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ORTHONORMAL POLYNOMIAL WAVELETS ON THE INTERVAL

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Abstract. We use special functions and orthonormal wavelet bases on the real line to construct wavelet-like bases. With these wavelets we can construct polynomial bases on the interval; moreover, we can use them for the numerical resolution of degenerate elliptic operators.

1. Introduction

In past years wavelet bases were usually constructed from some mother wavelets by dilations and translations. The purpose of this paper is to use special functions to construct wavelets. We are motivated by two facts. The first one is the construction of polynomial wavelets on an interval, and the second one is the numerical resolution of degenerate elliptic operators.

Chui and Mhaskar [2] used trigonometric polynomials and constructed Riesz bases. Kilgore and Prestin [12], as well as Fisher and Prestin [9], utilized Chebyshev polynomials and got a polynomial Riesz basis. Based on the eigenfunctions of the Sturm-Liouville eigenvalue problem for ordinary differential equations, Depczynski [8] also got a Riesz basis. In this paper we shall combine wavelets from a multiresolution analysis (MRA) and the Chebyshev polynomials to get polynomial wavelet-like bases that are orthonormal bases.

On the other hand, the wavelets of [8, 9, 12] were used for detection of singularities of functions, and our wavelets can be applied to resolve differential equations with degenerate coefficients. We point out that wavelets generated from some mother functions through translations and dilations cannot be applied directly to degenerate coefficient equations. The combination of special functions and wavelets takes advantages from both disciplines. In [10] Frazier and Zhang studied the Bessel operator $-g''(x)+\frac{\nu^2-1/4}{x^2}g(x)$ with singular coefficients. They used the Bessel function $J_\nu(x)$ and the Hankel transform. Their basis functions were obtained through the Hankel transform of the Meyer wavelets. These basis functions have explicit expressions in the Hankel domain just as the Meyer wavelets do in the Fourier domain. In [5] we used Hermite polynomials to analyze the Schrödinger equation

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with unbounded coefficients. In [1] multiresolution weighted norm equivalence was derived.

The organization of this paper is as follows. In section 2 we shall present our method, which will be carried out in an abstract setting. We shall derive a nesting property from one level to the next level. In section 3 we shall get a wavelet-like polynomial based on Chebyshev polynomials, which are from eigenfunctions of a Sturm-Liouville problem. Our method generalizes the Riesz constructions of [8, 9, 12] to orthonormal bases. In section 4 we give an application to the resolution of degenerate elliptic operators.

2. Construction on a Hilbert space

In this section we shall work with a separable Hilbert space H, which has an orthonormal basis $\{u_n(x)\}_{n=0}^{\infty}$. We denote its inner product by $\langle \cdot, \cdot \rangle_H$.

Suppose that the function $\psi(t)$ is a mother wavelet from some MRA [7] and that its Fourier transform satisfies $|\widehat{\psi}(\xi)| \le C(1+|\xi|)^{-\sigma}$ $(\sigma > 2)$ for some constant C and that $\{\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. Let the function $\varphi(t)$ be its corresponding scaling function.

For $j \in \mathbb{Z}^+$ and $k = 0, 1, \cdots, 2^j - 1$, we define two functions by

$$\Phi_{j,k}(x) = 2^{-j/2}u_0(x) + \sum_{n=1}^{\infty}\left[\widehat{\varphi_{j,k}}(2\pi n)u_{2n}(x) + \widehat{\varphi_{j,k}}(-2\pi n)u_{2n-1}(x)\right], \tag{1}$$

$$\Psi_{j,k}(x) = \sum_{n=1}^{\infty}\left[\widehat{\psi_{j,k}}(2\pi n)u_{2n}(x) + \widehat{\psi_{j,k}}(-2\pi n)u_{2n-1}(x)\right]. \tag{2}$$

Suppose that the function $f \in H$ has the spectral decomposition $f(x) = \sum_{n=0}^{\infty} f_n u_n(x)$. We introduce the periodic function

$$\underline{f}(s) = f_0 + \sum_{n=1}^{\infty}\left[f_{2n}e^{-i2\pi ns} + f_{2n-1}e^{i2\pi ns}\right].$$

It is obvious that $\underline{f} \in L^2(\mathbb{T})$. Moreover we have the following relationship.

Lemma 2.1. *Let $f \in H$. Then we have*

$$\langle f, \Phi_{j,k}\rangle_H = \langle \underline{f}, \varphi_{j,k}^{per}\rangle_{L^2(\mathbb{T})} \quad \text{and} \quad \langle f, \Psi_{j,k}\rangle_H = \langle \underline{f}, \psi_{j,k}^{per}\rangle_{L^2(\mathbb{T})},$$

where $\varphi_{j,k}^{per}$ is the periodization of the function $\varphi_{j,k}$ (cf. [7], Chapter 9). In the rest of the paper we shall also use the notation $\psi_{j,k}^{per}$.

Proof. By virtue of the Parseval identity, we have

$$\langle f, \Phi_{j,k}\rangle_H = 2^{-j/2} + \sum_{n=1}^{\infty}\left[\widehat{\varphi_{j,k}}(2\pi n)f_{2n} + \widehat{\varphi_{j,k}}(-2\pi n)f_{2n-1}\right] = \langle \underline{f}, \varphi_{j,k}^{per}\rangle_{L^2(\mathbb{T})},$$

which proves the first equation. The proof of the second equation is similar. □

Since $\{u_n\}_{n=0}^{\infty}$ is orthonormal and $\hat{\psi}(0) = 0$ we have

Lemma 2.2. $\langle u_0, \Psi_{j,k}(x)\rangle_H = 0$ *for $j \in \mathbb{Z}^+$ and $k = 0, 1, \cdots, 2^j - 1$.*

Lemma 2.2 shows that $\Psi_{j,k}(x)$ is orthogonal to the ground state u_0, which will be a constant in the Chebyshev system; hence the function $\Psi_{j,k}(x)$ oscillates and acts like a **wavelet**.

From the equations (1) and (2), for fixed j, their right-hand sides are 2^j periodic in the index k. Concerning their orthogonality we have the following results.

Lemma 2.3. *For $j, j' \in \mathbb{Z}$ and $k = 0, 1, \cdots, 2^j - 1, k' = 0, 1, \cdots, 2^{j'} - 1$, we have*

$$\langle \Phi_{j,k}, \Phi_{j,k'} \rangle_H = \delta_{k,k'}, \quad \langle \Psi_{j,k}, \Psi_{j',k'} \rangle_H = \delta_{j,j'} \delta_{k,k'}, \quad \langle \Phi_{j,k}, \Psi_{j,k'} \rangle_H = 0.$$

Proof. We prove only the first equation. The proofs for the rest are similar. From Parseval's identity, it follows that

$$\begin{aligned} \langle \Phi_{j,k}, \Phi_{j,k'} \rangle_H &= \sum_{n \in Z} \widehat{\varphi_{j,k}}(2\pi n) \overline{\widehat{\varphi_{j,k'}}(2\pi n)} \\ &= \langle \varphi_{j,k}^{per}, \varphi_{j,k'}^{per} \rangle_{L^2(\mathbb{T})} \\ &= \delta_{k,k'}. \end{aligned}$$

□

Suppose that the scaling function φ satisfies the two-scale equation

$$\varphi(t) = \sum_{k \in \mathbb{Z}} h_k \varphi(2t - k),$$

for some sequence $\{h_k\}_{k \in \mathbb{Z}}$ with

$$|h_k|^2 \leq K k^{-2} \tag{3}$$

for some constant K and all $|k| \geq 1$. We denote $g_k = (-1)^{k-1} h_{1-k}$.

To state the two-scale relation, we introduce two sequences $\{h_k^j\}$ and $\{g_k^j\}$. For $j \in \mathbb{Z}^+, k \in \mathbb{Z}$, let

$$h_k^j = \sum_{l \in \mathbb{Z}} h_{2^j l + k}, \quad g_k^j = \sum_{l \in \mathbb{Z}} g_{2^j l + k}. \tag{4}$$

From (3), h_k^j and g_k^j are well-defined for fixed j and k, and they are 2^j periodic in k.

We now state the two-scale relation:

Theorem 2.4. *For $j \in \mathbb{Z}^+, k = 0, 1, \cdots, 2^j - 1$, we have*

$$\Phi_{j,k}(x) = \frac{1}{\sqrt{2}} \sum_{l=0}^{2^{j+1}-1} h_{l-2k}^{j+1} \Phi_{j+1,l}(x)$$

and

$$\Psi_{j,k}(x) = \frac{1}{\sqrt{2}} \sum_{l=0}^{2^{j+1}-1} g_{l-2k}^{j+1} \Phi_{j+1,l}(x).$$

Proof. This can be carried out similarly as in [4], where the Meyer wavelet was used. □

Theorem 2.5. *Let $f \in H$. Then for any $j_0 \in \mathbb{Z}^+$,*

$$f(x) = \sum_{k=0}^{2^{j_0-1}} c_{j_0,k} \Phi_{j_0,k}(x) + \sum_{j \geq j_0} \sum_{k=0}^{2^j-1} d_{j,k} \Psi_{j,k}(x),$$

where $c_{j,k} = \langle f, \Phi_{j,k} \rangle_H, d_{j,k} = \langle f, \Psi_{j,k} \rangle_H$.

Proof. For any $N \geq j_0$, by virtue of Lemma 2.1 and the fact $\|f\|_H = \|\underline{f}\|_{L^2(\mathbb{T})}$ we have

$$\left\| f - \left[\sum_{k=0}^{2^{j_0}-1} \langle f, \Phi_{j_0,k} \rangle_H \Phi_{j_0,k} + \sum_{j=j_0}^{N} \sum_{k=0}^{2^j-1} \langle f, \Psi_{j,k} \rangle_H \Psi_{j,k} \right] \right\|_H^2$$

$$= \left\| \underline{f} - \left[\sum_{k=0}^{2^{j_0}-1} \langle \underline{f}, \varphi_{j_0,k}^{per} \rangle_{L^2(\mathbb{T})} \varphi_{j_0,k}^{per} + \sum_{j=j_0}^{N} \sum_{k=0}^{2^j-1} \langle \underline{f}, \psi_{j,k}^{per} \rangle_{L^2(\mathbb{T})} \psi_{j,k}^{per} \right] \right\|_{L^2(\mathbb{T})}^2 .$$

Since $\underline{f} \in L^2(\mathbb{T})$ from the results on periodic wavelets the right-hand side of the above equation converges to zero as $N \to \infty$ [7, 6]. □

The coefficients $c_{j,k}$ and $d_{j,k}$ defined in Theorem 2.5 inherit relations from the periodic wavelet system $\{\varphi_{j,k}^{per}, \psi_{j,k}^{per}\}$.

Theorem 2.6. *We have the decomposition algorithm*

$$c_{j,k} = \frac{1}{\sqrt{2}} \sum_{l=0}^{2^{j+1}-1} h_{l-2k}^{j+1} c_{j+1,l}, \quad d_{j,k} = \frac{1}{\sqrt{2}} \sum_{l=0}^{2^{j+1}-1} g_{l-2k}^{j+1} c_{j+1,l}, \quad k = 0, 1, \cdots, 2^j - 1,$$

and the reconstruction algorithm

$$c_{j+1,l} = \frac{1}{\sqrt{2}} \sum_{k=0}^{2^j-1} (h_{l-2k}^{j+1} c_{j,k} + g_{l-2k}^{j+1} d_{j,k}), \quad l = 0, 1, \cdots, 2^{j+1} - 1,$$

where the coefficients h_k^j, g_k^j are defined by (4).

Proof. Using Lemma 2.1 and results for the periodic function $\underline{f}$ with periodic wavelet system $\{\varphi_{j,k}^{per}, \psi_{j,k}^{per}\}$, we prove Theorem 2.6. □

3. Polynomial wavelets

In this section we apply the results of the previous section to the Chebyshev polynomials. We shall get a polynomial wavelet-like orthonormal system.

Consider the Chebyshev polynomials

$$T_n(t) = \cos(n \arccos t), \quad n = 0, 1, \cdots,$$

and let $u_0(t) = 1/\sqrt{\pi}, u_n(t) = \sqrt{2/\pi} T_n(t), n = 1, 2, \cdots$. Then $\{u_n(t)\}_{n=0}^{\infty}$ is an orthonormal basis of the weighted space $L_w^2([-1, 1])$ with inner product

$$(f, g) = \int_{-1}^{1} f(t) \overline{g(t)} (1 - t^2)^{-1/2} dt.$$

To get a polynomial basis, we choose the scaling function φ and wavelet ψ so that their Fourier transforms have compact support. An example is the Meyer wavelet. It is defined through Fourier transform by

$$\hat{\varphi}(\xi) = \begin{cases} 1, & |\xi| \leq \frac{2\pi}{3}, \\ \cos[\frac{\pi}{2} \nu (\frac{3}{2\pi} |\xi| - 1)], & \frac{2\pi}{3} < |\xi| < \frac{4\pi}{3}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\hat{\psi}(\xi) = \begin{cases} e^{-i\frac{\xi}{2}} \sin[\frac{\pi}{2}\nu(\frac{3}{2\pi}|\xi| - 1)], & \frac{2\pi}{3} \le |\xi| \le \frac{4\pi}{3}, \\ e^{-i\frac{\xi}{2}} \cos[\frac{\pi}{2}\nu(\frac{3}{4\pi}|\xi| - 1)], & \frac{4\pi}{3} < |\xi| < \frac{8\pi}{3}, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where the rising cutoff function $\nu(s)$ is continuously differentiable on $-\infty < s < +\infty$. We can take, for example, $\nu(s) = s - \sin(2\pi s)/(2\pi)$.

From the equations (1) and (2) the functions $\Phi_{j,k}(t)$ and $\Psi_{j,k}(t)$ will be polynomials. The first few of our polynomial wavelets are as follows. The scaling functions: $\Phi_{0,0}(t) = 1/\sqrt{\pi}, \Phi_{1,0}(t) = (t + 2t^2)/\sqrt{2\pi}, \Phi_{1,1}(t) = (2 - t - 2t^2)/\sqrt{2\pi}$, and the wavelet functions:

$$\Psi_{0,0}(t) = (1 + t - 2t^2)/\sqrt{\pi},$$
$$\Psi_{1,0}(t) = (-1 + i + (3 + i)t + (8 - 2i)t^2 - 4t^3 - 8t^4)/\sqrt{2\pi},$$
$$\Psi_{1,1}(t) = (-1 - i + (3 - i)t + (8 + 2i)t^2 - 4t^3 - 8t^4)/\sqrt{2\pi}.$$

In [12], Kilgore and Prestin constructed polynomial wavelets which are defined by $\Psi_{j,k}(t) = \sum_{n=2^j-1}^{2^{j+1}} c_{n,k} u_n(t)$, where the coefficient

$$c_{n,k} = u_n(\cos(2k + 1)\pi/2^{j+1})\epsilon_{j+1,n}.$$

Depczynski [8] considered the general Sturm-Liouville problem; in particular, he used $\Psi_{j,k}(t) = \sum_{n=2^j+1}^{2^{j+1}} c_{n,k} u_n(t)$, where the coefficient

$$c_{n,k} = \sqrt{2/(2^j + 1)} \cos(n\pi(2k + 1)/2^{j+1}).$$

These correspond to the dyadic Littlewood-Paley decomposition of the system $\{u_n\}$. They derived decomposition and reconstruction algorithms but got only a Riesz basis where the dimension of the scaling function space V_j at level j is $2^j + 1$. We used a smoothed dyadic Littlewood-Paley partition via the Meyer wavelet and chose an appropriate set of coefficients so that the dimension of V_j is exactly 2^j and the basis functions are mutually orthogonal. The wavelets in [8] have only approximate vanishing moments, whereas the wavelets constructed here have vanishing moments and their number increases with the level j.

4. Degenerate elliptic operator

The purpose of this section is to show that our generalized wavelets can be used for the numerical resolution of degenerate elliptic operators

$$Pu = -(1 - t^2)^{1/2} \frac{d}{dt} \left(a(t) \frac{du}{dt} \right), \tag{6}$$

where the function a is non-negative and satisfies

$$0 \le a_1(1 - t^2)^{1/2} \le a(t) \le a_2(1 - t^2)^{1/2}, \; t \in [-1, 1],$$

for some positive constants a_1, a_2.

The differential equation (6) is degenerate since its leading coefficient will be zero at $t = \pm 1$. Our wavelets can be used to resolve the degenerate elliptic equations with degenerate coefficients.

Jaffard [11] applied wavelets to study the second-order elliptic equation

$$-\frac{d}{dx}\left(a(x)\frac{dy}{dx}\right)+b(x)\frac{dy}{dx}=f(x),$$

with regular coefficients on a bounded domain, which is the perturbation of the model equation $-\frac{d^2y}{dx^2}=f(x)$.

We introduce the Chebyshev operator L defined on $C_0^\infty([-1,1])$ by

$$Lu=-(1-t^2)^{1/2}\frac{d}{dt}\left((1-t^2)^{1/2}\frac{du}{dt}\right).$$

For the positive operator L we define a function space $\mathbb{H}$ by

$$\mathbb{H}=\overline{\{u:\ u\in C_0^\infty([-1,1])\text{ such that }(Lu,u)<\infty\}}.$$

The space $\mathbb{H}$ is a Hilbert space, which is a subspace of $L_w^2([-1,1])$. We use the Chebyshev system $\{u_n\}_{n=0}^\infty$ and the wavelet system $\{\varphi,\psi\}$. It is not necessary to require them to have compact support in the frequency domain. We can characterize the space $\mathbb{H}$ via our wavelet system by

Lemma 4.1. *Suppose that $u\in\mathbb{H}$ is decomposed as*

$$u=c_{0,0}\Phi_{0,0}+\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}d_{j,k}\Psi_{j,k}.$$

Then for the Chebyshev operator L, we have

$$(Lu,u)\approx|c_{0,0}|^2+\sum_{j=0}^{\infty}4^j\sum_{k=0}^{2^j-1}|d_{j,k}|^2. \tag{7}$$

Proof. From the definition of $\Phi_{j,k}$ and $\Psi_{j,k}$ we have

$$u=c_{0,0}u_0+\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}d_{j,k}\sum_{n=1}^{\infty}\left[\widehat{\psi_{j,k}}(2\pi n)u_{2n}+\widehat{\psi_{j,k}}(-2\pi n)u_{2n-1}\right].$$

Hence, by applying the operator L, we have

$$Lu=c_{0,0}u_0+\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}d_{j,k}\sum_{n=1}^{\infty}\left[\widehat{\psi_{j,k}}(2\pi n)(2n+1)^2u_{2n}+\widehat{\psi_{j,k}}(-2\pi n)(2|n|-1)^2u_{2n-1}\right].$$

By the orthogonality of the Chebyshev system we get

(8)

$$\begin{aligned}(Lu,u)=|c_{0,0}|^2+\sum_{j,j'=0}^{\infty}\sum_{k=0}^{2^j-1}\sum_{k'=0}^{2^{j'}-1}d_{j,k}\overline{d_{j',k'}}\sum_{n=1}^{\infty}\Big[&(2n+1)^2\widehat{\psi_{j,k}}(2\pi n)\overline{\widehat{\psi_{j',k'}}(2\pi n)}\\&+(2|n|-1)^2\widehat{\psi_{j,k}}(-2\pi n)\overline{\widehat{\psi_{j',k'}}(-2\pi n)}\Big].\end{aligned}$$

Consider the pseudo-differential operator ℓ,

$$\ell u(x)=\sum_{n\in\mathbb{Z}}e^{i2\pi nx}l(n)\hat{u}(n),\qquad u\in C^\infty(\mathbb{T}),$$

with symbol given by

$$l(n) = \begin{cases} (2n+1)^2, & n \geq 0, \\ (2|n|-1)^2, & n < 0. \end{cases} \tag{9}$$

Then we have $l(n) \approx 1 + n^2$.

For the periodic function $w^{per} = c_{0,0} + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \sum_{n\in\mathbb{Z}} \widehat{\psi_{j,k}}(2\pi n) e^{i2\pi ns}$ we have after application of the pseudo-differential operator

$$\ell w^{per} = c_{0,0} + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \sum_{n\in\mathbb{Z}} l(n) \widehat{\psi_{j,k}}(2\pi n) e^{i2\pi ns}.$$

Hence we have

(10)

$$(\ell w^{per}, w^{per}) = |c_{0,0}|^2 + \sum_{j,j'=0}^{\infty} \sum_{k=0}^{2^j-1} \sum_{k'=0}^{2^{j'}-1} d_{j,k} \overline{d_{j',k'}} \sum_{n=1}^{\infty} \Big[(2n+1)^2 \widehat{\psi_{j,k}}(2\pi n) \times \overline{\widehat{\psi_{j',k'}}(2\pi n)} + (2|n|-1)^2 \widehat{\psi_{j,k}}(-2\pi n) \overline{\widehat{\psi_{j',k'}}(-2\pi n)} \Big].$$

On the other hand, by (9) for the first-order pseudo-differential operator we have

$$(\ell w^{per}, w^{per}) \approx \|w^{per}\|_1^2. \tag{11}$$

From the characterization for periodic functions [7] we have

$$\|w^{per}\|_1^2 \approx |c_{0,0}|^2 + \sum_{j=0}^{\infty} 4^j \sum_{k=0}^{2^j-1} |d_{j,k}|^2. \tag{12}$$

Combining (8), (10)–(12), we conclude that (7) holds. □

Lemma 4.2. *For the operator P and the operator L we have the equivalence*

$$(Pu, u) \approx (Lu, u), \forall u \in C_0^\infty([-1,1]).$$

Proof. By the assumption on the function a and integration by parts we have

$$\begin{aligned}
(Pu, u) &= \int_{-1}^{1} (Pu)(t)\overline{u(t)}(1-t^2)^{-1/2} dt \\
&= -\int_{-1}^{1} \frac{d}{dt}\left(a\frac{du}{dt}\right)\overline{u(t)} dt \\
&= -a\frac{du}{dt}\overline{u}|_{-1}^{1} + \int_{-1}^{1} a\frac{du}{dt}\overline{\frac{du}{dt}} dt \\
&= \int_{-1}^{1} a\frac{du}{dt}\overline{\frac{du}{dt}} dt \\
&\approx \int_{-1}^{1} (1-t^2)^{1/2} |\frac{du}{dt}|^2 dt \\
&= (Lu, u).
\end{aligned}$$

□

With Lemma 4.1 and Lemma 4.2 and the techniques in [10, 11] we can show that

Theorem 4.3. *For the Galerkin scheme of the degenerate elliptic equation*

$$Pu = g$$

we can simply pre-condition the resultant algebraic system by a diagonal matrix so that its condition number is uniformly bounded.

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