators on the space of bounded harmonic functions on the open unit disk. These operators are related to tight uniform algebras, the Dunford-Pettis property, and Bourgain algebras.

1. INTRODUCTION

Let X be a commutative Banach algebra and let Y be a closed subspace. We define a Hankel-type operator $S_g : Y \rightarrow X/Y$ with symbol $g \in X$ by $S_g f = gf + Y$ for all $f \in Y$. These operators are a natural generalization of classical Hankel operators on the Hardy space H^2 . The conditions for S_g to be compact, weakly compact and completely continuous have been investigated in various function spaces. The problem of whether all S_g are weakly compact on a uniform algebra is related to a tight algebra, and the problem of complete continuity appeared in the Dunford-Pettis property. The latter introduced a notion of Bourgain algebras which have been actively researched in analytic and harmonic function spaces on the open unit disk. Recently, Dudziak, Gamelin, and Gorkin [1] studied Hankel-type operators on analytic function spaces. But the problem of the compactness remains open in the case of the bounded harmonic function space on the open unit disk \mathbb{D} . In this paper we study this problem. See [3] and [5] as surveys for convenience.

Let $L^\infty(\mathbb{D})$ be the set of all bounded measurable functions on \mathbb{D} . Then $L^\infty(\mathbb{D})$ is a Banach algebra with the essential supremum norm $\|f\|_\infty = \text{ess. sup}\{|f(z)|; z \in \mathbb{D}\}$. Let $h^\infty := h^\infty(\mathbb{D})$ be the set of all bounded harmonic functions on \mathbb{D} . It follows that h^∞ is a closed subspace of $L^\infty(\mathbb{D})$. For $g \in L^\infty(\mathbb{D})$, we define the linear operator $S_g : h^\infty \rightarrow L^\infty(\mathbb{D})/h^\infty$ by $S_g f = gf + h^\infty$ for $f \in h^\infty$. Trivially $S_g : h^\infty \rightarrow L^\infty(\mathbb{D})/h^\infty$ is a bounded linear operator. Recall that S_g is said to be *compact* if S_g maps every bounded set into a relatively compact one, and that S_g is said to be *completely continuous* if S_g maps every weakly convergent sequence into a norm convergent one. In general, every compact operator is completely continuous.

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But the converse is not always true. We define the following sets:

$$h_c^\infty = \{g \in L^\infty(\mathbb{D}) : S_g \text{ is compact}\},$$

$$h_{cc}^\infty = \{g \in L^\infty(\mathbb{D}) : S_g \text{ is completely continuous}\}.$$

Note that $h_c^\infty \subset h_{cc}^\infty$. Although h^∞ is not an algebra, these are closed subalgebras of $L^\infty(\mathbb{D})$. Especially h_{cc}^∞ is called the Bourgain algebra of h^∞ .

The Bourgain algebra h_{cc}^∞ is characterized by Stroethoff, Yale, and the first author [4]. Let $C(\partial\mathbb{D})$ and $L^\infty(\partial\mathbb{D})$ be the spaces of continuous and bounded measurable functions on the unit circle $\partial\mathbb{D}$, respectively. For a function $f \in L^\infty(\partial\mathbb{D})$, we denote by \hat{f} the Poisson integral of f on \mathbb{D} , that is,

$$\hat{f}(z) = \int_0^{2\pi} f(e^{i\theta}) P_z(e^{i\theta}) d\theta / 2\pi,$$

where P_z is the Poisson kernel of $z \in \mathbb{D}$. Then $\hat{f} \in h^\infty$. For any nonempty subset B of $L^\infty(\partial\mathbb{D})$, we write $\hat{B} = \{\hat{f} : f \in B\}$. It is known that f in h^∞ has a boundary function f^* on $\partial\mathbb{D}$ and $\widehat{f^*} = f$ on \mathbb{D} , so that $h^\infty = \widehat{L^\infty(\partial\mathbb{D})}$. Let $H^\infty(\partial\mathbb{D})$ be the space of boundary functions of bounded analytic functions on \mathbb{D} . The algebra QC of bounded quasi-continuous functions on $\partial\mathbb{D}$ is given by

$$QC = (H^\infty(\partial\mathbb{D}) + C(\partial\mathbb{D})) \cap \overline{(H^\infty(\partial\mathbb{D}) + C(\partial\mathbb{D}))}.$$

Refer to [2] and [6] for more information. For $0 < r < 1$, we write $\mathbb{D}_r = \{|z| \leq r\}$. Put

$$V = \{g \in L^\infty(\mathbb{D}) : \|g\chi_{\mathbb{D} \setminus \mathbb{D}_r}\|_\infty \rightarrow 0 \text{ as } r \rightarrow 1^-\}.$$

The equality $h_{cc}^\infty = \widehat{QC} + V$ was given as Corollary 3 in [4]. In this paper, we show that

Theorem 1.1. $h_c^\infty = h_{cc}^\infty = \widehat{QC} + V$.

2. PROOF OF THEOREM 1.1

To prove Theorem 1.1, we use the following facts which can easily be checked:

- (i) $g \in h_c^\infty$ if and only if $\bar{g} \in h_c^\infty$,
- (ii) S_g is compact on h^∞ provided that for every sequence $\{f_n\}_n$ in h^∞ satisfying $\|f_n\|_\infty \leq 1$ and $f_n \rightarrow 0$ uniformly on compact subsets of \mathbb{D} , then there is a subsequence $\{f_{n_j}\}_j$ of $\{f_n\}_n$ such that $\|S_g f_{n_j}\|_\infty \rightarrow 0$ as $j \rightarrow \infty$.

Proof of Theorem 1.1. First, we show that

$$(2.1) \quad C(\overline{\mathbb{D}}) + V \subset h_c^\infty \subset h_{cc}^\infty = \widehat{QC} + V,$$

where $C(\overline{\mathbb{D}})$ is the space of continuous functions on the closed unit disk $\overline{\mathbb{D}}$. It is trivial that $V \subset h_c^\infty$. Since h_c^∞ is a closed subalgebra of $L^\infty(\mathbb{D})$, by (i) it is sufficient to prove that $z \in h_c^\infty$. Suppose that $f_n \in h^\infty$, $\|f_n\|_\infty \leq 1$ and $f_n \rightarrow 0$ uniformly on compact subsets of \mathbb{D} . We shall prove that $\|S_z f_{n_j}\|_\infty \rightarrow 0$ for some subsequence $\{f_{n_j}\}_j$ of $\{f_n\}_n$.

Write $f_n = u_n + iv_n$, where u_n and v_n are real-valued harmonic functions on \mathbb{D} . Denote by ${}^*u_n, {}^*v_n$ the harmonic conjugate functions of u_n, v_n , with ${}^*u_n(0) = {}^*v_n(0) = 0$, respectively. Set

$$g_n = \frac{1}{2}(u_n + i^*u_n + i(v_n + i^*v_n)), \quad h_n = \frac{1}{2}(u_n + i^*u_n - i(v_n + i^*v_n)).$$

Then $f_n = g_n + \overline{h_n}$, and it is known that $g_n, h_n \in H^2$, where H^2 is the Hardy space on \mathbb{D} ; see Chapter III of Garnett's book [2]. Identifying functions with boundary functions, and denoting by $\|\cdot\|_2$ the square integral norm on $\partial\mathbb{D}$, we know that

$$\|{}^*u_n\|_2 \leq \|u_n\|_2 \leq \|f_n\|_2 \leq \|f_n\|_\infty \leq 1, \quad \|{}^*v_n\|_2 \leq \|v_n\|_2 \leq 1.$$

Hence

$$(2.2) \quad \sup_n \|g_n\|_{H^2} \leq 2, \quad \sup_n \|h_n\|_{H^2} \leq 2,$$

where $\|\cdot\|_{H^2}$ is the Hardy space norm. By a normal family argument, there exist subsequences (which we do not relabel) of $\{g_n\}, \{h_n\}$ such that $g_n \rightarrow g, h_n \rightarrow h$ uniformly on compact subsets of \mathbb{D} for some analytic functions g and h . Since $f_n \rightarrow 0$ uniformly on compact subsets of \mathbb{D} and $f_n = g_n + \overline{h_n}, g = -\overline{h}$ on \mathbb{D} . Since g and h are analytic, $g = -\overline{h} = \alpha$ for some constant α with $|\alpha| \leq 2$.

Putting

$$k_n = zg_n + \overline{\left(\frac{h_n - h_n(0)}{z}\right)} - \alpha z,$$

then k_n is harmonic on \mathbb{D} and

$$(2.3) \quad k_n = z(g_n + \overline{h_n}) - z\overline{h_n} + \overline{\left(\frac{h_n - h_n(0)}{z}\right)} - \alpha z.$$

Since $f_n \in h^\infty, z(g_n + \overline{h_n}) - \alpha z = zf_n - \alpha z \in L^\infty(\mathbb{D})$. It is clear that $z\overline{h_n} - \overline{(h_n - h_n(0))/z}$ is bounded near the origin of \mathbb{D} . By (2.2), we have

$$(2.4) \quad |h_n(z)| \leq \|h_n\|_{H^2} \frac{1}{\sqrt{1-|z|^2}} \leq \frac{2}{\sqrt{1-|z|^2}} \quad \text{for } z \in \mathbb{D}.$$

Since

$$\left|z\overline{h_n}(z) - \overline{\left(\frac{h_n(z) - h_n(0)}{z}\right)}\right| \leq \frac{|h_n(z)|(1-|z|^2)}{|z|} + \frac{|h_n(0)|}{|z|},$$

$z\overline{h_n} - \overline{(h_n - h_n(0))/z}$ is bounded on \mathbb{D} . Thus we get $k_n \in h^\infty$.

To show that $\|zf_n - k_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, let ϵ be an arbitrary small positive number and $\delta = \sqrt{1-\epsilon^2}$. By (2.3), we have

$$\|zf_n - k_n\|_\infty = \|z\overline{(h_n + \overline{\alpha})} - \overline{\left(\frac{h_n + \overline{\alpha} - (h_n(0) + \overline{\alpha})}{z}\right)}\|_\infty.$$

Since $h_n \rightarrow -\overline{\alpha}$ uniformly on \mathbb{D}_δ ,

$$(2.5) \quad \sup_{z \in \mathbb{D}_\delta} \left|z\overline{(h_n(z) + \overline{\alpha})} - \overline{\left(\frac{h_n(z) + \overline{\alpha} - (h_n(0) + \overline{\alpha})}{z}\right)}\right| \rightarrow 0$$

as $n \rightarrow \infty$. For $|z| > \delta$, we have

$$\begin{aligned} & \left|z\overline{(h_n(z) + \overline{\alpha})} - \overline{\left(\frac{h_n(z) + \overline{\alpha} - (h_n(0) + \overline{\alpha})}{z}\right)}\right| \\ &= \left|\frac{1-|z|^2}{\overline{z}}\overline{(h_n(z) + \overline{\alpha})} - \frac{\overline{h_n(0) + \overline{\alpha}}}{\overline{z}}\right| \\ &\leq \frac{\epsilon\|h_n + \overline{\alpha}\|_{H^2}}{\delta} + \frac{|h_n(0) + \overline{\alpha}|}{\delta} \quad \text{by (2.4)} \\ &< \frac{4\epsilon}{\delta} + \frac{|h_n(0) + \overline{\alpha}|}{\delta} \quad \text{by (2.2) and } |\alpha| \leq 2. \end{aligned}$$

Combining with (2.5),

$$\limsup_{n \rightarrow \infty} \|zf_n - k_n\|_\infty \leq \frac{4\epsilon}{\delta} = \frac{4\epsilon}{\sqrt{1 - \epsilon^2}}.$$

Since ϵ is arbitrary,

$$\lim_{n \rightarrow \infty} \|S_z f_n\|_\infty \leq \lim_{n \rightarrow \infty} \|zf_n - k_n\|_\infty = 0.$$

Thus by (ii), $z \in h_c^\infty$, and it holds that

$$C(\overline{\mathbb{D}}) + V \subset h_c^\infty \subset h_{cc}^\infty = \widehat{QC} + V.$$

Next we show that for any $g \in QC$, $\hat{g} \in h_c^\infty$, that is, $S_{\hat{g}}$ is compact on h^∞ . To show this, note the equality $h^\infty = L^\infty(\partial\mathbb{D})$. Let $\{f_n\}_n$ be a sequence in h^∞ satisfying that $\|f_n\|_\infty \leq 1$ and $\hat{f}_n \rightarrow 0$ uniformly on compact subsets of \mathbb{D} . We have $\|S_{\hat{g}}\hat{f}_n + h^\infty\| \leq \|\hat{g}\hat{f}_n - \widehat{gf}_n\|_\infty$ and

$$|\hat{g}(z)\hat{f}_n(z) - \widehat{gf}_n(z)| \leq \int_0^{2\pi} |\hat{g}(z) - g(e^{i\theta})| |P_z(\theta)| d\theta / 2\pi.$$

Since $g \in QC$, we have

$$(2.6) \quad \sup_{r < |z| < 1} \sup_n |\hat{g}(z)\hat{f}_n(z) - \widehat{gf}_n(z)| \rightarrow 0 \quad \text{as } r \rightarrow 1;$$

see [2, 4, 6]. Then for any $\epsilon > 0$, there exists $R, 0 < R < 1$, such that

$$(2.7) \quad \sup_{R < |z| < 1} \sup_n |\hat{g}(z)\hat{f}_n(z) - \widehat{gf}_n(z)| < \epsilon.$$

On the other hand, by the assumption, $\sup_{z \in \mathbb{D}_R} |\hat{g}(z)\hat{f}_n(z)| \rightarrow 0$ as $n \rightarrow \infty$. Now we define $g_t(e^{i\theta}) = \hat{g}(te^{i\theta})$ for $0 < t < 1$. Then $g_t \in C(\partial\mathbb{D})$, $\hat{g}_t(z) = \hat{g}(tz)$ for $z \in \mathbb{D}$, and

$$\begin{aligned} \sup_{z \in \mathbb{D}_R} |\widehat{gf}_n(z) - \widehat{g_t f_n}(z)| &\leq \sup_{z \in \mathbb{D}_R} \int_0^{2\pi} |g(e^{i\theta}) - g_t(e^{i\theta})| |f_n(e^{i\theta})| |P_z(e^{i\theta})| d\theta / 2\pi \\ &\leq \frac{1+R}{1-R} \int_0^{2\pi} |g(e^{i\theta}) - g_t(e^{i\theta})| d\theta / 2\pi. \end{aligned}$$

By the Lebesgue dominated convergence theorem,

$$\lim_{t \rightarrow 1} \sup_n \sup_{z \in \mathbb{D}_R} |\widehat{gf}_n(z) - \widehat{g_t f_n}(z)| = 0.$$

Take a positive number t_0 such that $t_0 < 1$ and

$$(2.8) \quad \sup_n \sup_{z \in \mathbb{D}_R} |\widehat{gf}_n(z) - \widehat{g_{t_0} f_n}(z)| < \epsilon.$$

By (2.1), $\hat{g}_{t_0} \in C(\overline{\mathbb{D}}) \subset h_c^\infty$, so that by considering a subsequence we may assume that $\|\hat{g}_{t_0}\hat{f}_n + h^\infty\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Then

$$(2.9) \quad \|\hat{g}_{t_0}\hat{f}_n - \widehat{g_{t_0} f_n} + h^\infty\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We want to prove that

$$(2.10) \quad \sup_{z \in \mathbb{D}_R} |\widehat{g_{t_0} f_n}(z)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let $\{h_n\}_n$ be a sequence in h^∞ satisfying

$$(2.11) \quad \|\hat{g}_{t_0} \hat{f}_n - \widehat{g_{t_0} f_n} + h_n\|_\infty = \|\hat{g}_{t_0} \hat{f}_n - \widehat{g_{t_0} f_n} + h^\infty\|_\infty.$$

Since $g_t \in C(\partial\mathbb{D}) \subset QC$, by (2.6)

$$\|\hat{g}_{t_0} \hat{f}_n - \widehat{g_{t_0} f_n} + h_n\|_\infty \geq \limsup_{|z| \rightarrow 1} |h_n(z)| = \|h_n\|_\infty.$$

By (2.9) $\|h_n\|_\infty \rightarrow 0$, and by (2.11)

$$(2.12) \quad \|\hat{g}_{t_0} \hat{f}_n - \widehat{g_{t_0} f_n}\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We have

$$\|\hat{g}_{t_0} \hat{f}_n - \widehat{g_{t_0} f_n}\|_\infty \geq \sup_{z \in \mathbb{D}_R} |\widehat{g_{t_0} f_n}(z) - |\hat{g}_{t_0}(z) \hat{f}_n(z)|.$$

By the assumption, $\sup_{z \in \mathbb{D}_R} |\hat{g}_{t_0}(z) \hat{f}_n(z)| \rightarrow 0$ as $n \rightarrow \infty$. Then by (2.12), we get (2.10).

Combining (2.8) and (2.10), we have

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{D}_R} |\widehat{g f_n}(z)| \leq \epsilon.$$

This shows that $\sup_{z \in \mathbb{D}_R} |\widehat{g f_n}(z)| \rightarrow 0$ as $n \rightarrow \infty$. Since $\sup_{z \in \mathbb{D}_R} |\hat{g}(z) \hat{f}_n(z)| \rightarrow 0$ by our assumption, we get

$$\sup_{z \in \mathbb{D}_R} |\hat{g}(z) \hat{f}_n(z) - \widehat{g f_n}(z)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Combining with (2.7), we have

$$\limsup_{n \rightarrow \infty} \|\hat{g} \hat{f}_n - \widehat{g f_n}\|_\infty \leq \epsilon.$$

Consequently we obtain

$$\|S_{\hat{g}} \hat{f}_n + h^\infty\| \leq \|\hat{g} \hat{f}_n - \widehat{g f_n}\|_\infty \rightarrow 0,$$

that is, $S_{\hat{g}}$ is compact and so $\widehat{QC} \subset h_c^\infty$. We complete the proof. □

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