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A MINIMAL PAIR OF K-DEGREES

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ABSTRACT. We construct a minimal pair of K-degrees. We do this by showing the existence of an unbounded nondecreasing function f which forces K-triviality in the sense that $\gamma \in 2^\omega$ is K-trivial if and only if for all n, $K(\gamma \upharpoonright n) \leq K(n) + f(n) + \mathcal{O}(1)$.

1. Introduction and notation

K-reducibility is defined with the intention of measuring the relative randomness of infinite binary strings, which we refer to as reals. This reducibility was defined using a function, K, that assigns to each finite binary string the length of its shortest description, in a sense we will specify. The idea being that if a string is random, there should not be any short way of describing it. The precise definition of K is given below, though the proofs presented in this paper use only the two properties of K listed at the end of this section.

The *prefix-free Kolmogorov complexity* of a string $\sigma \in 2^{<\omega}$ is defined to be the length of the shortest program $p \in 2^{<\omega}$ such that $U(p) = \sigma$, where U is a universal prefix-free Turing machine. That is, U is universal for machines V with the property that if $V(\tau)\downarrow$, then $V(\tau')\uparrow$ for all $\tau' \supset \tau$. We denote the Kolmogorov complexity of σ by $K(\sigma)$. This definition is independent of the choice of universal machine U, up to an additive constant. The advantage of restricting to prefix-free machines is that otherwise the Kolmogorov complexity would contain extra information about the length of the string. For more background on Kolmogorov complexity, see Li and Vitányi [LV97], and Downey and Hirschfeldt [DH].

Prefix-free Kolmogorov complexity is used to define a notion of randomness for real numbers. A real $\gamma \in 2^\omega$ is *K-random* (or Levin-Gaćs-Chaitin random) if for all n, $K(\gamma \upharpoonright n) \geq n - \mathcal{O}(1)$. This notion has been extensively studied and coincides with other notions of randomness based on measure theory or unpredictability [DH], [DHNT]. We can also use K to define what it means for a real to be far from being random. We say a real is *K-trivial* if for all n, $K(\gamma \upharpoonright n) \leq K(n) + \mathcal{O}(1)$; that is, every initial segment is as simple as possible. But what of relative randomness of reals? K-reducibility was introduced to study notions of relative randomness. For

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two reals α and β in 2^ω we let

$$\alpha \leq_K \beta \iff (\forall n)\ K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + \mathcal{O}(1),$$

i.e., if there exists a constant C such that $(\forall n)\ K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + C$. The K-degrees are defined as equivalence classes under this quasi-ordering.

As is usual when considering a reducibility, we want to understand the structure of the K-degrees. We know that the K-degrees have a bottom element that corresponds to the K-degree of the K-trivial reals. Yu, Ding, and Downey showed that there are uncountably many K-degrees, indeed $2^{\aleph_0}$ many among the K-random reals ([YDD04], see [DHNT]). When restricting attention to c.e. reals (reals with nice approximations), Downey, Hirschfeldt, and LaForte have shown density and existence of join [DHL04]. A result of Solovay is that K-reducibility does not imply Turing reducibility (see [DH]).

A natural question to ask when studying a reducibility is if there exists a minimal pair. Rod Downey and Denis Hirschfeldt asked this question for the K-degrees. That is, they asked whether there exist non-K-trivial reals α and β in 2^ω such that whenever $\gamma \in 2^\omega$ is such that $\gamma \leq_K \alpha$ and $\gamma \leq_K \beta$, then γ is K-trivial. Here we answer this question affirmatively with a simple and elegant construction of a minimal pair. We do this by first constructing an unbounded nondecreasing function f which forces K-triviality in the sense that a real γ is K-trivial if and only if $(\forall n)\ K(\gamma \upharpoonright n) \leq K(n) + f(n) + \mathcal{O}(1)$. This function will likely be useful in showing other results about K-reducibility.

If a real is K-trivial, then there is some constant which witnesses its K-triviality. We say a real γ is *K-trivial(C)* if for all n, $K(\gamma \upharpoonright n) \leq K(n) + C$, where $K(n) = K(0^n)$. Then, we have that γ is K-trivial if and only if it is K-trivial(C) for some C. We say that γ *appears to be K-trivial(C) at n* if for all $m \leq n$, $K(\gamma \upharpoonright m) \leq K(m) + C$. We say that γ *stops appearing K-trivial(C) at n* if it appears K-trivial(C) at $n-1$ but not at n. Throughout the paper, γ will always denote a real, i.e. $\gamma \in 2^\omega$.

The properties of K that we will use are the following.

Property 1 (Zambella—see [DHNS03])**.** *For every C, there are only finitely many reals that are K-trivial(C).*

Property 2. *For any $\sigma \in 2^{<\omega}$, $\sigma^\frown 0^\omega$ is K-trivial, and hence K-trivial(C) for some C.*

2. Construction of a minimal pair

Theorem 1. *There exists a minimal pair of K-degrees.*

To prove our theorem, we will use the following lemma, which is interesting in itself and may have other applications.

Lemma 1. *There exists an unbounded nondecreasing function f such that for all reals $\gamma \in 2^\omega$, the following are equivalent.*

(1) *γ is K-trivial.*
(2) *For almost every n, $K(\gamma \upharpoonright n) \leq K(n) + f(n)$.*
(3) *$(\forall n)\ K(\gamma \upharpoonright n) \leq K(n) + f(n) + \mathcal{O}(1)$.*

Before proving Lemma 1, we show how Theorem 1 follows from it.

Proof of Theorem 1. Let f be as in Lemma 1. We will construct two non-K-trivial reals α and β such that $\min\{K(\alpha \upharpoonright n), K(\beta \upharpoonright n)\} \leq K(n)+f(n)$. This will give us a minimal pair because if $\gamma \leq_K \alpha$ and $\gamma \leq_K \beta$, then $K(\gamma \upharpoonright n) \leq K(n)+f(n)+\mathcal{O}(1)$, and hence γ is K-trivial.

We construct α and β as the limits of two sequences of finite strings, $\{\alpha_s\}_{s\in\omega}$ and $\{\beta_s\}_{s\in\omega}$, which satisfy that, for every s, $\alpha_s \subset \alpha_{s+1}$, $\beta_s \subset \beta_{s+1}$ and $|\alpha_s| = |\beta_s|$. We denote $|\alpha_s|$ by n_s. To get $\min\{K(\alpha \upharpoonright n), K(\beta \upharpoonright n)\} \leq K(n) + f(n)$, we ensure that if $n_s \leq n < n_{s+1}$, then $K(\alpha \upharpoonright n) \leq K(n) + f(n)$ if s is odd, and $K(\beta \upharpoonright n) \leq K(n)+f(n)$ if s is even. To make α and β non-K-trivial, we ensure that for every s there is some n, $n_s \leq n < n_{s+1}$, such that either $K(\alpha \upharpoonright n) > K(n) + s$, or $K(\beta \upharpoonright n) > K(n) + s$ depending or whether s is even or odd.

Construction. Stage 0: Let $\alpha_0 = \beta_0 = \emptyset$. Stage $s+1$: Suppose first that s is even. Let $\alpha'_{s+1} \supset \alpha_s$ be such that $K(\alpha'_{s+1}) \geq K(|\alpha'_{s+1}|) + s$. Such an α'_{s+1} must exist because not every extension of α_s is K-trivial$(s-1)$. Let C_{s+1} be such that $\alpha'_{s+1}\widehat{\ }0^\omega$ is K-trivial(C_{s+1}). Choose $n_{s+1} > |\alpha'_{s+1}|$ such that $f(n_{s+1}) \geq C_{s+1}$. Finally, let $\alpha_{s+1} = \alpha'_{s+1}\widehat{\ }0^\omega \upharpoonright n_{s+1}$ and $\beta_{s+1} = \beta_s\widehat{\ }0^\omega \upharpoonright n_{s+1}$. If s is odd do the same as above but with the roles of α and β reversed.

It is clear from the construction that for s even there is some n, $n_s \leq n < n_{s+1}$, such that $K(\alpha \upharpoonright n) > K(n)+s$, namely $|\alpha'_{s+1}|$. Also, for every n, $n_{s+1} \leq n < n_{s+2}$,

$$\begin{aligned} K(\alpha \upharpoonright n) = K(\alpha_{s+2} \upharpoonright n) &= K(\alpha_{s+1}\widehat{\ }0^\omega \upharpoonright n) = K(\alpha'_{s+1}\widehat{\ }0^\omega \upharpoonright n) \\ &\leq K(n) + C_{s+1} \leq K(n) + f(n_{s+1}) \leq K(n) + f(n), \end{aligned}$$

and analogously for s odd. □

Proof of Lemma 1. Clearly (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) for any unbounded nondecreasing function. We now show that (3) $\Rightarrow$ (1). That is, we construct an unbounded nondecreasing function f such that, for any real γ, if $(\forall n)\ K(\gamma \upharpoonright n) \leq K(n) + f(n) + \mathcal{O}(1)$, then γ is K-trivial.

We first define an unbounded nondecreasing function f_0 such that $(\forall n)$ $K(\gamma \upharpoonright n) \leq K(n) + f_0(n)$ implies that γ is K-trivial(0). We do this by defining a sequence $n_0 < n_1 < n_2 < \cdots$, and letting $f_0(n) = k$ for every n such that $n_{k-1} < n \leq n_k$ (where $n_{-1} = -1$).

As there are only finitely many reals that are K-trivial(2), we can choose n_0 such that any γ that is K-trivial(2), but not K-trivial(0), has stopped appearing K-trivial(0) by n_0. Suppose now that we have already defined n_k. Let n_{k+1} be such that any γ that is K-trivial$(k+3)$, but not K-trivial(0), has stopped appearing K-trivial(0) by n_{k+1}. We can do this because there are only finitely many reals that are K-trivial$(k+3)$. Except when $k = 0$, we also require n_{k+1} to be such that any γ which stopped appearing K-trivial(0) at some m, $n_{k-1} < m \leq n_k$, does not appear to be K-trivial$(k+1)$ by n_{k+1}. Note that such n_{k+1} has to exist. Indeed, by definition of n_{k-1}, $\gamma \upharpoonright m$ can have no K-trivial$(k+1)$ real extending it. So by König's Lemma, the tree of apparently K-trivial$(k+1)$ extensions of $\gamma \upharpoonright m$ must be finite.

We claim that f_0 is as wanted. Suppose that γ is a real such that $(\forall n)$ $K(\gamma \upharpoonright n) \leq K(n) + f_0(n)$; we want to show that actually $(\forall n)\ K(\gamma \upharpoonright n) \leq K(n)$. Clearly γ appears to be K-trivial(0) up to length n_0. Assume for a contradiction that γ is not K-trivial(0). Let $k > 0$ be least such that γ stops appearing K-trivial(0) at some m, $n_{k-1} < m \leq n_k$. Then by definition of n_{k+1}, γ stops appearing K-trivial$(k+1)$ by n_{k+1}. That means that there is some $m \leq n_{k+1}$ such that $K(\gamma \upharpoonright m) \geq K(m) + k + 2 > K(m) + f_0(m)$, a contradiction.

There is nothing special about 0 in this proof. In the same way we can construct, for each i, a function f_i such that $f_i(0) = i$ and $(\forall n)\ K(\gamma \upharpoonright n) \leq K(n) + f_i(n)$ implies that γ is K-trivial(i). Just choose n_0 such that any γ that is K-trivial$(i+2)$, but not K-trivial(i), has stopped appearing K-trivial(i) by n_0. Then given n_k, let n_{k+1} be such that any γ that is K-trivial$(i + k + 3)$, but not K-trivial(i), has stopped appearing K-trivial(i) by n_{k+1}. For $k \neq 0$, also require n_{k+1} to be such that any γ which stopped appearing K-trivial(i) at some m, $n_{k-1} < m \leq n_k$, does not appear to be K-trivial$(i + k + 1)$ by n_{k+1}. Let $f_i(n) = i + k$ for every n such that $n_{k-1} < n \leq n_k$.

For each $n \in \omega$, let $f(n) = \min\{f_{2i}(n) - i : i \in \omega\}$, which exists because $(\forall i, n) f_{2i}(n) - i \geq i$. Note that f is a nondecreasing function. It is also unbounded because for each j, if we let n be such that $(\forall i < j) f_{2i}(n) > 2j$, then $j \leq f(n)$. Now, suppose that γ is a real such that $(\forall n)\ K(\gamma \upharpoonright n) \leq K(n) + f(n) + i$ for some i. Then $(\forall n)\ K(\gamma \upharpoonright n) \leq K(n) + f_{2i}(n)$, and hence γ is K-trivial$(2i)$. So every γ such that $K(\gamma \upharpoonright n) \leq K(n) + f(n) + \mathcal{O}(1)$ is K-trivial. $\square$

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