PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 5, Pages 1355–1358
S 0002-9939(05)08081-0
Article electronically published on October 6, 2005

GLOBAL SOLUTIONS TO SPECIAL LAGRANGIAN EQUATIONS

YU YUAN

(Communicated by Jon G. Wolfson)

ABSTRACT. We show that any global solution to the special Lagrangian equations with the phase larger than a critical value must be quadratic.

1. INTRODUCTION

In this note, we show that any global solution u in $\mathbb{R}^n$ to the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = c \tag{1.1}$$

with phase $|c| > \frac{\pi}{2}(n-2)$ must be a quadratic polynomial, which states the λ_i's are the eigenvalues of the Hessian D^2u. Recall the Bernstein theorem, where any global solution to the minimal surface equation $div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$ in $\mathbb{R}^7$ must be a linear function.

Equation (1.1) stems from the special Lagrangian geometry [HL]. The Lagrangian graph $(x, \nabla u(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the phase or the argument of the complex number $\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_n\right)$ is constant c, that is, u satisfies equation (1.1), and it is special if and only if $(x, \nabla u(x))$ is a minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ [HL, Theorem 2.3, Proposition 2.17]. To be precise, we state

Theorem 1.1. *Let u be a smooth solution in $\mathbb{R}^n$ to* (1.1) *with $|c| > \frac{\pi}{2}(n-2)$. Then u is quadratic.*

Fu [F] proved Theorem 1.1 in the two-dimensional case. Indeed (1.1) with $c = \frac{\pi}{2}$ in the 2-d case also has the Monge-Ampère form $\det D^2u = 1$, and Jörgens already showed Theorem 1.1 in this special case earlier on (cf. [N]).

Other Bernstein-Liouville type results concerning (1.1) are in order. Borishenko [B] showed that any convex solution with linear growth to (1.1) with $c = k\pi$ is linear. The author [Y] proved that any convex solution to (1.1) must be quadratic. For $c = k\pi$ in $n = 3$ case, (1.1) has another form

$$\triangle u = \det D^2u. \tag{1.2}$$

Received by the editors September 9, 2004 and, in revised form, November 29, 2004.
2000 *Mathematics Subject Classification.* Primary 35J60.
The author was partially supported by an NSF grant and a Sloan Research Fellowship. The author was a visiting fellow at the Australian National University while this work was done.

It was proved in [BCGJ] that any strictly convex solution to (1.2) with quadratic growth must be quadratic. Under the assumption that the Hessian is bounded and $\lambda_i\lambda_j \geq -\frac{3}{2}$, it was also shown in [TW] that any global solution to (1.1) is quadratic.

The heuristic idea of the proof of Theorem 1.1 is to find a better graph representation of $(x, \nabla u(x))$ so that the Hessian of the new potential is bounded, and the new potential function satisfies a convex uniformly elliptic equation. By Krylov-Evan's [K], [E] interior Hölder estimates on the Hessian, we draw the conclusion.

As there are nontrivial global harmonic solutions to (1.1) with $c = 0$ in the case $n = 2$, we guess (1.1) with $c = \frac{\pi}{2}(n-2)$ also has nontrivial global solutions in the higher-dimensional case. Observe that in the case $n = 3$ and $c = \frac{\pi}{2}$, (1.1) also takes the interesting form $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1$.

2. Proof

Step 1. We first find a better graph representation of M through Lewy rotation (cf. [N]) so that the Hessian of the potential function is bounded. By symmetry we only consider the case $c > \frac{\pi}{2}(n-2)$. Let $\sum_{i=1}^n \theta_i = \frac{\pi}{2}(n-2) + \delta$, where $\theta_i = \arctan \lambda_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\delta \in (0, \pi)$. Note that

$$-\frac{\pi}{2} + \frac{(n-1)}{n}\delta < \theta_i - \frac{\delta}{n} < \frac{\pi}{2} - \frac{\delta}{n}. \tag{2.1}$$

The first inequality follows from $\frac{\pi}{2}(n-2) + \delta < \theta_i + \frac{\pi}{2}(n-1)$. We rotate the $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ coordinate system to $(\bar{x}, \bar{y})$ by $\frac{\delta}{n}$, namely, $\bar{x} = x\cos\frac{\delta}{n} + y\sin\frac{\delta}{n}$, $\bar{y} = -x\sin\frac{\delta}{n} + y\cos\frac{\delta}{n}$. In terms of complex variables $z = x + \sqrt{-1}y$, that is, we identify $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$, the rotation takes the form $\bar{z} = e^{-\sqrt{-1}\delta/n}z$. Then M has a new parametrization

$$\begin{cases} \bar{x} = x\cos\frac{\delta}{n} + \nabla u(x)\sin\frac{\delta}{n}, \\ \bar{y} = -x\sin\frac{\delta}{n} + \nabla u(x)\cos\frac{\delta}{n}. \end{cases}$$

By (2.1), $M = (x, \nabla u(x))$ is still a graph over the whole $\bar{x}$ space $\mathbb{R}^n$. Further the rotation belongs to $U(n)$. Then M is also a special Lagrangian graph $(\bar{x}, \nabla\bar{u}(\bar{x}))$, where $\bar{u}$ is a smooth function [HL, p. 87, Proposition 2.17]. Let $\bar{\lambda}_i$ be the eigenvalues of $D^2\bar{u}$. Then $\bar{\theta}_i = \arctan\bar{\lambda}_i = \theta_i - \frac{\delta}{n} \in \left(-\frac{\pi}{2} + \frac{(n-1)}{n}\delta, \frac{\pi}{2} - \frac{\delta}{n}\right)$. That is,

$$\left|D^2\bar{u}\right| \leq C(\delta).$$

Finally $\bar{u}$ satisfies the equation

$$\sum_{i=1}^n \arctan\bar{\lambda}_i = \frac{\pi}{2}(n-2). \tag{2.2}$$

Step 2. We proceed with the following lemma, which is Lemma 8.1 in [CNS] when n is even and $c = \frac{\pi}{2}(n-2)$.

Lemma 2.1. *Let $f(\lambda_1, \cdots, \lambda_n) = \sum_{i=1}^n \arctan\lambda_i$ and let $\Gamma = \{\lambda | f(\lambda) = c\}$ with $|c| \geq \frac{\pi}{2}(n-2)$. Then Γ is convex.*

Proof. We skip the case $n = 1$. By symmetry we just consider the case $c \geq 0$. Set $c = \frac{\pi}{2}(n-2) + \delta$ with $\delta \in [0, \pi)$. We may assume that $\theta_i = \arctan\lambda_i \geq 0$ for

$i = 1, \cdots, n-1$. The normal of Γ is $\nabla f = \left(\cos^2\theta_1, \cdots, \cos^2\theta_n\right)$. Let

$$A \triangleq -\frac{1}{2} D^2 f = \begin{bmatrix} \tan\theta_1 \cos^4\theta_1 & & \\ & \cdots & \\ & & \tan\theta_n \cos^4\theta_n \end{bmatrix}.$$

Take any tangent vector $T = (t_1, \cdots, t_n) \in T_\lambda \Gamma$, that is,

$$\sum_{i=1}^{n} t_i \cos^2\theta_i = 0.$$

We show that $A(T,T) \geq 0$.

Case a) $\theta_n \geq 0$. Certainly it is true.

Case b) $\theta_n < 0$. First we know that $\theta_i > 0$ for $i = 1, \cdots, n-1$ and $\delta < \frac{\pi}{2}$. Next we have

$$A(T,T) = \sum_{i=1}^{n-1} \tan\theta_i \cos^4\theta_i t_i^2 + \tan\theta_n \cos^4\theta_n t_n^2.$$

Now we use the trick in [CNS, p. 299],

$$\begin{aligned} (-t_n \cos\theta_n)^2 &= \left(\sum_{i=1}^{n-1} t_i \cos^2\theta_i\right)^2 \\ &\leq \left(\sum_{i=1}^{n-1} t_i^2 \cos^4\theta_i \tan\theta_i\right)\left(\sum_{i=1}^{n-1} \cot\theta_i\right). \end{aligned}$$

Then

$$\tan\theta_n \cos^4\theta_n t_n^2 \geq \left(\sum_{i=1}^{n-1} t_i^2 \cos^4\theta_i \tan\theta_i\right)\left(\sum_{i=1}^{n-1} \cot\theta_i\right)\tan\theta_n$$

and

$$\begin{aligned} A(T,T) &\geq \left(\sum_{i=1}^{n-1} t_i^2 \cos^4\theta_i \tan\theta_i\right)\left[1 + \left(\sum_{i=1}^{n-1} \cot\theta_i\right)\tan\theta_n\right] \\ &= \left(\sum_{i=1}^{n-1} t_i^2 \cos^4\theta_i \tan\theta_i\right)\left(\sum_{i=1}^{n} \cot\theta_i\right)\tan\theta_n. \end{aligned}$$

Let $\alpha_i = \frac{\pi}{2} - \theta_i$. We have

$$\begin{aligned} &\frac{\pi}{2} < \pi - \delta = \alpha_1 + \cdots + \alpha_n < \pi, \\ &0 < \alpha_1, \cdots, \alpha_{n-1} < \frac{\pi}{2} < \alpha_n, \end{aligned}$$

and

$$\sum_{i=1}^{n} \cot\theta_i = \sum_{i=1}^{n-1} \tan\alpha_i + \tan\alpha_n.$$

It follows that $\tan\alpha_n < 0$ and

$$\frac{\tan(\alpha_1 + \cdots + \alpha_{n-1}) + \tan\alpha_n}{1 - \tan(\alpha_1 + \cdots + \alpha_{n-1})\tan\alpha_n} = \tan(\alpha_1 + \cdots + \alpha_n) < 0.$$

Then $\tan(\alpha_1 + \cdots + \alpha_{n-1}) + \tan\alpha_n < 0$. Note that $\alpha_1 + \cdots + \alpha_{n-1} = \pi - \delta - \alpha_n < \frac{\pi}{2}$; we have

$$\begin{aligned}\tan(\alpha_1 + \cdots + \alpha_{n-1}) &\geq \tan\alpha_1 + \tan(\alpha_2 + \cdots + \alpha_{n-1}) \\ &\cdots \\ &\geq \tan\alpha_1 + \tan\alpha_2 + \cdots + \tan\alpha_{n-1}.\end{aligned}$$

So $\sum_{i=1}^n \cot\theta_i = \sum_{i=1}^n \tan\alpha_i < 0$ and $A(T,T) \geq 0$. Therefore Γ is convex (w.r.t. the normal ∇f). □

Remark 1. The level set Γ is no longer convex or concave when $|c| < \frac{\pi}{2}(n-2)$.

Step 3. The final argument is standard. We now have global solution $\bar{u}$ with bounded Hessian on the convex level set Γ, more precisely a convex level set in the symmetric matrix space (cf. [CNS, p. 276]). In another word, $\bar{u}$ satisfies (2.2), which is now uniformly elliptic. By Krylov-Evans theorem ([K],[E])

$$\left[D^2\bar{u}\right]_{C^\beta(B_r)} \leq C(n,\delta)\frac{\left\|D^2\bar{u}\right\|_{L^\infty(B_{2r})}}{r^\beta} \leq \frac{C(n,\delta)}{r^\beta},$$

where $\beta = \beta(n,\delta) \in (0,1)$. Let r go to $+\infty$; we see that $D^2\bar{u}$ is a constant matrix. Thus $(\bar{x}, \nabla\bar{u})$ is a plane, and consequently u is quadratic.

References

[BCGJ] Bao, J.-G., Chen, J.-Y., Guan, B., and Ji, M., *Liouville property and regularity of a Hessian quotient equation,* Amer. J. Math. **125** (2003), 301–316. MR1963687 (2004b:35079)

[B] Borisenko, A. A., *On a Liouville-type theorem for the equation of special Lagrangian submanifolds,* (Russian) Mat. Zametki **52** (1992), 22–25; English translation in Math. Notes **52** (1992), 1094–1096 (1993). MR1201944 (93k:53055)

[CNS] Caffarelli, L. A., Nirenberg, L., and Spruck, J., *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian,* Acta Math. **155** (1985), 261-301. MR0806416 (87f:35098)

[E] Evans, L. C., *Classical solutions of fully nonlinear, convex, second-order elliptic equations,* Comm. Pure Appl. Math. **35** (1982), no. 3, 333–363. MR0649348 (83g:35038)

[F] Fu, L., *An analogue of Bernstein's theorem,* Houston J. Math. **24** (1998), 415–419. MR1686614 (2000c:53006)

[HL] Harvey, R. and Lawson, H. B., Jr., *Calibrated geometry,* Acta Math. **148** (1982), 47–157. MR0666108 (85i:53058)

[K] Krylov, N. V., *Boundedly nonhomogeneous elliptic and parabolic equations,* Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 487–523 in Russian; English translation in Math. USSR Izv. **20**(1983), 459–492. MR0661144 (84a:35091)

[N] Nitsche, J. C. C., *Elementary proof of Bernstein's theorem on minimal surfaces,* Ann. of Math. **66** (1957), 543–544. MR0090833 (19:878f)

[TW] Tsui, M.-P. and Wang, M.-T., *A Bernstein type result for special Lagrangian submanifolds*, Math. Res. Lett. **9** (2002), 529–535. MR1928873 (2003k:53059)

[Y] Yuan, Y., *A Bernstein problem for special Lagrangian equations,* Invent. Math. **150** (2002), 117–125. MR1930884 (2003k:53060)

Department of Mathematics, Box 354350, University of Washington, Seattle, Washington 98195

E-mail address: yuan@math.washington.edu