

AN IMAGE PROBLEM FOR COMPACT OPERATORS

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ABSTRACT. Let \mathcal{X} be a separable Banach space and $(\mathcal{X}_n)_n$ a sequence of closed subspaces of \mathcal{X} satisfying $\mathcal{X}_n \subset \mathcal{X}_{n+1}$ for all n . We first prove the existence of a dense-range and injective compact operator K such that each $K\mathcal{X}_n$ is a dense subset of \mathcal{X}_n , solving a problem of Yahaghi (2004). Our second main result concerns isomorphic and dense-range injective compact mappings between dense sets of linearly independent vectors, extending a result of Grivaux (2003).

1. INTRODUCTION

Let \mathcal{X} be an infinite-dimensional separable real or complex Banach space and denote by $\mathcal{L}(\mathcal{X})$ the algebra of all bounded and linear mappings from \mathcal{X} to \mathcal{X} . A chain of subspaces of \mathcal{X} is defined to be a sequence (at most countable) $(\mathcal{X}_n)_{n \geq 0}$ of closed subspaces of \mathcal{X} such that $\mathcal{X}_0 = \{0\}$ and $\mathcal{X}_n \subset \mathcal{X}_{n+1}$ for all $n \geq 0$. The identity map on \mathcal{X} is denoted by Id .

Section 2 of this paper is devoted to the construction of injective and dense-range compact operators $K \in \mathcal{L}(\mathcal{X})$ such that $K\mathcal{X}_n$ is a dense subset of \mathcal{X}_n for all n . Note that for separable Hilbert spaces, using an orthonormal basis \mathcal{B} and diagonal compact operators relative to \mathcal{B} , it is not difficult to construct injective, dense-range and normal compact operators such that each $K\mathcal{X}_n$ is a dense subset of \mathcal{X}_n . Nevertheless, in the case of an arbitrary Banach space, the construction is not so obvious. In [6], Yahaghi obtained partial results. He proved that there exist dense-range compact operators K such that $K\mathcal{X}_n$ is a dense subset of \mathcal{X}_n and there exist injective compact operators such that $K\mathcal{X}_n \subset \mathcal{X}_n$.

Section 3 of the paper was motivated by Lemma 2.1 of [1], asserting that, on a general separable Banach space \mathcal{X} , two dense sets of linearly independent vectors of \mathcal{X} are isomorphic. We construct injective, dense-range and compact mappings between dense sets of linearly independent vectors in \mathcal{X} .

In our construction we shall make use of biorthogonal sequences in Banach spaces. Recall that a sequence (x_n, x_n^*) in \mathcal{X} , where each x_n lies in \mathcal{X} and x_n^* in \mathcal{X}^* , is *biorthogonal* if $x_n^*(x_m) = 0$ for $m \neq n$ while $x_n^*(x_n) = 1$ for all n . It is *fundamental* if the linear span of the vectors x_n is dense, and it is *total* if the only vector x such that $x_n^*(x) = 0$ for all n is the zero vector. Every separable Banach space possesses fundamental total biorthogonal sequences, and they can even be

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found such that $(\|x_n\| \|x_n^*\|)$ is bounded [3]. We refer to [2] for a detailed account of this subject.

2. INJECTIVE AND DENSE-RANGE COMPACT OPERATORS PRESERVING CHAINS

Let $(\mathcal{X}_n)_{n \geq 0}$ be a chain of subspaces of \mathcal{X} . Our aim is to construct an injective and dense-range compact operator K such that $K\mathcal{X}_n$ is a dense subset of \mathcal{X}_n for all $n \geq 0$. The first result is elementary, but is the key to more elaborate constructions.

Proposition 2.1. *Let \mathcal{X} be a separable Banach space. Then there exists a compact operator on \mathcal{X} that is injective and has dense range.*

Proof. There exists a fundamental and total biorthogonal sequence $(x_n, x_n^*)_{n \geq 1}$ for \mathcal{X} with $\|x_n\| = 1$ for each n (see [2]). We may assume if we like that the (x_n^*) are also uniformly bounded, although this is not needed here. Define the compact operator K by

$$Kx = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{x_n^*(x)x_n}{\|x_n^*\|}.$$

Clearly the range of K contains all finite linear combinations of the vectors x_n and is dense since the (x_n) are a fundamental system. If $Kx = 0$, then $x_k^*(Kx) = 0$ for each k , which implies that $x_k^*(x) = 0$ for each k and hence $x = 0$ since the (x_n^*) are total. \square

The following theorem gives a positive answer to a question posed by Yahaghi [6, p. 1063].

Theorem 2.1. *Let \mathcal{X} be an infinite-dimensional separable Banach space and \mathcal{Y} a closed subspace of \mathcal{X} . Then there is a dense-range and injective compact operator K such that $K\mathcal{Y}$ is a dense subspace of \mathcal{Y} .*

Proof. Take fundamental total biorthogonal sequences (y_n, y_n^*) and (w_n, w_n^*) in the separable Banach spaces \mathcal{Y} and \mathcal{X}/\mathcal{Y} respectively. Using the Hahn–Banach theorem we may suppose that y_n^* is extended to an element of \mathcal{X}^* , and we keep the same notation.

Note that the quotient mapping $p : \mathcal{X} \rightarrow \mathcal{X}/\mathcal{Y}$ is a contraction, and its adjoint is a mapping $p^* : (\mathcal{X}/\mathcal{Y})^* \rightarrow \mathcal{X}^*$. Define $x_n^* = p^*(w_n^*)$ for each n .

Let us suppose that both \mathcal{Y} and \mathcal{X}/\mathcal{Y} are infinite-dimensional: if one is finite-dimensional the same construction works, with some of the sums being finite. Without loss of generality $\|y_n\| = 1$ and $\|w_n\| = 1$ for all n , so we can choose $x_n \in \mathcal{X}$ with $p(x_n) = w_n$ and $\|x_n\| \leq 2$.

Define

$$Kx = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{y_n^*(x)y_n}{\|y_n^*\|} + \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{x_n^*(x)x_n}{\|x_n^*\|}.$$

Clearly K is a compact operator and, if $y \in \mathcal{Y}$, then $x_n^*(y) = (p^*w_n^*)(y) = w_n^*(py) = 0$ for each n , and so K maps \mathcal{Y} into \mathcal{Y} . Also the image space $K\mathcal{Y}$ is dense in \mathcal{Y} as in Proposition 2.1.

Note that Ky_n is a nonzero multiple of y_n and that Kx_n is a vector in \mathcal{Y} plus a nonzero multiple of x_n , since

$$x_m^*(x_n) = w_m^*(px_n) = w_m^*(w_n) = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

It remains to show that $K\mathcal{X}$ is dense in \mathcal{X} and that K is injective. If $x \in \mathcal{X}$, then for each $\varepsilon > 0$ we can find a finite linear combination $w = \sum c_n w_n = p(\sum c_n x_n)$ with $\|p(x) - w\| < \varepsilon/3$; thus there is a vector $y \in \mathcal{Y}$ with $\|x - \sum c_n x_n - y\| < 2\varepsilon/3$, and finally a finite sum $\sum b_n y_n$ with $\|y - \sum b_n y_n\| < \varepsilon/3$, giving

$$\|x - (\sum c_n x_n + \sum b_n y_n)\| < \varepsilon.$$

Now $\sum c_n x_n + \sum b_n y_n - K(\sum n^2 \|x_n^*\| c_n x_n)$ lies in \mathcal{Y} , and $K\mathcal{Y}$ is dense in \mathcal{Y} , and we deduce that K has dense range.

Suppose finally that $Kx = 0$. Applying x_m^* to Kx we deduce that $x_m^*(x) = 0$ for each m , and so $x \in \mathcal{Y}$. Hence

$$Kx = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{y_n^*(x)y_n}{\|y_n^*\|} = 0.$$

Applying y_m^* we deduce that $y_m^*(x) = 0$ for each m and hence $x = 0$. \square

The above proof could be shortened, but would become less elementary, by applying stronger results on the extension of biorthogonal sequences. We shall use the above proof technique, together with the more difficult results, in order to deal with chains of subspaces, as in the next theorem; this also provides stronger forms of results given by Yahaghi [6].

Theorem 2.2. *Let \mathcal{X} be a separable infinite-dimensional Banach space and let $(\mathcal{X}_n)_{n \geq 0}$ be a chain of subspaces of \mathcal{X} . Then there is a dense range and injective compact operator K such that $K\mathcal{X}_n$ is a dense subspace of \mathcal{X}_n for all n .*

Proof. We shall use the known result that a uniformly bounded fundamental total biorthogonal sequence for a subspace of a Banach space can be extended to a uniformly bounded fundamental total biorthogonal sequence for the larger space (cf. [4, 5]). Assuming this result, we construct the operator K given by

$$Kx = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \frac{x_{nk}^*(x)x_{nk}}{\|x_{nk}\|},$$

where $\{(x_{jk}, x_{jk}^*|_{\mathcal{X}_n}) : j \leq n, k \geq 1\}$ is a uniformly bounded fundamental total biorthogonal sequence in each \mathcal{X}_n . (Our notation assumes that each $\mathcal{X}_{n+1}/\mathcal{X}_n$ is infinite-dimensional; an obvious modification should be made if not.)

If the union of the \mathcal{X}_n is dense, then this is the operator required, as it can be shown in a fashion similar to Proposition 2.1 that this operator is compact and injective, mapping \mathcal{X}_n densely into \mathcal{X}_n . Otherwise, let \mathcal{Y} denote the closure of the union of the spaces \mathcal{X}_n . As in the proof of Theorem 2.1, we may find a biorthogonal sequence (w_n, w_n^*) for \mathcal{X}/\mathcal{Y} with $\|w_n\| = 1$ for all n . Writing p for the quotient mapping from \mathcal{X} onto \mathcal{X}/\mathcal{Y} , we choose $x_n \in \mathcal{X}$ with $p(x_n) = w_n$ and $\|x_n\| \leq 2$ and

let $x_n^* = p^*(w_n^*) \in \mathcal{X}^*$. The desired operator is given by

$$K'x = Kx + \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{x_n^*(x)x_n}{\|x_n^*\|},$$

and its properties are verified as in the proof of Theorem 2.1. \square

3. ISOMORPHIC AND COMPACT MAPPINGS BETWEEN DENSE INDEPENDENT SETS

We first recall an ingenious lemma of S. Grivaux, which motivates what follows.

Lemma 3.1 (Lemma 2.1 in [1]). *Let $V = \{v_n : n \geq 1\}$ and $W = \{w_n : n \geq 1\}$ be dense linearly independent sets in a Banach space \mathcal{X} . For any $\varepsilon > 0$, there exists an isomorphism L of \mathcal{X} with $\|\text{Id} - L\| < \varepsilon$ such that $L(V) = W$.*

A simple way to construct a countable dense linearly independent set in a separable infinite-dimensional Banach space \mathcal{X} is to begin by choosing a linearly independent set $U = \{u_n : n \geq 1\}$ of normalized vectors in \mathcal{X} such that the linear span of U is dense. Next, let $T = \{t_m : m \geq 1\}$ be an enumeration of the set of finite linear combinations of vectors in U with coefficients in \mathbb{Q} (or $\mathbb{Q} + i\mathbb{Q}$ if the Banach space is complex). Then T is dense, and hence so is the set $S = \{t_m + u_{f(m)}/m : m \geq 1\}$, for any $f : \mathbb{N} \rightarrow \mathbb{N}$. We can guarantee that S is an independent set by choosing the sequence $(f(m))$ to be increasing sufficiently rapidly.

In order to obtain an analogous version of Lemma 3.1 replacing isomorphism by injective and dense-range compact operator, we first need the following observation.

Proposition 3.1. *If (v_n) is a dense independent set in \mathcal{X} and $T : \mathcal{X} \rightarrow \mathcal{X}$ is a bounded linear operator that is injective with dense range, then (Tv_n) is also a dense independent set in \mathcal{X} .*

Proof. Take any $x \in \mathcal{X}$ and choose vectors $y_k = Tx_k \in T\mathcal{X}$ such that $\|y_k - x\| \rightarrow 0$. Choose v_{n_k} such that $\|v_{n_k} - x_k\| \rightarrow 0$. Then

$$\|Tv_{n_k} - x\| \leq \|T\|\|v_{n_k} - x_k\| + \|y_k - x\| \rightarrow 0.$$

Thus the (Tv_n) form a dense set. Also, if $\sum_{n=1}^N a_n Tv_n = 0$, then $\sum_{n=1}^N a_n v_n = 0$, since T is injective, so $a_1 = a_2 = \dots = a_N = 0$, and hence the (Tv_n) are independent. \square

Combining these results, we obtain the following.

Corollary 3.1. *Let $V = \{v_n : n \geq 1\}$ and $W = \{w_n : n \geq 1\}$ be dense linearly independent sets in a separable Banach space \mathcal{X} . Then there is an injective compact operator K (necessarily with dense range) such that $K(V) = W$.*

Proof. Let T be any compact injective operator on \mathcal{X} with dense range, for example the one given in Proposition 2.1. By Proposition 3.1, the vectors (Tv_n) form a dense independent set U in \mathcal{X} . Now by Lemma 3.1 there is an isomorphism L such that $L(U) = W$. In other words, if we define $K = LT$, then K is compact and injective with dense range, and $K(V) = LT(V) = L(U) = W$, as required. \square

In the spirit of the results of Section 2, we now look at mappings involving subspaces. The main result of this section is the following theorem.

Theorem 3.1. *Let \mathcal{Y} be a closed subspace of a separable infinite-dimensional Banach space \mathcal{X} , and let V_1, W_1 be countable, dense, linearly independent subsets of \mathcal{Y} . Let V_2, W_2 be countable subsets of \mathcal{X} that are linearly independent over \mathcal{Y} (i.e., the sets of quotients $p(V_2)$ and $p(W_2)$ are independent in \mathcal{X}/\mathcal{Y}), such that $V_1 \cup V_2$ and $W_1 \cup W_2$ are dense (independent) subsets of \mathcal{X} . Then there is an invertible operator L mapping V_1 onto W_1 and V_2 onto W_2 ; there is also a compact, injective, dense-range operator K with the same mapping properties.*

Proof. For the construction of L we modify the argument of Grivaux in [1]. Let $V_1 = \{v_1, v_2, \dots\}$, $W_1 = \{w_1, w_2, \dots\}$ and $V_2 = \{t_1, t_2, \dots\}$ and $W_2 = \{u_1, u_2, \dots\}$. We construct a convergent sequence of invertible operators on \mathcal{X} that all map \mathcal{Y} onto \mathcal{Y} , beginning with $L_0 = \text{Id}$. The sequence continues $I_1, J_1, K_1, L_1, I_2, J_2, K_2, L_2, I_3, \dots$. If at any stage an operator already satisfies the additional mapping conditions required of its successor, then its successor is defined to be the same operator once more; otherwise the new operator is defined as a small perturbation of its predecessor (for example, with $\|T_{k+1} - T_k\| \leq 2^{-k}$, where T_k denotes the k th operator constructed).

We proceed as follows:

- (1) Let $I_1 = L_0 + x_1^* \otimes x_1$ be such that $I_1 v_1 = w_{p_1}$ for some p_1 ; to achieve this choose x_1^* with $x_1^*(v_1) = 1$ and take $x_1 \in \mathcal{Y}$ small such that $v_1 + x_1 = w_{p_1}$.
- (2) Let $J_1 = (I_1^{-1} + y_1^* \otimes y_1)^{-1}$ be such that $J_1 v_1 = w_{p_1}$ and $J_1 v_{q_1} = w_1$ for some q_1 ; to achieve this choose y_1^* with $y_1^*(w_{p_1}) = 0$, $y_1^*(w_1) = 1$, and take $y_1 \in \mathcal{Y}$ small such that $I_1^{-1} w_1 + y_1 = v_{q_1}$.
- (3) Let $K_1 = J_1 + z_1^* \otimes z_1$ be such that $K_1 v_1 = w_{p_1}$, $K_1 v_{q_1} = w_1$, and $K_1 t_1 = u_{r_1}$ for some r_1 ; to achieve this choose z_1^* such that $z_1^*(y) = 0$ for all $y \in \mathcal{Y}$, $z_1^*(t_1) = 1$, and take $z_1 \in \mathcal{X}$ small such that $J_1 t_1 + z_1 = u_{r_1}$.
- (4) Let $L_1 = (K_1^{-1} + a_1^* \otimes a_1)^{-1}$ be such that $L_1 v_1 = w_{p_1}$, $L_1 v_{q_1} = w_1$, $L_1 t_1 = u_{r_1}$, and $L_1 t_{s_1} = u_1$ for some s_1 ; to achieve this choose a_1^* with $a_1^*(y) = 0$ for all $y \in \mathcal{Y}$, $a_1^*(u_{r_1}) = 0$, $a_1^*(u_1) = 1$, and take $a_1 \in \mathcal{X}$ small such that $K_1^{-1} u_1 + a_1 = t_{s_1}$.

We then cycle around through I_2, J_2, K_2 and on to L_2 , and continue in this fashion for $k = 2, 3, \dots$

- (1) We take $x_k^*(v_k) = 1$ and $x_k^*(x) = 0$ for $y \in \{v_j, v_{q_j}, t_j, t_{s_j} : j < k\}$, and $x_k \in \mathcal{Y}$ is sufficiently small; then we can ensure that $I_k = L_{k-1} + x_k^* \otimes x_k$ satisfies $I_k v_k = w_{p_k}$ for some p_k .
- (2) We take $y_k^*(w_k) = 1$ and $y_k(x) = 0$ for $x \in \{w_{p_j}, w_j, u_{r_j}, u_j : j < k\} \cup \{w_{p_k}\}$, and $y_k \in \mathcal{Y}$ is sufficiently small; then we can ensure that $J_k = (I_k^{-1} + y_k^* \otimes y_k)^{-1}$ satisfies $J_k v_{q_k} = w_k$ for some q_k .
- (3) We take $z_k^*(t_k) = 1$ and $z_k^*(x) = 0$ for $x \in \mathcal{Y} \cup \{t_j, t_{s_j} : j < k\}$, and $z_k \in \mathcal{X}$ is sufficiently small; then we can ensure that $K_k = J_k + z_k^* \otimes z_k$ satisfies $K_k t_k = u_{r_k}$ for some r_k .
- (4) We take $a_k^*(u_k) = 1$ and $a_k^*(x) = 0$ for $x \in \mathcal{Y} \cup \{u_{r_j}, u_j : j < k\} \cup \{u_{r_k}\}$, and $a_k \in \mathcal{X}$ is sufficiently small; then we can ensure that $L_k = (K_k^{-1} + a_k^* \otimes a_k)^{-1}$ satisfies $L_k t_{s_k} = u_k$ for some s_k .

Provided that the vectors x_k, y_k, z_k and a_k are chosen sufficiently small at each stage, these choices ensure that each operator is invertible and maps \mathcal{Y} onto \mathcal{Y} , and that the sequence converges in norm to an operator L with the desired properties.

Finally, to obtain K we proceed as in the proof of Corollary 3.1 using an operator $K = LT$, where T is constructed as in Theorem 2.2. Note that TV_2 will be independent over \mathcal{Y} , since if a vector u satisfies $Tu \in \mathcal{Y}$, then $\sum_{n=1}^{\infty} \frac{1}{n^2} x_n^*(u)x_n \in \mathcal{Y}$, and so $\sum_{n=1}^{\infty} \frac{1}{n^2} w_n^*(pu)w_n = 0$ in \mathcal{X}/\mathcal{Y} , so that $pu = 0$ and $u \in \mathcal{Y}$. \square

Remark 3.1. (1) It is not sufficient that $V_1 \cup V_2$ and $W_1 \cup W_2$ be independent; if $L\mathcal{Y} = \mathcal{Y}$, then we also require independence of V_2 and W_2 over \mathcal{Y} in order to prescribe that $LV_2 = W_2$.

(2) In the situation when we have a chain of subspaces $\mathcal{X}_1 \subset \mathcal{X}_2 \subset \mathcal{X}_3 \subset \dots \subset \mathcal{X}$, it seems to be possible to make analogous constructions of operators preserving chains of dense independent subsets $V_1 \subset V_2 \subset V_3 \dots$ and $W_1 \subset W_2 \subset W_3 \dots$. However, the simplest construction appears to need a complicated diagonal induction, prescribing the image or pre-image in turn of, say,

$$v_{11}, w_{11}, v_{21}, w_{21}, v_{12}, w_{12}, v_{31}, w_{31}, v_{22}, w_{22}, v_{13}, w_{13}, \dots,$$

where $v_{ij} \in V_i$ and $w_{ij} \in W_i$. We omit the details.

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REFERENCES

- [1] S. Grivaux. Construction of operators with prescribed behaviour. *Arch. Math. (Basel)*, 81(3):291–299, 2003. MR2013260 (2004g:47011)
- [2] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces. I*. Springer-Verlag, Berlin, 1977. Sequence spaces, *Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92*. MR0500056 (58:17766)
- [3] R. I. Ovsepian and A. Pelczyński. On the existence of a fundamental total and bounded biorthogonal sequence in every separable Banach space, and related constructions of uniformly bounded orthonormal systems in L^2 . *Studia Math.*, 54(2):149–159, 1975. MR0394137 (52:14942)
- [4] I. Singer. *Bases in Banach spaces. II*. Springer-Verlag, Berlin-New York, 1981. MR0610799 (82k:46024)
- [5] P. Terenzi. Extension of uniformly minimal M -basic sequences in Banach spaces. *J. London Math. Soc. (2)*, 27(3):500–506, 1983. MR0697142 (85b:46016)
- [6] B. R. Yahaghi. On injective or dense-range operators leaving a given chain of subspaces invariant. *Proc. Amer. Math. Soc.*, 132(4):1059–1066, 2004. MR2045421 (2005a:47011)

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