PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 5, Pages 1257–1263
S 0002-9939(05)08090-1
Article electronically published on October 4, 2005

FINITENESS CONDITIONS AND INFINITE MATRIX RINGS

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(Communicated by Martin Lorenz)

Abstract. For a unital ring R, $\mathrm{RCFM}_\alpha(R)$ denotes the ring of row and column finite matrices over R indexed by α. We give necessary and sufficient structural conditions on $\mathrm{RCFM}_\alpha(R)$ which are equivalent to R being, respectively, Quasi-Frobenius, left artinian, and left noetherian.

In this paper R denotes an (associative and unital) ring and α is an infinite set. We use the following notation, where "matrices" means "matrices indexed by α with entries in R":

$$\begin{array}{rcl} A = \mathrm{RFM}_\alpha(R) & = & \text{ring of row finite matrices,} \\ B = \mathrm{RCFM}_\alpha(R) & = & \text{ring of row and column finite matrices,} \\ B_0 = \mathrm{FM}_\alpha(R) & = & \text{ring of finite matrices.} \end{array} \tag{1}$$

At first sight it might seem that the rings A and B are too big to reflect properties of the ring R, and still more unexpected that A or B could encode finiteness conditions of R. However, already in [6, 12] it is shown that the ring A reflects some finiteness conditions of R. There is a long tradition in the study of the ring theoretical properties of the ring A (among others see [2, 6, 9, 12]). Recently several authors have shown interest in the study of the ring B (see, e.g., [5, 7, 10]). In this paper we study the properties of the ring B under the assumption that R satisfies some finiteness condition (quasi-Frobenius, artinian, noetherian).

The relationship between a ring R and A comes essentially from the adjoint pair $\mathrm{Hom}_R(F,-) : R-\mod \leftrightarrows A-\mod : F\otimes_A -$, where F is a free left R-module of rank $|\alpha|$ and A is canonically identified with $\mathrm{End}_R(F)$. An interesting exception may be found in the computation of the Jacobson radical (see [13]), where the amount of matrix manipulation exceeds adjunction techniques. If one wants to relate the rings R and B one can also use the adjoint pair $\mathrm{Hom}_R(F,-) : R-\mod \leftrightarrows B-\mod : F\otimes_B -$. The difference in the performance of the adjoint pairs for A and B relies on the fact that while A is in the image of $\mathrm{Hom}_R(F,-)$, namely $A \simeq \mathrm{End}_R(F)$, this is not the case for B. However, we may still use some "adjoint-like" techniques to relate some special objects in the category of R-modules and some special matrices or ideals of B and keeping in mind that B is the ring of continuous endomorphisms of F in a certain topology [11].

We start with some notation. If $i,j\in\alpha$, then e_{ij} denotes the element of B_0 having 1 in the (i,j)-th entry and zeroes elsewhere, and if $a\in A$, then $a(i,j)$ denotes the (i,j)-th entry of a. Set $e_i = e_{ii} = e_{\{i\}}$. If F is a subset of α, then

Received by the editors September 2, 2004 and, in revised form, November 19, 2004.
2000 *Mathematics Subject Classification.* Primary 16S50; Secondary 16P20, 16P40.
Both authors were partially supported by the D.G.I. of Spain and Fundación Séneca of Murcia.

set $e_F = \sum_{i\in F} e_i$. A careful study of the arithmetic of the rings A, B and B_0 and their idempotents e_F for F finite leads to the following lemma.

Lemma 1. 1. *B_0 is a two-sided ideal of B and a right ideal of A.*
2. *For every $x \in B_0$ there is $e = e^2 \in B_0$ such that $x = ex = xe$.*
3. *The map $\rho : A \to \mathrm{End}({}_{B_0}B_0)$ that associates $a \in A$ with the endomorphism ρ_a of B_0 given by $\rho_a(x) = xa$ is a ring isomorphism.*

We need the following module theoretical lemma (see [4, Exercise 18.17]).

Lemma 2. *A left R-module M is quasi-injective if and only if $f(M) \subseteq M$ for every endomorphism f of the injective hull of M.*

Recall that R is said to be quasi-Frobenius (QF) if R is right and left artinian, and there exists a duality between the categories of finitely generated left and right R-modules. Further, R is said to be quasi-continuous if $e(R) \subseteq R$ for every idempotent endomorphism e of the injective hull of ${}_RR$.

In [6] it is shown that R is QF if and only if A is left self-injective. In contrast, the ring B cannot be even left or right quasi-continuous [7] for any ring R. Our first result shows how to compute the injective hull of ${}_BB$ if R is QF. Note that as a consequence of the next theorem any of the equivalent conditions 2-6 hold for some infinite set α if and only if they hold for every infinite set.

Theorem 3. *The following conditions are equivalent for a ring R and an infinite set α:*

1. *R is quasi-Frobenius.*
2. *$\mathrm{RFM}_\alpha(R)$ is left self-injective.*
3. *$\mathrm{FM}_\alpha(R)$ is an injective object in the category $\mathrm{FM}_\alpha(R)-\mathrm{mod}$ consisting of those $\mathrm{FM}_\alpha(R)$-modules M such that $\mathrm{FM}_\alpha(R)M = M$.*
4. *$\mathrm{FM}_\alpha(R)$ is quasi-injective as a left $\mathrm{RCFM}_\alpha(R)$-module.*
5. *$\mathrm{RFM}_\alpha(R)$ is injective as a left $\mathrm{RCFM}_\alpha(R)$-module.*
6. *$\mathrm{RFM}_\alpha(R)$ is the injective hull of the regular left $\mathrm{RCFM}_\alpha(R)$-module $\mathrm{RCFM}_\alpha(R)$.*

Proof. Let A, B and B_0 be as in (1) and let E denote the injective hull of ${}_BB$.

The equivalence between conditions 1 and 2 was proved in [6].

The equivalence between conditions 5 and 6 follows from the fact that B_0 is essential in ${}_BA$ and hence $E = E({}_BA) = E({}_BB_0)$.

The fact that condition 3 implies 4 follows from the equality $B_0 = BB_0$.

Identifying A with $\mathrm{End}_R(R^{(\alpha)})$ one can consider $R^{(\alpha)}$ as a $R-B_0$-bimodule and it is well known that the functor $F = B_0\mathrm{Hom}_R(R^{(\alpha)}, -) : R-\ \mathrm{mod}\ \to B_0-$ mod is an equivalence of categories such that $F(R^{(\alpha)}) = B_0$. (Alternatively one can prove this by using the results of [1] or [3].) Now the equivalence between conditions 1 and 3 follows from the fact that R is quasi-Frobenius if and only if ${}_RR^{(\alpha)}$ is injective; this is a direct consequence of Theorems 24.18 and 24.20 of [8].

Now we prove that condition 4 implies 6. Assume that ${}_BB_0$ is quasi-injective. We have already seen that $A \subseteq E$. Let $e \in E$ and consider the map $f : B_0 \to E$ given by $f(x) = xe$. By assumption and Lemmas 1 and 2 there is an $a \in A$ such that $xe = xa$ for every $x \in B_0$. We claim that $e = a$ and this completes the proof. Indeed, if $e \neq a$, then $B(e-a) \neq 0$ and since B_0 is essential in ${}_BE$ there is $0 \neq x \in B_0 \cap B(e-a)$. Thus there is $x_0 \in B_0$ and $b \in B$ such that $x = x_0x = b(e-a)$ and hence $x = x_0x = x_0b(e-a) = 0$, a contradiction.

Finally we prove that condition 5 implies 3. Assume that ${}_BA$ is injective and let $f : N \to B_0$ be a homomorphism of B_0-modules with N a submodule of B_0. By using $B_0N = N$ and $B_0 = B_0B = BB_0$ one deduces that f is a homomorphism of B-modules and hence f extends to an endomorphism g of ${}_BA$. If $x \in B_0$, then there is $e \in B_0$ such that $x = ex$ and hence $g(x) = eg(x) \in B_0$. Therefore g restricts to an endomorphism of ${}_BB_0$ which extends f. This shows that ${}_BB_0$ is quasi-injective by Lemma 2. □

For every $i \in \alpha$, consider the map $\pi_i : B \to R^{(\alpha)}$ that associates to $a \in B$, the i-th row of a. Let I be a left ideal of B. Since $\pi_j(a) = \pi_i(e_{ij} \cdot a)$, $\pi_i(I)$ does not depend on i. Write $\pi(I) = \pi_i(I)$ for $i \in \alpha$ arbitrary. Clearly $\pi(I)$ is a submodule of ${}_RR^{(\alpha)}$. (There is an alternative definition of $\pi(I)$ for I a left ideal of B as $R^{(\alpha)}I$, where $R^{(\alpha)}$ is considered as an R-B-bimodule in the natural matricial way.) Thus if $a \in B$, then $\pi(Aa)$ is the submodule of $R^{(\alpha)}$ generated by the rows of a. If M is a submodule of ${}_RR^{(\alpha)}$, then $\iota(M) = \{a \in B : \pi(a) \in M,$ for every $i \in \alpha\}$ is a left ideal of B.

Definition 4. We say that a left ideal I of B is *closed* if for every $a \in B \setminus I$ there is $i \in I$ such that $\pi_i(a) \notin \pi(I)$.

Lemma 5. *A left ideal I of B is closed if and only if $I = \iota(M)$ for some submodule M of ${}_RR^{(\alpha)}$. In particular the left annihilators $l_B(X)$ in B of subsets X of B and the left ideals of the form $B(J) = \{a \in B : a(i,j) \in J$ for every $i,j\}$, where J is a left ideal of R, are closed left ideals of B.*

Proof. Clearly $I \subseteq \iota(\pi(I))$ and I is closed if and only if the equality holds. Further if M is a submodule of $R^{(\alpha)}$, then $\iota(\pi(\iota(M))) = \iota(M)$. This proves the first statement. The second is a consequence of the first because $B(J) = \iota(J^{(\alpha)})$ and if $X \subseteq B$, then $l_B(X) = \iota(M)$ where M is the annihilator in $R^{(\alpha)}$ of X. □

Proposition 6. *The following conditions are equivalent for a left ideal J of R:*

1. $\mathrm{RCFM}_\alpha(J) = \{a \in \mathrm{RCFM}_\alpha(R) : a(i,j) \in J$ *for every* $i,j\}$ *is a cyclic left ideal of* $\mathrm{RCFM}_\alpha(R)$.
2. $\mathrm{RCFM}_\alpha(J)$ *is a finitely generated left ideal of* $\mathrm{RCFM}_\alpha(R)$.
3. J *is a finitely generated left ideal of* R.

Proof. Let A, B and B_0 as in (1). The equivalence between conditions 1 and 2 follows from the fact that B has single basis number, i.e. ${}_BB \simeq {}_BB^n$ for every positive integer n. If ${}_BB(J)$ is generated by X, then J is generated by the entries of the first row of the elements of X. This proves that condition 2 implies 3.

Let $x_1, \dots, x_n$ generate ${}_RJ$. Since α is infinite there is a natural identification between $B(J)$, $\mathrm{RCFM}_\alpha(M_{n,1}(K))$ and $\mathrm{RCFM}_\alpha(M_{1,n}(R))$ that we are going to use without specific mention. Let $a \in B(J)$ be the following "diagonal" matrix (diagonal in $M_\alpha(M_{n,1}(J))$):

$$g = \begin{pmatrix} X & 0 & 0 & \cdots \\ 0 & X & 0 & \cdots \\ 0 & 0 & X & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{with} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

If $b \in B(J)$, then for every $x, y \in J$ there is $c(x,y) \in M_{1,n}(R)$ such that the (x,y)-th entry $b(x,y)$ of b is $c(x,y)X$ and we may take $c(x,y) = 0$ if $b(x,y) = 0$. Then $c = (c(x,y))_{x,y\in\alpha} \in B$ and $b = ca$. This proves that $B(J) = Ba$. □

Remark 7. If RCFM is replaced by RFM in conditions 1 and 2 of Proposition 6 then they are still equivalent, since $\mathrm{RFM}_\alpha(R)$ has a single basis number. However they are not equivalent to condition 3. For example, if J is the ideal generated by a subset X of R, then $\mathrm{RFM}_X(J)$ is the cyclic left ideal generated by the matrix having the elements of X in one column and zeroes in the remaining columns.

The Jacobson radical of R is denoted by $J(R)$. The ring R is said to be a Baer ring if any left (equivalently, right) annihilator in R of a subset of R is a direct summand.

Theorem 8. *If α is an infinite set, then a unital ring R is left artinian if and only if $J(\mathrm{RCFM}_\alpha(R))$ is cyclic as a left ideal of $\mathrm{RCFM}_\alpha(R)$, and*

$$\mathrm{RCFM}_\alpha(R)/J(\mathrm{RCFM}_\alpha(R))$$

is a Baer ring. In this case $J(\mathrm{RCFM}_\alpha(R)) = \mathrm{RCFM}_\alpha(J(R))$.

Proof. Assume that R is left artinian. By Proposition 6, $\mathrm{RCFM}_\alpha(J(R))$ is a cyclic left ideal of $\mathrm{RCFM}_\alpha(R)$. By [7, Theorem 14], $\mathrm{RCFM}_\alpha(R)/J(\mathrm{RCFM}_\alpha(R))$ is a Baer ring, and $J(\mathrm{RCFM}_\alpha(R)) = \mathrm{RCFM}_\alpha(J(R))$, so that $J(\mathrm{RCFM}_\alpha(R))$ is a cyclic left ideal of $\mathrm{RCFM}_\alpha(R)$.

Conversely, by [7, Theorem 14], R is a perfect ring and $J(\mathrm{RCFM}_\alpha(R)) = \mathrm{RCFM}_\alpha(J(R))$. By hypothesis and Proposition 6, $J(R)$ is a finitely generated left ideal of R. Then by [4, Ex. 28.9] R is left artinian. □

By Proposition 6 if every closed left ideal of B is finitely generated, then R is left noetherian. Our next result proves the converse for α countable.

Theorem 9. *The following conditions are equivalent for a unital ring R:*

1. *R is left noetherian.*
2. *Every closed left ideal of $\mathrm{RCFM}_{\mathbb{N}}(R)$ is finitely generated.*
3. *Every closed left ideal of $\mathrm{RCFM}_{\mathbb{N}}(R)$ is cyclic.*

Proof. By Proposition 6 we only have to prove that condition 1 implies 3.

Let $I = \iota(M)$ be a closed left ideal of $B = \mathrm{RCFM}_{\mathbb{N}}(R)$. For every non-negative integer n let $p_n : R^{(\mathbb{N})} \to R^{(\mathbb{N})}$ be the projection on the first n coordinates and let

$$M_n = M \cap \mathrm{Im}\, p_n \quad \text{and} \quad K_n = M \cap \ker p_n.$$

Note that

$$0 = M_0 \leq M_1 \leq M_2 \leq \ldots \leq \bigcup_{n\geq 0} M_n = M = K_0 \geq K_1 \geq K_2 \geq \ldots \geq \bigcap_{n\geq 0} K_n = 0.$$

If M is finitely generated, say $M = \langle x_1, \ldots, x_k \rangle$, then I is generated by the matrix

$$a = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ 0 \\ \vdots \end{pmatrix}.$$

(Note that the rows of a are elements of $R^{(\mathbb{N})}$ and hence $a \in B$.) Thus assume that M is not finitely generated. This implies that $M \neq M_n$ for every $n \geq 0$.

If $n \geq 0$, then $p_n(M)$ is finitely generated and hence $p_n(M) = p_n(\langle m_1, \ldots, m_k \rangle)$ for some $m_1, \ldots, m_k \in M$. Since M is not finitely generated there is $m \in M \setminus \langle m_1, \ldots, m_k \rangle$. On the other hand $p_n(m) = \sum_{i=1}^k r_i p_n(m_i)$ for some $r_i \in R$ and thus $0 \neq m - \sum_{i=1}^k r_i m_i \in K_n$.

The argument of the previous paragraph shows that $K_n \neq 0$ for every $n \geq 0$, and using this fact we recursively construct a sequence $(m_n, k_n, S_n)_{n \geq 0}$ as follows:

$$\begin{aligned} k_0 &= 0, \text{ the integer;} \\ S_0 &= 0, \text{ the trivial submodule of } M; \\ m_n &= \min\{m : M_m \cap K_{k_n} \neq 0\}; \\ k_{n+1} &= \min\{k : (M_{m_n} + \textstyle\sum_{i=0}^n S_i) \cap K_k = 0\} \end{aligned}$$

and S_{n+1} is a finitely generated submodule of K_{k_n} such that

$$\left(M_{m_{n+1}} + \sum_{i=0}^n S_i \right) \cap K_{k_n} \subseteq S_{n+1} \subseteq K_{k_n} \quad \text{and} \quad (S_{n+1} + K_{k_{n+1}})/K_{k_{n+1}} = K_{k_n}/K_{k_{n+1}}. \tag{2}$$

The existence of m such that $M_m \cap K_{k_n} \neq 0$ is warranted by the fact that $K_{k_n} \neq 0$; the existence of k such that $(M_{m_n} + \sum_{i=0}^n S_i) \cap K_k = 0$ follows from the fact the $M_n + \sum_{i=0}^n S_i$ is finitely generated and hence it is embedded in K_k for some k; finally the existence of S_{n+1} finitely generated and satisfying (2) follows from the fact that $M_{m_{n+1}} \cap K_{k_n}$ and $K_{k_n}/K_{k_{n+1}}$ are finitely generated. Since $M_{m_n} \cap K_{k_n} \neq 0$, $M_{m_n} \cap K_{k_{n+1}} \neq 0$ and $M_{m_{n+1}} \cap K_{k_{n+1}} \neq 0$, $m_n < m_{n+1}$ and $k_n < k_{n+1}$ for every n, i.e. (m_n) and (k_n) are increasing sequences.

Claim: For every $n, r \geq 0$, $(M_{m_n} + \sum_{i=0}^n S_i) \cap K_{k_r} \subseteq \sum_{i \geq r+1}^{n+1} S_i$.
In particular, $K_{k_r} \subseteq \sum_{i \geq r+1} S_i$ for every $r \leq 0$.

If $n - r < 0$, then $(M_{m_n} + \sum_{i=0}^n S_i) \cap K_{k_r} \subseteq (M_{m_n} + \sum_{i=0}^n S_i) \cap K_{k_{n+1}} = 0$ and the Claim is obvious, so we assume that $n - r \geq 0$ and argue by induction on $n - r$. If $n - r = 0$, then $(M_{m_n} + \sum_{i=0}^n S_i) \cap K_{k_r} = (M_{m_r} + \sum_{i=0}^r S_i) \cap K_{k_r} \subseteq S_{r+1}$ by construction. Assume that $n - r > 0$ and let $m \in (M_{m_n} + \sum_{i=0}^n S_i) \cap K_{k_r}$. Then there is $s_{r+1} \in S_{r+1}$ such that $n = m - s_{r+1} \in K_{k_{r+1}}$. Thus $n \in (M_{m_n} + \sum_{i=0}^n S_i) \cap K_{k_{r+1}}$, by induction hypothesis $n \in \sum_{i \geq r+2}^{n+1} S_i$ and so $m \in \sum_{i \geq r+1}^{n+1} S_i$. This proves the Claim.

For every n let X_n be a finite generating set of S_n and construct the matrix

$$a = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \end{pmatrix}.$$

That is, the first rows of a are formed by the elements of X_1, in some order, the next rows are formed by the elements of X_2, etc. Since $S_n \subseteq K_{k_n}$ and (k_n) is a strictly increasing sequence, each column of a has only finitely many non-zero entries. That is, $a \in B$ and because I is a closed left ideal and every row of a belongs to $M = \pi(I)$, one deduces that $a \in I$.

Let u_n be the cardinality of X_n and let $v_n = \sum_{i=1}^n u_n$. By the claim if $x \in K_r$, then $x = ya$ for some $y \in \ker p_{v_r}$. If $x \in I$, then there is a strictly increasing sequence $(r_n)_n$ such that $\pi_m(x) \in K_{k_n}$ for every $m \geq r_n$. Thus if $r_n \leq m < r_{n+1}$, then $\pi_m(x) = y_m a$ with $y_m \in \ker p_{v_r}$ and hence $x = ya$, where y is the row finite matrix defined by setting $\pi_m(y) = y_m$. Since $M = \sum_{n\geq 1} S_n$ is not finitely generated the sequence (v_n) is non-decreasing and non-bounded and this implies that $y \in B$. Thus $x \in Ba$, and this proves that $I = Ba$. □

Note that the proof of Theorem 9 does not apply if the index set is not countable. We do not know whether or not the closed ideals of $B_\alpha(R)$ are cyclic for R noetherian and α a non-countable set. A consequence of Theorem 9 is the following.

Corollary 10. *If R is left noetherian, then $\mathrm{RCFM}_{\mathbb{N}}(R)$ is left coherent.*

Proof. Let I be a finitely generated left ideal of B. Since B has single basis number there is a surjective homomorphism $f : B \to I$ of left B-modules. Then $I = Ba$ for some $a \in B$ and hence Ker f is the left annihilator of a. Thus Ker f is a closed left ideal of B. By Theorem 9, Ker f is finitely generated and we conclude that I is finitely presented. □

We conjecture that the converse of Corollary 10 is false in general. However, we provide a partial converse in the next proposition.

Proposition 11. *Let α be an infinite set and let R be a unital ring. If $\mathrm{RCFM}_\alpha(R)$ is left coherent, then R is left coherent and satisfies acc on direct summands.*

Proof. Assume that $B = \mathrm{RCFM}_\alpha(R)$ is left coherent. To prove that R is left coherent we show that if $a_1, \dots, a_r \in R$, then the kernel of the homomorphism $f : R^n \to R$ given by $f(x_1, \dots, x_n) = \sum_{i=1}^n x_i a_i$ is finitely generated. Let $J = \{j_1, \dots, j_n\}$ be a subset of α of cardinality n and $a \in B$ be given by

$$a(x,y) = \begin{cases} 1, & \text{if } x = y \notin J, \\ a_i, & \text{if } x = j_i \text{ and } y = j_1, \\ 0, & \text{otherwise.} \end{cases}$$

By the construction of a, the left annihilator $l_B(a)$ of a in B is formed by the element $b \in B$ such that $b(x,y) = 0$ if $y \notin J$ and $(b(x,j_1), \dots, b(x,j_n)) \in \ker f$ for every $x \in \alpha$. By hypothesis there is $b \in B$ such that $Bb = l_B(a)$ and hence it is easy to see that $\ker f$ is generated by the elements of the form $b(x) = (b(x,j_1), \dots, b(x,j_n))$ with $x \in \alpha$. Since $b \in B$, $b(x) = 0$ for almost all $x \in \alpha$ and hence $\ker f$ is finitely generated as wanted.

Now we prove that R satisfies acc on direct summands. Otherwise R has an infinite countable set $\{f_1, f_2, \dots\}$ of non-zero orthogonal idempotents. Let $J = \{j_1, j_2, \dots\}$ be an infinite countable subset of α and consider the matrix $a \in B$ given as follows:

$$a(x,y) = \begin{cases} 1, & \text{if } x = y \notin J, \\ 1 - f_n, & \text{if } x = y = j_n, \\ -f_{n+1}, & \text{if } x = j_{n+1} \text{ and } y = j_n, \\ f_{n+1} - 1, & \text{if } x = j_{n+2} \text{ and } y = j_n, \\ 0, & \text{otherwise.} \end{cases}$$

By hypothesis $l_B(a) = Bb$ for some $b \in B$. Let $I = \sum_{n\in\mathbb{N}} Rf_i$ and let K be the left ideal of R generated by the entries in the j_1-column of b. Note that K is finitely generated while I is not. We are going to obtain a contradiction by showing that $I = K$.

To prove $I \subseteq K$, consider, for any $n \in \mathbb{N}$, the matrix $m_n \in B$ having f_n in the entries (j_1, j_i) for $1 \leq i \leq n$ and 0 in any other entry. Note that $m_n a = 0$ and hence $m_n \in Bb$. Thus $f_n \in K$.

The reverse inclusion follows by showing that all the entries of b belong to I. (In fact the same holds for any element in $l_B(a)$). Fix $x \in \alpha$ and let $Y = \{y \in \alpha : b(x,y) \neq 0\}$. We have to prove that $b(x,y) \in I$ for every $y \in Y$. This is obvious if $Y = \emptyset$, so assume that this is not the case. Having in mind that $ba = 0$ one has that $Y \subseteq J$. Let n be the maximum positive integer such that $b(x, j_n) \neq 0$. We prove that $b(x, j_m) \in I$ for $1 \leq m \leq n$ by induction on $m - n$. First $0 = (ba)(x, j_n) = b(x, j_n)(1 - f_n)$ and hence $b(x, j_n) \in Rf_n \subseteq I$. Second $0 = (ba)(x, j_{n-1}) = b(x, j_{n-1})(1-f_{n-1}) - b(x, j_n)f_n$ and hence $b(x, j_{n-1}) \in Rf_{n-1} + Rf_n \subseteq I$. Finally, if $1 \leq m \leq n-2$, then $0 = (ba)(x, j_m) = b(x, j_m)(1 - f_{j_m}) - b(x, j_{m+1})f_{m+1} + b(x, j_{m+1})(f_{m+1} - 1)$ and hence $b(x, j_m) \in I$, because by the induction hypothesis $b(x, j_{m+1}) \in I$. □

Corollary 12. *If α is an infinite cardinal and R is a ring, then R is semisimple artinian if and only if R is von Neumman regular and* $\mathrm{RCFM}_\alpha(R)$ *is left coherent.*

Proof. This is a direct consequence of Proposition 11 and the fact that R is semisimple artinian if and only if R is von Neumann regular and satisfies acc on direct summands [8, Theorem 19.26 A]. □

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