

## STABLY AND ALMOST COMPLEX STRUCTURES ON BOUNDED FLAG MANIFOLDS

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ABSTRACT. We study the enumeration problem of stably complex structures on bounded flag manifolds arising from omniorientations, and determine those induced by almost complex structures. We also enumerate the stably complex structures on these manifolds which bound, therefore representing zero in the complex cobordism ring  $\Omega_*^U$ .

### 1. INTRODUCTION

The geometry of torus actions on varieties and manifolds has long been studied in various mathematical disciplines. The starting point lies in the discovery of a relationship between polyhedral and algebraic geometry. This relationship involves the theory of toric varieties, sometimes known as torus embeddings.

The underlying combinatorial data for constructing a toric variety is constrained by the fact that it depends on the geometric realization of a fan as well as its combinatorial type. In certain respects, this seems to be too strong a requirement, and may be weakened under suitable circumstances to create smooth manifolds carrying similar properties to toric varieties. These ideas had led to the discovery of quasitoric manifolds by Davis and Januszkiewicz [9]. The basic combinatorial ingredients for constructing quasitorics are more flexible than the existence of a smooth fan, and require a simple convex polytope  $P$  and a collection of primitive vectors attached to the facets of  $P$ . Of course, such a flexible prerequisite has some disadvantages; for instance, the manifolds which are obtained need not even be almost complex. However, they always carry a stably complex structure [5].

The bounded flag manifolds fit into both settings and provide beautiful examples. The quasitoric structure of these manifolds has already been investigated in [5], and we will display them as toric varieties associated to fans arising from crosspolytopes. These manifolds were originally constructed by Bott and Samelson, and were introduced into complex cobordism by Ray [12].

The geometric and computational flavor of the toric geometry enables us to translate the topological problems into combinatorial ones and vice versa. Under these circumstances, we would like to investigate the enumeration problem of

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stably complex structures on bounded flag manifolds compatible with their toric structures.

We first recall how to associate a stably complex structure to a quasitoric manifold by following [5]. Let  $M^{2n}$  be a quasitoric manifold over  $P^n$  and let  $\mathcal{F} = \{F_1, \dots, F_m\}$  be the set of facets of  $P^n$ . Then for each  $F_i$ , the pre-image  $\pi^{-1}(F_i)$  is a submanifold  $M_i^{2(n-1)} \subset M^{2n}$  with isotropy group a circle  $T(F_i)$  in  $T^n$ . Since there is a one-to-one correspondence (up to a sign) between the set of primitive vectors in  $\mathbb{Z}^n$  and the subcircles in  $T^n$ , we obtain the *characteristic map* of  $M^{2n}$  given by

$$\begin{aligned} \lambda: \mathcal{F} &\rightarrow \mathbb{Z}^n, \\ F_i &\mapsto \lambda(F_i) := \lambda_i, \end{aligned}$$

where  $\lambda_i$  generates the circle  $T(F_i)$  in  $T^n$ . We note that the map  $\lambda$  is well defined only up to a sign, and if the sign of each  $\lambda_i$  is chosen, we then call  $\lambda$  a *dicharacteristic map* of  $M^{2n}$ . Therefore, there are  $2^m$  dicharacteristic maps in total attached to  $M^{2n}$ . On the other hand, each such choice for  $\lambda_i$  determines an orientation of the normal bundle  $\nu_i$  of  $M_i^{2(n-1)}$ , so an orientation for  $M_i^{2(n-1)}$ . Conversely, an *omniorientation* of  $M^{2n}$  consists of a choice of an orientation for every submanifold  $M_i^{2(n-1)}$ , which in turn settles a sign for each vector  $\lambda_i$ . Thus, every omniorientation is equipped with a unique dicharacteristic map and vice versa. Buchstaber and Ray [5] were able to show that any omniorientation of  $M^{2n}$  induces a stably complex structure on it by means of the following isomorphism:

$$(1.1) \quad \tau(M^{2n}) \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \dots \oplus \rho_m,$$

where the bundles  $\rho_i$  are called *facial bundles* of  $M^{2n}$ , and they are obtained as the pull-back of the line bundles corresponding to the Thom classes defined by  $\nu_i$ 's along the Pontryagin-Thom collapse. We note that the change of the omniorientation provides a new stably complex structure. However, two different omniorientations may induce the same stably complex structure; in other words, they can be homotopic. Therefore, it would be interesting to find out the number of different such structures on a given quasitoric manifold  $M^{2n}$ . We offer a solution to such a problem in the case of the bounded flag manifolds.

## 2. BOUNDED FLAG MANIFOLDS

The geometry of bounded flag manifolds plays an important role in complex cobordism, namely that they generate the dual of the Landweber-Novikov algebra [4]. One way of constructing these manifolds can be obtained by taking iterated 2-sphere bundles, and the other way may be characterized by the set of certain flags in a complex space.

We begin with introducing some notation. We follow combinatorial convention by writing  $[n]$  for the set of natural numbers  $\{1, 2, \dots, n\}$ , and an interval in the poset  $[n]$  has the form  $[a, b]$  for some  $1 \leq a \leq b \leq n$  which consists of all  $k$  satisfying  $a \leq k \leq b$ . Throughout,  $\omega_1, \dots, \omega_{n+1}$  will denote the standard basis vectors in  $\mathbb{C}^{n+1}$ , and we write  $\mathbb{C}_I$  and  $\mathbb{C}P_I$  for the subspace spanned by the vectors  $\{\omega_i : i \in I\}$  and the projectivization of  $\mathbb{C}_I$  respectively, where  $I \subset [n + 1]$ .

**Definition 2.1.** A flag  $U: 0 < U_1 < \dots < U_n < \mathbb{C}^{n+1}$  is called *bounded* if  $\mathbb{C}_{[i-1]} < U_i$  for each  $1 \leq i \leq n$ . The set of all bounded flags in  $\mathbb{C}^{n+1}$  is called

the bounded flag manifold, which is an  $n$ -dimensional smooth complex manifold and will be denoted by  $B(\mathbb{C}^{n+1})$  (or simply by  $B_n$ ).

We note that as a consequence of the definition, each bounded flag  $U$  is equivalent to a sequence of lines  $L_k < \mathbb{C}_k \oplus L_{k+1}$  for  $2 \leq k \leq n$ , where  $L_{n+1} = \mathbb{C}_{n+1}$ .

We consider complex line bundles  $\eta_i$  and  $\eta_i^\perp$  over  $B_n$  classified respectively by the maps  $q_{n-i+1}$  and  $r_{n-i+1}$  defined by  $q_{n-i+1}(U) = L_{n-i+1}$  and  $r_{n-i+1}(U) = L_{n-i+1}^\perp$ , where  $L_k^\perp$  denotes the orthogonal complement of  $L_k$  in  $\mathbb{C}_k \oplus L_{k+1}$ . It then follows that

$$(2.1) \quad \eta_i \oplus \eta_i^\perp \cong \eta_{i-1} \oplus \mathbb{C}$$

for every  $i$ , where we take  $\eta_0$  as a trivial line bundle. As detailed in [12], there is an isomorphism

$$(2.2) \quad \tau(B_n) \oplus \mathbb{R}^2 \cong \bigoplus_{i=0}^{n-1} \eta_i \oplus \mathbb{C},$$

giving a stably complex structure on  $B_n$ . However, each  $B_n$  can be identified with the total space of the sphere bundle of  $\eta_{n-1} \oplus \mathbb{R}$  over  $B_{n-1}$ , and the above  $U$ -structure extends over the associated 3-disk bundle; hence,  $B_n$  represents zero in the complex cobordism ring  $\Omega_*^U$ .

We let  $x_1, \dots, x_n \in H^2(B_n; \mathbb{Z})$  denote the respective first Chern classes of  $\eta_1, \dots, \eta_n$ .

**Theorem 2.2** ([4]). *The integral cohomology ring  $H^*(B_n)$  is generated by  $x_1, \dots, x_n$ , and these are subject only to the relations  $x_1^2 = 0$  and  $x_i^2 = x_i x_{i-1}$  for each  $2 \leq i \leq n$  and for all  $n > 0$ .*

By following Batyrev's construction [1], we display the bounded flag manifold  $B_n$  as a toric variety and then investigate its quasitoric structure. Let  $\Sigma(n)$  be a fan spanned by the crosspolytope  $Q^n$  in  $\mathbb{R}^n$  with the set of vertices

$$V(Q) = \{e_1, -e_1, e_2, -e_1 - e_2, \dots, e_n, -e_1 - \dots - e_n\},$$

where  $e_1, \dots, e_n$  are the canonical basis vectors of  $\mathbb{R}^n$ . The associated smooth toric variety  $M_n$  of  $\Sigma(n)$  can be obtained as a quotient of  $(\mathbb{C}^2 \setminus 0)^n$  by the action of the algebraic subtorus  $H(n)$  of  $\mathbb{C}_\times^{2n}$  consisting of elements of the form

$$(a_1, a_1 a_2^{-1}, \dots, a_k, a_k a_{k+1}^{-1}, \dots, a_{n-1}, a_{n-1} a_n^{-1}, a_n, a_n),$$

where  $\mathbb{C}_\times = \mathbb{C} \setminus \{0\}$ , and  $H(n)$  acts on  $(\mathbb{C}^2 \setminus 0)^n$  diagonally;

$$(2.3) \quad (x_1, y_1; \dots; x_n, y_n) \cdot (a_1, \dots, a_n) \\ := (x_1 a_1, y_1 a_1 a_2^{-1}; \dots; x_{n-1} a_{n-1}, y_{n-1} a_{n-1} a_n^{-1}; x_n a_n, y_n a_n).$$

Let  $[x, y]$  denote the equivalence class of  $(x, y) \in (\mathbb{C}^2 \setminus 0)^n$  under the defined action so that  $M_n = \{[x, y] : (x, y) \in (\mathbb{C}^2 \setminus 0)^n\}$ . For a given  $(x, y) \in (\mathbb{C}^2 \setminus 0)^n$ , we let  $l_{n+1} := \omega_{n+1}$  and define  $l_i := x_i \omega_i + y_i l_{i+1}$  for  $1 \leq i \leq n$ . We then obtain a bounded flag:

$$U(x, y): 0 < L_1 < \mathbb{C}_1 \oplus L_2 < \dots < \mathbb{C}_{[n-1]} \oplus L_n < \mathbb{C}_{[n+1]},$$

where each line  $L_i$  is spanned by the vector  $l_i$ . Conversely, for any bounded flag  $U \in B_n$ , we can find  $(x, y) \in (\mathbb{C}^2 \setminus 0)^k$  such that  $U = U(x, y)$ ; however, such a vector is not always unique. On the other hand, if we define  $\Gamma: M_n \rightarrow B_n$  by  $\Gamma_n([x, y]) := U(x, y)$ , it can be verified that  $\Gamma_n$  is a diffeomorphism of complex

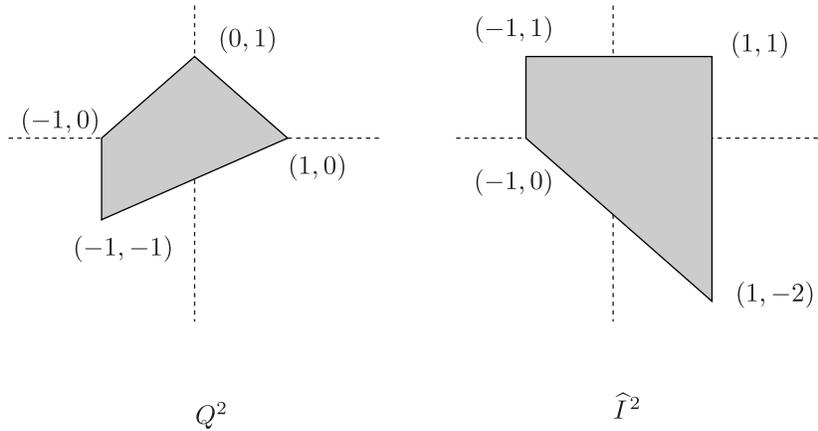


FIGURE 1.

manifolds, which displays the bounded flag manifold  $B_n$  as a toric variety [6]. In order to exhibit  $B_n$  as a quasitoric manifold, we will use the identification  $B_n \cong M_n$ . We first note that following the torodial structure of  $M_n$ , the group  $\mathbb{C}_\times^n$  acts on  $M_n$  by

$$(2.4) \quad ((a_1, \dots, a_n), [x, y]) \mapsto [(a_1x_1, y_1), \dots, (a_nx_n, y_n)],$$

where  $(a_1, \dots, a_n) \in \mathbb{C}_\times^n$  and  $[x, y] \in M_n$ . Since  $\mathbb{C}_\times^n$  contains the compact torus  $T^n$ , the group  $T^n$  also acts on  $M_n$  in the same way. It is easy to observe that this action is locally standard. Secondly, the polar of  $Q^n$  is the cube  $\hat{I}^n$  (see Figure 1 when  $n = 2$ ) which is defined by the following half-spaces:

$$\hat{H}_i^0 := \{x \in \mathbb{R}^n : x_i \leq 1\} \quad \text{and} \quad \hat{H}_i^1 := \{x \in \mathbb{R}^n : x_1 + \dots + x_i \geq -1\},$$

for  $1 \leq i \leq n$ , with corresponding facets  $\hat{C}_i^\gamma = \hat{I}^n \cap \hat{H}_i^\gamma$ , where  $\gamma = 0$  or  $1$ . To define the projection  $\pi: M_n \rightarrow \hat{I}^n$ , we first choose a “canonical representative” for each equivalence class  $[x, y] \in M_n$ . So, for any given  $[x, y] \in M_n$ , we consider a vector  $(a, b) \in [x, y]$  satisfying  $|a_k|^2 + |b_k|^2 = 1 + |b_{k-1}|^2$  for each  $1 \leq k \leq n$ , where we set  $b_0 = 1$ . The existence and uniqueness of such a vector  $(a, b)$  for any  $[x, y]$  is obvious. We then define

$$(2.5) \quad \begin{aligned} \pi: M_n &\longrightarrow \hat{I}^n, \\ [a, b] &\mapsto (1 - |a_1|^2, 1 - |a_2|^2, \dots, 1 - |a_n|^2) \end{aligned}$$

for any  $[a, b] \in M_n$ . On the other hand, it easily follows that the facial submanifolds  $\pi^{-1}(\hat{C}_k^\gamma)$  corresponding to the codimension-one faces of  $\hat{I}^n$  are  $M_k^\gamma$ , where

- (i)  $M_k^0 := \{[x, y] \in M_n \mid x_k = 0\}$  if  $\gamma = 0$ ,
- (ii)  $M_k^1 := \{[x, y] \in M_n \mid y_k = 0\}$  if  $\gamma = 1$

for each  $1 \leq j \leq k$ . Note that the submanifold  $M_k^0$  is a copy of  $B_{n-1}$  whose flags lie in  $\mathbb{C}_{[n+1] \setminus \{k\}}$ , and similarly,  $M_k^1$  is a copy of  $B_{k-1} \times B_{n-k}$  whose flags lie in  $\mathbb{C}_{[k]} \times \mathbb{C}_{[k+1, n+1]}$ . If we write  $T_k$  for the  $k$ th coordinate circle in  $T^n$ , while  $T_\delta$  denotes the diagonal circle, we then obtain that the stabilizer of  $M_k^\gamma$  in  $T^n$  is given by  $T_k$  if  $\gamma = 0$  and  $T_\delta < T^k$  if  $\gamma = 1$ , where the latter is embedded in  $T^n$  via the first  $k$

coordinates, for  $1 \leq k \leq n$ . Therefore, we deduce a characteristic map, which may be represented by the dicharacteristic map  $\lambda: \{\widehat{C}_k^\gamma: \gamma = 0, 1 \text{ and } 1 \leq k \leq n\} \rightarrow \mathbb{Z}^n$  for the action of  $T^n$  on  $M_n$ , defined by

$$(2.6) \quad \lambda(\widehat{C}_k^\gamma) := \begin{cases} e_k, & \text{if } \gamma = 0, \\ -e_1 - \dots - e_k, & \text{if } \gamma = 1, \end{cases}$$

for each  $1 \leq k \leq n$ . Moreover, we may readily identify the facial bundles  $\rho(\widehat{C}_k^0)$  and  $\rho(\widehat{C}_k^1)$  with  $\bar{\eta}_{n-k+1}$  and  $\eta_{n-k+1}^\perp$  over  $B_n$  respectively, where  $\bar{\eta}$  denotes the conjugate of  $\eta$ . Thus, the omniorientation corresponding to (2.6) induces the stably complex structure  $\tau(B_n) \oplus \mathbb{R}^{2n} \cong \bigoplus_{k=1}^n \bar{\eta}_k \oplus \eta_k^\perp$ . We note that since the above omniorientation arises from the toric variety structure of  $B_n$ , it is the realification of the complex isomorphism

$$(2.7) \quad \tau(B_n) \oplus \mathbb{C}^n \cong \bigoplus_{k=1}^n \bar{\eta}_k \oplus \eta_k^\perp.$$

A second dicharacteristic map  $\lambda'$  arises by setting  $\lambda'(\widehat{C}_k^0) := -e_k$  and  $\lambda'(\widehat{C}_k^1) := \lambda(\widehat{C}_k^1)$  for  $1 \leq k \leq n$ . Then, the corresponding omniorientation induces the stably complex structure  $\tau(B_n) \oplus \mathbb{R}^{2n} \cong \bigoplus_{k=1}^n \eta_k \oplus \eta_k^\perp$ . When combined with the canonical trivialization  $\eta_n \oplus \bigoplus_{k=1}^n \eta_k^\perp \cong \mathbb{C}^{n+1}$ , this reduces to the bounding structure of (2.2).

### 3. STABLY AND ALMOST COMPLEX STRUCTURES

We write  $BO$  and  $BU$  respectively for the real and complex infinite Grassmannians and specify a realification map  $r: BU \rightarrow BO$ . Let  $N^n$  be a smooth, closed and connected manifold. We assume that the stable tangent bundle is represented by a map  $\tau^S(N): N^n \rightarrow BO$ , which we fix henceforth.

**Definition 3.1.** Let  $\xi$  be a real vector bundle over  $N^n$ . A *complex structure* (respectively, a *stably complex structure*) on  $\xi$  is a specific bundle isomorphism (resp. stable isomorphism)  $f: \xi \rightarrow \beta_{\mathbb{R}}$ , where  $\beta_{\mathbb{R}}$  is the real bundle underlying some complex vector bundle  $\beta$  over  $N$ .

**Definition 3.2.** A complex structure on  $\tau^S(N)$ , which is given by a lift  $\tau_U$  to  $BU$ , is said to be a *stably complex structure* on  $N^n$ .

If  $\tau_U$  and  $\tau'_U$  are two stably complex structures on  $N^n$ , we say that they are *equivalent* (or *homotopic*), whenever they are homotopic through lifts of  $\tau^S(N)$ . When  $\tau_U$  is fixed, it leads to a complementary lift of the stable normal bundle  $\nu^S(N)$ , and conversely; this correspondence preserves homotopy classes.

Similarly, a complex structure on the tangent bundle  $\tau(N)$  of  $N^n$  is known as an *almost complex structure* on  $N^n$ . When  $N^n$  is itself complex, it therefore admits an associated almost complex structure, which in turn induces a *canonical* stably complex structure. On the other hand, an arbitrary stably complex structure need not restrict to  $\tau(N)$ , and an arbitrary almost complex structure need not arise from any complex structure on  $N^n$ . For example, if we consider the toric manifolds  $\overline{\mathbb{C}P}^2$  or the connected sum  $\mathbb{C}P^2 \# \mathbb{C}P^2$ , where  $\overline{\mathbb{C}P}^2$  is obtained from  $\mathbb{C}P^2$  by reversing its orientation, then neither  $\overline{\mathbb{C}P}^2$  nor  $\mathbb{C}P^2 \# \mathbb{C}P^2$  admit almost complex structures.

We note that from the fibration

$$(3.1) \quad O/U \xrightarrow{f} BU \xrightarrow{r} BO,$$

we can obtain the associated exact Bott sequence for  $B_n$ :

$$(3.2) \quad \dots \rightarrow \tilde{K}O^{-1}(B_n) \rightarrow \tilde{K}O^{-2}(B_n) \xrightarrow{\chi} \tilde{K}^0(B_n) \xrightarrow{r} \tilde{K}O^0(B_n) \rightarrow \tilde{K}O^1(B_n) \rightarrow \dots,$$

which links the real and complex  $K$ -theory through the realification homomorphism  $r$ . Here,  $\chi$  is induced by  $f$  and may be identified with  $z^{-1} \cdot c$  by composing the complexification homomorphism with multiplication by  $z^{-1}$ , where  $K_* = \mathbb{Z}[z, z^{-1}]$  is the coefficient ring. We showed in [7] that the exact sequence (3.2) reduces to the short exact sequence

$$(3.3) \quad 0 \rightarrow \tilde{K}O^{-2}(B_n) \xrightarrow{\chi} \tilde{K}^0(B_n) \xrightarrow{r} \tilde{K}O^0(B_n) \rightarrow 0,$$

and if  $\mathcal{K}_n$  denotes the kernel of  $r: \tilde{K}^0(B_n) \rightarrow \tilde{K}O^0(B_n)$ , then the group  $\mathcal{K}_n \cong \tilde{K}O^{-2}(B_n)$  can be identified with the set of stably complex structures on  $B_n$ , since  $\chi$  is a monomorphism in this case [13].

Since the number of facets of  $\hat{I}^n$  is equal to  $2n$ , we have the possibility of choosing  $2^{2n}$  different omniorientations on  $B_n$ . However, two different omniorientations on  $B_n$  may induce the same stably complex structure. Therefore, our aim now is to find the number of different stably complex structures on  $B_n$  arising from omniorientations.

We should first introduce some notation. We define

$$\begin{aligned} U(i, j^\perp) &:= \eta_i \oplus \eta_j^\perp, & D(i, j^\perp) &:= \bar{\eta}_i \oplus \bar{\eta}_j^\perp, \\ R(i, j^\perp) &:= \eta_i \oplus \bar{\eta}_j^\perp, & L(i, j^\perp) &:= \bar{\eta}_i \oplus \eta_j^\perp, \end{aligned}$$

for any  $1 \leq i, j \leq n$ . As a consequence of the isomorphisms  $\eta_i \oplus \eta_i^\perp \cong \eta_{i-1} \oplus \mathbb{C}$  for all  $1 \leq i \leq n$ , we have the following identities:

$$(3.4) \quad \begin{aligned} U(i, i^\perp) &\cong U(i-1, 0) = R(i-1, 0), \\ D(i, i^\perp) &\cong D(i-1, 0) = L(i-1, 0), \\ R(i, i^\perp) \oplus L(i, 0) &\cong L(i-1, 0) \oplus R(i, 0), \\ L(i, i^\perp) \oplus R(i, 0) &\cong R(i-1, 0) \oplus L(i, 0) \end{aligned}$$

for  $1 \leq i \leq n$ , where  $\eta_0$  denotes the trivial line bundle as before.

**Proposition 3.3.** *Any stably complex structure on  $B_n$  arising from an omniorientation is of the form*

$$(3.5) \quad \tau(B_n) \oplus \mathbb{R}^{2n} \cong \bigoplus_{i=1}^n X_i(i, i^\perp),$$

where  $X_i \in \{U, D, R, L\}$  for  $1 \leq i \leq n$ .

*Proof.* Any omniorientation on  $B_n$  may be obtained from the dicharacteristic map  $\lambda$  of (2.6) by changing the signs of some of the vectors  $\lambda(\widehat{C}_k^\gamma)$ , yielding the result.  $\square$

As we have already mentioned, some of these stably complex structures may be equivalent so that they are represented by the same element in  $K(B_n)$ . To distinguish those that are not, we first construct a subset of those stably complex structures on  $B_n$  described by Proposition 3.3. To do that, we define

$$\begin{aligned} \Delta_i^U &:= \{R(i-1, 0), L(i-1, 0), R(i, i^\perp), L(i, i^\perp)\}, \\ \Delta_i^R &:= \Delta_i^U \setminus \{L(i-1, 0)\} \quad \text{and} \quad \Delta_i^L := \Delta_i^U \setminus \{R(i-1, 0)\} \end{aligned}$$

for any  $1 \leq i \leq n$ . We note that  $U(1, 1^\perp) \cong D(1, 1^\perp) \cong R(0, 0) \cong L(0, 0) \cong \mathbb{C}^2$ . Let  $\mathcal{Y}_n$  denote the set of  $n$ -tuples  $(Y_1, \dots, Y_n)$ , where  $Y_i \in \Delta_i^U$  and if  $Y_i = R(i, i^\perp)$ , then  $Y_{i+1} \in \Delta_{i+1}^R$ , and correspondingly, if  $Y_i = L(i, i^\perp)$ , then  $Y_{i+1} \in \Delta_{i+1}^L$  for  $2 \leq i \leq n-1$ .

**Theorem 3.4.** *For each stably complex structure  $\bigoplus_{i=1}^n X_i(i, i^\perp)$  on  $B_n$ , there exists an  $n$ -tuple  $(Y_1, \dots, Y_n)$  in  $\mathcal{Y}_n$  such that*

$$(3.6) \quad \bigoplus_{i=1}^n X_i(i, i^\perp) \cong \bigoplus_{i=1}^n Y_i.$$

*Proof.* We should mention that this is an isomorphism of complex vector bundles, and it may be proven by induction on  $n$  with the help of identities (3.4).  $\square$

In fact, we will prove that the set  $\mathcal{Y}_n$  can be identified with the set of actual stably complex structures arising from omniorientations. To see that, we need a technical lemma.

**Lemma 3.5.** *For any  $1 \leq i \leq n$ , we have*

$$(3.7) \quad \bar{\eta}_i^\perp - \eta_i^\perp = (\bar{\eta}_{i-1} - \eta_{i-1}) + (\eta_i - \bar{\eta}_i)$$

in  $\mathcal{K}_n$ .

*Proof.* From the isomorphism  $\eta_i \oplus \eta_i^\perp \cong \eta_{i-1} \oplus \mathbb{C}_{n-i+1}$ , we see that the first Chern class of  $\eta_i^\perp$  is given by  $c_1(\eta_i^\perp) = c_1(\eta_{i-1}) - c_1(\eta_i)$ . If we define  $\mu_i^\perp := \eta_i^\perp - 1 \in \tilde{K}^0(B_n)$ , and apply the Chern character, we obtain that  $\mu_i^\perp = \mu_{i-1} - \mu_i$  for  $1 \leq i \leq n$ . Therefore,

$$\begin{aligned} \bar{\eta}_i^\perp - \eta_i^\perp &= \bar{\mu}_i^\perp - \mu_i^\perp = \bar{\mu}_{i-1} - \bar{\mu}_i - \mu_{i-1} + \mu_i \\ &= (\bar{\mu}_{i-1} - \mu_{i-1}) + (\mu_i - \bar{\mu}_i) = (\bar{\eta}_{i-1} - \eta_{i-1}) + (\eta_i - \bar{\eta}_i). \end{aligned}$$

$\square$

Now let us fix the stably complex structure

$$\tau(B_n) \oplus \mathbb{R}^{2n} \cong \bigoplus_{i=1}^n U(i, i^\perp) \cong \bigoplus_{i=1}^{n-1} R(i, 0) \oplus \mathbb{C}^2,$$

and write  $\tau_n^F$  for its representative in  $\tilde{K}^0(B_n)$ . On the other hand, we denote the class in  $\tilde{K}^0(B_n)$  corresponding to the bundle  $\bigoplus_{i=1}^n Y_i$  by  $\tau_Y$  for any  $(Y_1, \dots, Y_n) \in \mathcal{Y}_n$ .

**Theorem 3.6.** *For any  $(Y_1, \dots, Y_n) \in \mathcal{Y}_n$ , there exists  $\alpha_Y = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$  such that*

$$(3.8) \quad \tau_Y = \tau_n^F + \alpha_1 \cdot (\eta_1 - \bar{\eta}_1) + \dots + \alpha_n \cdot (\eta_n - \bar{\eta}_n).$$

Moreover, if  $(Y_1, \dots, Y_n)$  and  $(W_1, \dots, W_n)$  are two different  $n$ -tuples in  $\mathcal{Y}_n$ , then we have  $\alpha_Y \neq \alpha_W$ .

*Proof.* Let  $(Y_1, \dots, Y_n)$  be given in  $\mathcal{Y}_n$ . We first need to show that there exist integers  $\alpha_{i-1}^i$  and  $\alpha_i^i$  such that

$$(3.9) \quad \eta_i + \eta_i^\perp + \alpha_{i-1}^i \cdot (\eta_{i-1} - \bar{\eta}_{i-1}) + \alpha_i^i \cdot (\eta_i - \bar{\eta}_i) = Y_i$$

for each  $1 \leq i \leq n$ . However, the existence of such integers is obvious, and they are given by

$$(\alpha_{i-1}^i, \alpha_i^i) := \begin{cases} (0, 0), & \text{if } Y_i = R(i-1, 0), \\ (-1, 0), & \text{if } Y_i = L(i-1, 0), \\ (-1, 1), & \text{if } Y_i = R(i, i^\perp), \\ (0, -1), & \text{if } Y_i = L(i, i^\perp), \end{cases}$$

for  $1 \leq i \leq n$  (see also Lemma 3.5). Then, the first assertion follows at once from (3.9) if we define  $\alpha_i := \alpha_i^i + \alpha_i^{i+1}$  for  $1 \leq i \leq n-1$  and  $\alpha_n = \alpha_n^n$ . For the second one, we apply induction on  $n$ .  $\square$

Therefore, combining Theorems 3.4 and 3.6, we obtain that the actual set of stably complex structures on  $B_n$  arising from omniorientations is indexed by the elements of the set  $\mathcal{Y}_n$ . It is therefore natural to ask how many they are. To answer such a question, we first need to bring the notion of “generating trees” into use. Extensive material on the subject may be found in [16].

We claim that any element of the set  $\mathcal{Y}_n$  is indexed by a unique path in a generating tree  $\Lambda_n$ , which may be described by the following succession rules:

$$(3.10) \quad \begin{aligned} \text{Root: } & (a) \\ \text{Rules: } & (a) \rightarrow (a)(a)(b) \\ & (b) \rightarrow (a)(a)(b)(b). \end{aligned}$$

Now, we label the bundles  $R(i, i^\perp)$  and  $L(i, i^\perp)$  by  $(a)$  for  $1 \leq i \leq n$ , and correspondingly, the bundles  $R(i, 0)$  and  $L(i, 0)$  have the same label  $(b)$  for any  $0 \leq i \leq n$ . In particular, we assume that the root, being in the zeroth level, is labeled by  $(a)$ . On the other hand, since we do not distinguish the labels of  $R(i, i^\perp)$  and  $L(i, i^\perp)$ , we have to insist that whenever a node in the  $i$ -th level is labeled by  $(a)$ , then its children should be from the set  $\Delta_{i+1}^R$  or  $\Delta_{i+1}^L$  according to whether  $(a)$  represents  $R(i, i^\perp)$  or  $L(i, i^\perp)$ . Therefore, the level-number  $(a)_n$  gives the number of  $n$ -tuples  $(Y_1, \dots, Y_n)$  in  $\mathcal{Y}_n$  for which  $Y_n = R(n, n^\perp)$  or  $L(n, n^\perp)$ , and similarly,  $(b)_n$  gives the number of such  $n$ -tuples with  $Y_n = R(n-1, 0)$  or  $L(n-1, 0)$ . Moreover, the number of elements in  $\mathcal{Y}_n$  is given by  $L_n$ ; that is,  $|\mathcal{Y}_n| = L_n$ .

**Lemma 3.7.** *The  $n$ th level-numbers of the generating tree  $\Lambda_n$  are given as follows:*

$$(3.11) \quad \begin{aligned} (a)_n &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} 2^{n-i}, & (b)_n &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i+1} 2^{n-i-1}, \\ L_n &= \sum_{i=0}^{\lceil \frac{n}{2} \rceil} \binom{n+1}{2i} 2^{n-i} \end{aligned}$$

for all  $n \geq 1$ , where  $\lfloor x \rfloor$  is the greatest integer less than or equal to  $x$  and respectively,  $\lceil x \rceil$  is the least integer greater than or equal to  $x$ .

*Proof.* The transition matrix of  $\Lambda_n$  is given by

$$T_n := \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

Since the zeroth level-numbers are  $\binom{1}{0}$ , the  $n$ th level numbers of  $\Lambda_n$  are given by the solution of the equation

$$\begin{pmatrix} (a)_n \\ (b)_n \end{pmatrix} = (T_n)^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

from which we obtain the result.  $\square$

**Corollary 3.8.** *The number of stably complex structures on  $B_n$  arising from omniorientations equals  $L_n$  for any  $n \geq 1$ .*

**Example 3.9.** Consider the case when  $n = 2$ . Then, the number of stably complex structures on  $B_2$  arising from omniorientations is equal to  $L_2 = 10$ , and they are given as follows:

$$\begin{aligned} \mathbb{C}^2 \oplus R(1, 0) &= \eta_1 \oplus \mathbb{C}^3, \\ \mathbb{C}^2 \oplus L(1, 0) &= \bar{\eta}_1 \oplus \mathbb{C}^3, \\ \mathbb{C}^2 \oplus L(2, 2^\perp) &= \bar{\eta}_2 \oplus \eta_2^\perp \oplus \mathbb{C}^2, \\ \mathbb{C}^2 \oplus R(2, 2^\perp) &= \eta_2 \oplus \bar{\eta}_2^\perp \oplus \mathbb{C}^2, \\ R(1, 1^\perp) \oplus R(1, 0) &= 2\eta_1 \oplus \bar{\eta}_1^\perp \oplus \mathbb{C}, \\ L(1, 1^\perp) \oplus L(1, 0) &= 2\bar{\eta}_1 \oplus \eta_1^\perp \oplus \mathbb{C}, \\ L(1, 1^\perp) \oplus L(2, 2^\perp) &= \bar{\eta}_1 \oplus \eta_1^\perp \oplus \bar{\eta}_2 \oplus \eta_2^\perp, \\ R(1, 1^\perp) \oplus R(2, 2^\perp) &= \eta_1 \oplus \bar{\eta}_1^\perp \oplus \eta_2 \oplus \bar{\eta}_2^\perp, \\ L(1, 1^\perp) \oplus R(2, 2^\perp) &= \bar{\eta}_1 \oplus \eta_1^\perp \oplus \eta_2 \oplus \bar{\eta}_2^\perp, \\ R(1, 1^\perp) \oplus L(2, 2^\perp) &= \eta_1 \oplus \bar{\eta}_1^\perp \oplus \bar{\eta}_2 \oplus \eta_2^\perp. \end{aligned}$$

For the rest of this section, we will try to answer two separate questions. The first question deals with the determination of those stably complex structures on  $B_n$  induced by almost complex structures. We recall that in our restricted case, a stably complex structure is induced by an almost complex structure if and only if its top Chern class is equal to the Euler class of the manifold [15]. The second one concerns cobordism calculations related to each class in  $\Omega_*^U$  determined by  $B_n$  with a fixed stably complex structure arising from an omniorientation.

We first need to introduce some notation. Let  $\mathcal{Y}_n^{\mathbb{C}}$  denote a subset of  $\mathcal{Y}_n$  consisting of  $n$ -tuples of the form  $(Y_1, \dots, Y_n)$  such that  $Y_i$  is either  $R(i, i^\perp)$  or  $L(i, i^\perp)$  for any  $1 \leq i \leq n$ . The set  $\mathcal{Y}_n^{\mathbb{C}}$  corresponds to a sub-generating tree  $\Lambda_n^{\mathbb{C}}$  in  $\Lambda_n$  with the succession rule:

$$(3.12) \quad \begin{aligned} \text{Root: } &(a) \\ \text{Rules: } &(a) \rightarrow (a)(a), \end{aligned}$$

and the number of elements in  $\mathcal{Y}_n^{\mathbb{C}}$  is clearly equal to  $2^n$ .

**Definition 3.10.** We say that an  $n$ -tuple  $(Y_1, \dots, Y_n)$  in  $\mathcal{Y}_n^{\mathbb{C}}$  has the binary representation  $w = w_1 \dots w_n \in \mathcal{S}^n$  if

$$Y_k = \begin{cases} L(k, k^\perp), & \text{if } w_k = 1, \\ R(k, k^\perp), & \text{if } w_k = 0, \end{cases}$$

for each  $1 \leq k \leq n$ , where  $\mathcal{S}^n$  denotes the set of binary words of length  $n$  on  $\mathcal{S} = \{0, 1\}$ . In this case, we sometimes call  $w = w_1 \dots w_n$  the binary representation of the stably complex structure  $\tau_n(Y) := \bigoplus_{k=1}^n Y_k$ .

**Example 3.11.** Consider the  $n$ -tuple  $(L(1, 1^\perp), \dots, L(n, n^\perp))$  in  $\mathcal{Y}_n^{\mathbb{C}}$ , and recall that the tangent bundle of  $B_n$  considered as a complex manifold satisfies  $\tau(B_n) \oplus \mathbb{C}^n \cong \bigoplus_{k=1}^n L(k, k^\perp) =: \tau_n(L)$  (compare to (2.7)). Now,  $w_L := 11 \dots 1 \in \mathcal{S}^n$  is the binary representation of the bundle  $\tau_n(L)$ .

In order to assist our cobordism calculations and decide which stably complex structures among  $\tau_n(Y)$  are induced by almost complex structures, we next determine the top Chern class of  $\tau_n(Y)$ , where  $(Y_1, \dots, Y_n) \in \mathcal{Y}_n$ . We recall that the first Chern classes of the bundles  $\eta_i$  and  $\eta_i^\perp$  are given by  $x_i$  and  $x_{i-1} - x_i$  respectively for  $1 \leq i \leq n$ , where  $x_0 = 0$ .

**Theorem 3.12.** For any given  $n$ -tuple  $(Y_1, \dots, Y_n)$  in  $\mathcal{Y}_n^{\mathbb{C}}$  with the binary representation  $w = w_1 \dots w_n$ , the top Chern class of  $\tau_n(Y)$  is  $c_n(\tau_n(Y)) = (-1)^{\lambda_n} 2^n x_n^n$ , where  $\lambda_n := w_1 + \dots + w_n$ .

*Proof.* We first need to observe that

$$(3.13) \quad c_n(\tau_n(Y)) = c_{n-1}(\tau_{n-1}(Y)) \cdot c_1(Y_n),$$

where  $\tau_k(Y) := \bigoplus_{l=1}^k Y_l$  for any  $1 \leq k \leq n$ . To see that, let  $c(\tau_n(Y))$  denote the total Chern class of  $\tau_n(Y)$ . Then it follows that

$$c(\tau_n(Y)) = c(Y_1) \cdots c(Y_{n-1}) \cdot c(Y_n) = c(\tau_{n-1}(Y)) \cdot c(Y_n).$$

On the other hand, we have  $c(R(k, k^\perp)) = 1 - x_{k-1} + 2x_k$  and  $c(L(k, k^\perp)) = 1 + x_{k-1} - 2x_k$  for any  $1 \leq k \leq n$ . Therefore, we obtain  $c(\tau_n(Y)) = c(\tau_{n-1}(Y)) \cdot (1 \pm x_{n-1} \mp 2x_n)$ ; hence, (3.13) is obvious. We prove the claim by induction on  $n$ . When  $n = 1$ , there is nothing to prove, since there are exactly two elements in  $\mathcal{Y}_1^{\mathbb{C}}$  given by  $R(1, 1^\perp)$  and  $L(1, 1^\perp)$  with  $c_1(R(1, 1^\perp)) = 2x_1$  and  $c_1(L(1, 1^\perp)) = -2x_1$ . Suppose that the claim holds for some  $n > 1$  and for all  $1 \leq k \leq n$ . Let  $(Y_1, \dots, Y_{n+1})$  be any  $(n + 1)$ -tuple in  $\mathcal{Y}_{n+1}^{\mathbb{C}}$  with the binary representation  $w = w_1 \dots w_{n+1}$ . Without loss of generality, we may assume that  $Y_{n+1} = L(n + 1, (n + 1)^\perp)$  so that  $w_{n+1} = 1$ . Thus, we obtain by (3.13) and the induction assumption that

$$\begin{aligned} c_{n+1}(\tau_{n+1}(Y)) &= c_n(\tau_n(Y)) \cdot c_1(L(n + 1, (n + 1)^\perp)) \\ &= ((-1)^{\lambda_n} 2^n x_n^n) \cdot (x_n - 2x_{n+1}) \\ &= (-1)^{\lambda_{n+1}} 2^{n+1} x_{n+1}^{n+1}, \end{aligned}$$

where  $x_n^{n+1} = 0$  and  $\lambda_{n+1} = \lambda_n + 1$  since  $w_{n+1} = 1$ . □

**Example 3.13.** Consider the element  $(L(1, 1^\perp), R(2, 2^\perp), R(3, 3^\perp), L(4, 4^\perp))$  in  $\mathcal{Y}_4^{\mathbb{C}}$ . Its binary representation is given by  $w = 1001$ . To simplify the notation, we may denote the corresponding bundle by  $\tau_4(Y_w)$ ; that is,

$$\tau_4(Y_w) := L(1, 1^\perp) \oplus R(2, 2^\perp) \oplus R(3, 3^\perp) \oplus L(4, 4^\perp).$$

Then, applying Theorem 3.12, the top Chern class of  $\tau_4(Y_w)$  is given by

$$c_4(\tau_4(Y_w)) = 16x_4^4 = 16x_1x_2x_3x_4.$$

**Corollary 3.14.** *For any  $(Y_1, \dots, Y_n) \in \mathcal{Y}_n$  if  $(Y_1, \dots, Y_n) \notin \mathcal{Y}_n^{\mathbb{C}}$ , then the top Chern class of  $\tau_n(Y)$  is equal to zero.*

*Proof.* Let  $(Y_1, \dots, Y_n)$  be such an  $n$ -tuple. Then, there exists at least one  $1 \leq i \leq n$  such that  $Y_i = R(i-1, 0)$  or  $L(i-1, 0)$  with  $c(Y_i) = 1+x_{i-1}$  or  $1-x_{i-1}$  respectively. It is easy to observe that, in the total Chern class of  $\tau_n(Y)$ , there does not exist a nonzero monomial of degree  $n$ .  $\square$

**Theorem 3.15.** *A stably complex structure  $\tau_n(Y) = \bigoplus_{i=1}^n Y_i$  on  $B_n$  is induced by an almost complex structure if and only if  $(Y_1, \dots, Y_n) \in \mathcal{Y}_n^{\mathbb{C}}$  and  $\lambda_n \equiv n \pmod{2}$  for any  $n \geq 1$ . Moreover, the number of such structures is equal to  $2^{n-1}$ .*

*Proof.* Recall that when we consider  $B_n$  as a complex manifold, the tangent bundle  $\tau(B_n)$  satisfies  $\tau(B_n) \oplus \mathbb{C}^n \cong \bigoplus_{k=1}^n L(k, k^\perp) = \tau_n(L)$ . Therefore, appealing to Theorem 3.12, the Euler class of  $\tau(B_n)$  is given by

$$e(\tau(B_n)) = c_n(\tau_n(L)) = (-1)^n 2^n x_n^n.$$

On the other hand, the Euler characteristic of  $B_n$  is  $2^n$ ; hence, the Kronecker product  $\langle x_n^n, \kappa_n \rangle$  equals  $(-1)^n$ , where  $\kappa_n \in H_{2n}(B_n; \mathbb{Z})$  is the fundamental class of  $B_n$  induced by  $B_n \xrightarrow{\tau(B_n)} BU(n) \rightarrow BSO(2n)$ . Now, the stably complex structure  $\tau_n(Y)$  is induced by an almost complex structure if and only if  $c_n(\tau_n(Y)) = e(\tau(B_n))$ . Therefore, the claims follow from Corollary 3.14 and Theorem 3.12.  $\square$

We next move on to calculations in the complex cobordism ring  $\Omega_*^U$  determined by  $B_n$  with a fixed stably complex structure of the form  $\tau_n(Y)$  for some  $(Y_1, \dots, Y_n)$  in  $\mathcal{Y}_n$ . If  $(p_1, \dots, p_k)$  is a partition of  $n$ , we then write  $c_{p_1} \cdots c_{p_k}[\tau_n(Y)]$  for the corresponding Chern number. It is well known that the Chern numbers of a stably complex manifold determine the cobordism class.

We have already represented  $B_n$  with a bounding structure. We will prove that it is not the only one with this property.

Let  $\mathcal{Y}_n^B$  denote a subset of  $\mathcal{Y}_n$  consisting of  $(Y_1, \dots, Y_n)$  such that  $Y_n$  is either  $R(n-1, 0)$  or  $L(n-1, 0)$ . The number of elements in  $\mathcal{Y}_n^B$  is given by the level number  $(b)_n$  in  $\Lambda_n$  (see Lemma 3.7).

**Theorem 3.16.** *The bounded flag manifold  $B_n$  with the stably complex structure  $\tau_n(Y)$  represents zero in  $\Omega_*^U$  if and only if  $(Y_1, \dots, Y_n) \in \mathcal{Y}_n^B$ .*

*Proof.* Assume that  $(Y_1, \dots, Y_n) \in \mathcal{Y}_n^B$  is given. It follows that the first Chern class of  $Y_n$  is either  $x_{n-1}$  or  $-x_{n-1}$ . Therefore, none of the monomials in the total Chern class of  $\tau_n(Y)$  contain  $x_n$  as a factor, so that the product

$$c_{p_1}(\tau_n(Y)) \cdots c_{p_k}(\tau_n(Y))$$

is zero in  $H^{2n}(B_n; \mathbb{Z})$  for any partition  $(p_1, \dots, p_k)$  of  $n$ . Thus, the first assertion follows. For the second one, if  $(Y_1, \dots, Y_n) \notin \mathcal{Y}_n^B$ , then we have at least  $c_1 \cdots c_1[\tau_n(Y)] \neq 0$ .  $\square$

## REFERENCES

1. V. Batyrev, *Quantum cohomology rings of toric manifolds*, Astérisque, Société Mathématique de France, vol. 218, 1993, pp. 9-34. MR1265307 (95b:32034)
2. A. Brønsted, *An Introduction to Convex Polytopes*, Springer-Verlag, New York, 1983. MR0683612 (84d:52009)
3. V. Buchstaber and T. Panov, *Torus Actions, Combinatorial Topology and Homological Algebra*, AMS University Lecture Series, vol.24, 2002. MR1897064 (2003e:57039)

4. V. Buchstaber and N. Ray, *Flag manifolds and the Landweber-Novikov algebra*, *Geometry and Topology*, vol. 2, 1998, pp. 79-101. MR1623426 (99c:57064)
5. V. Buchstaber and N. Ray, *Tangential structures on toric manifolds, and connected sums of polytopes*, *International Mathematical Research Notices*, vol. 4, 2001, pp. 193-219. MR1813798 (2002b:57043)
6. Y. Civan, *The Topology of Families of Toric Manifolds*, Ph.D. Thesis, University of Manchester, 2001.
7. Y. Civan, *KO-groups of bounded flag manifolds*, *Turkish Journal of Mathematics*, vol. 26, 2002, pp. 447-463. MR1944933 (2003m:55005)
8. Y. Civan and N. Ray, *Homotopy decompositions and real K-theory of Bott towers*, *K-theory*, vol. 34, 2005, pp. 1-33.
9. M. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, *Duke Math. Journal*, vol. 62, 1991, pp. 417-451. MR1104531 (92i:52012)
10. G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Springer-Verlag, New York, GTM 168, 1996. MR1418400 (97i:52012)
11. T. Oda, *Convex Bodies and Algebraic Geometry*, Springer-Verlag, New York, 1988. MR0922894 (88m:14038)
12. N. Ray, *On a construction in bordism theory*, *Proceedings of the Edinburgh Mathematical Society*, **29**, 1986, pp. 413-422. MR0865274 (88a:57065)
13. N. Ray, R. Switzer and L. Taylor, *Normal structures and bordism theory with applications to  $MSp_*$* , *Memoires of the American Mathematical Society*, vol. 193-12, 1977. MR0461520 (57:1505)
14. R.E. Stong, *Notes on Cobordism Theory*, Princeton University Press, New Jersey, 1968. MR0248858 (40:2108)
15. E. Thomas *Complex structures on real vector bundles*, *American Journal of Mathematics*, vol. 89, 1967, pp. 887-908. MR0220310 (36:3375)
16. J. West, *Generating trees and forbidden subsequences*, *Discrete Mathematics*, vol. 157, 1996, pp. 363-374. MR1417303 (98d:05013)

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