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INVERTIBILITY OF LINEAR COMBINATIONS OF TWO IDEMPOTENTS

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ABSTRACT. Let P and Q be two idempotents on a Hilbert space. In this note, we prove that the invertibility of the linear combination $\lambda_1 P + \lambda_2 Q$ is independent of the choice of λ_i, $i = 1, 2$, if $\lambda_1\lambda_2 \neq 0$ and $\lambda_1 + \lambda_2 \neq 0$.

Let $\mathcal{H}$ be a Hilbert space, and let all bounded linear operators on $\mathcal{H}$ be denoted by $\mathcal{B}(\mathcal{H})$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be idempotent if $P^2 = P$. The set $\mathcal{P}$ of all idempotents in $\mathcal{B}(\mathcal{H})$ is invariant under similarity; that is, if $P \in \mathcal{P}$ and $S \in \mathcal{B}(\mathcal{H})$ is an invertible operator, then $S^{-1}PS$ is still an idempotent since $(S^{-1}PS)^2 = S^{-1}PSS^{-1}PS = S^{-1}P^2S = S^{-1}PS$. An idempotent P is called an orthogonal projection if $P^2 = P = P^*$, where P^* is the adjoint of P. Moreover, for an idempotent $P \in \mathcal{P}$, there exists an invertible operator $U \in \mathcal{B}(\mathcal{H})$ such that $U^{-1}PU$ is an orthogonal projection. In fact, if $P \in \mathcal{P}$, then P can be written in the form of

$$P = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp$, where $\mathcal{R}(M)$ denotes the range of the operator M. In this case, we have

$$\begin{pmatrix} I & P_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{P} = \begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix}$ is invertible and $\tilde{P}^{-1} = \begin{pmatrix} I & P_1 \\ 0 & I \end{pmatrix}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $(Ax, x) \geq 0$ for all $x \in \mathcal{H}$. If A is positive, then $A^{\frac{1}{2}}$ denotes the positive square root of A.

In recent years, a number of researchers have considered questions concerning the idempotents and linear combinations of idempotents (see [1]-[8]). Particularly, some researchers pay much attention to the study of linear combinations of two idempotents ([1], [5]). For example, if P_i, $i = 1, 2$, are idempotents in the finite-dimensional space C^n, J. K. Baksalary and O. M. Baksalary ([1]) have proved that the nonsingularity of $P_1 + P_2$ is equivalent to the nonsingularity of any linear combination $c_1P_1 + c_2P_2$, where $c_1 + c_2 \neq 0$. In the present note, we will study the

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invertibility of linear combinations of two idempotents on an infinite-dimensional Hilbert space. We obtain the main result, which is similar to [1], but the idea of the proof is different from [1].

Theorem 1. *Let P and Q in $\mathcal{B}(\mathcal{H})$ be two idempotents. If λ_1 and λ_2 are nonzero complex numbers and $\lambda_1+\lambda_2 \neq 0$, then the invertibility of $\lambda_1 P+\lambda_2 Q$ is independent of the choice of λ_i, $i = 1, 2$.*

To prove Theorem 1, we need some lemmas which are well known, so the proofs are omitted.

Lemma 2. *Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be a bounded linear operator on $\mathcal{H} \oplus \mathcal{K}$. Then A is a positive operator if and only if $A_{11} \geq 0$, $A_{22} \geq 0$, $A_{12} = A_{21}^*$ and there exists a contraction D from $\mathcal{K}$ into $\mathcal{H}$ such that*

$$A = \begin{pmatrix} A_{11} & A_{11}^{\frac{1}{2}} D A_{22}^{\frac{1}{2}} \\ A_{22}^{\frac{1}{2}} D^* A_{11}^{\frac{1}{2}} & A_{22} \end{pmatrix}.$$

Lemma 3. *Let $A \in \mathcal{B}(\mathcal{H})$ be invertible and $\widetilde{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$. Then $\widetilde{A}$ is invertible if and only if $D - CA^{-1}B$ is invertible.*

Lemma 4. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then the following statements hold:*

(1) *$\mathcal{R}(A) \subseteq \mathcal{R}(A^{\frac{1}{2}})$ and $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^{\frac{1}{2}})}$, where $\overline{K}$ denotes the closure of K;*
(2) *$\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$;*
(3) *$\mathcal{R}(A) = \mathcal{H}$ if and only if A is invertible.*

Proof. Let P and Q be two idempotents. By the discussion above, since $\lambda_1 P + \lambda_2 Q$ is invertible if and only if $\lambda_1 S^{-1}PS + \lambda_2 S^{-1}QS$ is invertible, to consider the invertibility of $\lambda_1 P + \lambda_2 Q$, without loss of generality, we can assume that one of P and Q is an orthogonal projection. For example, assume that Q is an orthogonal projection. Of course, Q is a positive operator. In this case, by Lemma 2, P and Q have the following operator matrix forms:

$$P = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} Q_1 & Q_1^{\frac{1}{2}} D Q_2^{\frac{1}{2}} \\ Q_2^{\frac{1}{2}} D^* Q_1^{\frac{1}{2}} & Q_2 \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp$, where Q_1 and Q_2 are positive operators on $\mathcal{R}(P)$ and $\mathcal{R}(P)^\perp$, respectively, and D is a contraction operator from $\mathcal{R}(P)^\perp$ into $\mathcal{R}(P)$.

Suppose $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. If $\lambda_1 P + \lambda_2 Q$ is invertible, that is, the operator matrix

$$\lambda_1 P + \lambda_2 Q = \begin{pmatrix} \lambda_1 I + \lambda_2 Q_1 & \lambda_1 P_1 + \lambda_2 Q_1^{\frac{1}{2}} D Q_2^{\frac{1}{2}} \\ \lambda_2 Q_2^{\frac{1}{2}} D^* Q_1^{\frac{1}{2}} & \lambda_2 Q_2 \end{pmatrix}$$

is invertible, then $\mathcal{R}((\lambda_2 Q_2^{\frac{1}{2}} D^* Q_1^{\frac{1}{2}}, \lambda_2 Q_2)) = \mathcal{R}(P)^\perp$. By Lemma 4, $\mathcal{R}(Q_2) \subseteq \mathcal{R}(Q_2^{\frac{1}{2}})$ and observing that $\mathcal{R}((\lambda_2 Q_2^{\frac{1}{2}} D^* Q_1^{\frac{1}{2}}, \lambda_2 Q_2)) \subseteq \mathcal{R}(Q_2)^{\frac{1}{2}} \subseteq \mathcal{R}(P)^\perp$, then

$$\mathcal{R}(Q_2^{\frac{1}{2}}) = \mathcal{R}(P)^\perp.$$

By Lemma 4 again we have

$$\mathcal{R}(Q_2) = \mathcal{R}(P)^\perp.$$

This shows that Q_2 is invertible. In this case, by Lemma 3, $\lambda_1 P + \lambda_2 Q$ is invertible if and only if

$$\lambda_1 I + \lambda_2 Q_1 - (\lambda_1 P_1 Q_2^{-\frac{1}{2}} + \lambda_2 Q_1^{\frac{1}{2}} D) D^* Q_1^{\frac{1}{2}}$$

is invertible.

Since Q_1 is a positive contraction on $\mathcal{R}(P)$ and Q_2 is an invertible positive contraction on $\mathcal{R}(P)^\perp$, then Q_1 as an operator on $\mathcal{R}(P)$ and Q_2 as an operator on $\mathcal{R}(P)^\perp$ have the following operator matrix forms:

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q_{11} \end{pmatrix}, Q_2 = \begin{pmatrix} Q_{22} & 0 \\ 0 & I \end{pmatrix}$$

with respect to the space decomposition

$$\mathcal{R}(P) = \mathcal{N}(Q_1) \oplus \mathcal{N}(I - Q_1) \oplus (\mathcal{R}(P) \ominus (\mathcal{N}(Q_1) \oplus \mathcal{N}(I - Q_1)))$$

and the space decomposition

$$\mathcal{R}(P)^\perp = (\mathcal{R}(P)^\perp \ominus \mathcal{N}(I - Q_2)) \oplus \mathcal{N}(I - Q_2),$$

respectively.

Then denote $\mathcal{H}_0 = \mathcal{N}(Q_1)$, $\mathcal{H}_1 = \mathcal{N}(I - Q_1)$, $\mathcal{H}_2 = \mathcal{R}(P) \ominus (\mathcal{N}(Q_1) \oplus \mathcal{N}(I - Q_1))$, $\mathcal{H}_3 = \mathcal{R}(P)^\perp \ominus \mathcal{N}(I - Q_2)$ and $\mathcal{H}_4 = \mathcal{N}(I - Q_2)$, Q_{11} and $I - Q_{11}$ are injective positive contractions on $\mathcal{H}_2$ and Q_{22} is an invertible positive contraction on $\mathcal{H}_3$ with $1 \notin \sigma_p(Q_{22})$, where $\sigma_p(M)$ denotes the point spectrum of the operator M. In this case, P and Q have the following matrix representations:

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} & 0 \\ 0 & 0 & Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \tag{1}$$

and

$$P = \begin{pmatrix} I & 0 & 0 & P_{11} & P_{12} \\ 0 & I & 0 & P_{21} & P_{22} \\ 0 & 0 & I & P_{31} & P_{32} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

with respect to the space decomposition $\mathcal{H} = \bigoplus_{i=0}^4 \mathcal{H}_i$. If we let

$$Q_0 = \begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix},$$

then Q being an orthogonal projection implies that Q_0 is also an orthogonal projection on $\mathcal{H}_2 \oplus \mathcal{H}_3$. That is, $Q_0 = Q_0^2$. Hence we have

$$\begin{aligned} Q_0^2 &= \begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix} \\ &= \begin{pmatrix} Q_{11}^2 + Q_{11}^{\frac{1}{2}} D_1 Q_{22} D_1^* Q_{11}^{\frac{1}{2}} & Q_{11}^{\frac{3}{2}} D_1 Q_{22}^{\frac{1}{2}} + Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{3}{2}} \\ Q_{22}^{\frac{3}{2}} D_1^* Q_{11}^{\frac{1}{2}} + Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{3}{2}} & Q_{22}^2 + Q_{22}^{\frac{1}{2}} D_1^* Q_{11} D_1 Q_{22}^{\frac{1}{2}} \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix}$$
$$= \begin{pmatrix} Q_{11}^2 + Q_{11}^{\frac{1}{2}} D_1 Q_{22} D_1^* Q_{11}^{\frac{1}{2}} & Q_{11}^{\frac{3}{2}} D_1 Q_{22}^{\frac{1}{2}} + Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{3}{2}} \\ Q_{22}^{\frac{3}{2}} D_1^* Q_{11}^{\frac{1}{2}} + Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{3}{2}} & Q_{22}^2 + Q_{22}^{\frac{1}{2}} D_1^* Q_{11} D_1 Q_{22}^{\frac{1}{2}} \end{pmatrix}.$$

Comparing the two sides of the above equation, we have

$$\begin{cases} Q_{11} = Q_{11}^2 + Q_{11}^{\frac{1}{2}} D_1 Q_{22} D_1^* Q_{11}^{\frac{1}{2}}, \\ Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} = Q_{11}^{\frac{3}{2}} D_1 Q_{22}^{\frac{1}{2}} + Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{3}{2}}, \\ Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} = Q_{22}^{\frac{3}{2}} D_1^* Q_{11}^{\frac{1}{2}} + Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{3}{2}}, \\ Q_{22} = Q_{22}^2 + Q_{22}^{\frac{1}{2}} D_1^* Q_{11} D_1 Q_{22}^{\frac{1}{2}}. \end{cases}$$

Observing that Q_{11}, $I_{\mathcal{H}_2} - Q_{11}$, Q_{22} and $I_{\mathcal{H}_3} - Q_{22}$ are injective, we get

$$\begin{cases} I_{\mathcal{H}_2} = Q_{11} + D_1 Q_{22} D_1^*, \\ D_1 = Q_{11} D_1 + D_1 Q_{22}, \\ D_1^* = Q_{22} D_1^* + D_1^* Q_{11}, \\ I_{\mathcal{H}_3} = Q_{22} + D_1^* Q_{11} D_1, \end{cases} \tag{3}$$

where $I_{\mathcal{H}_i}$ denotes the identity on $\mathcal{H}_i$, $i = 2, 3$. From the last of equations (3), we see that $I_{\mathcal{H}_3} - Q_{22} = D_1^* Q_{11} D_1$, and hence $Q_{11} D_1$ is injective. From the second of equations (3) we see that

$$(I_{\mathcal{H}_2} - Q_{11}) D_1 = D_1 Q_{22}.$$

On the other hand, from the first of equations (3) we get

$$(I_{\mathcal{H}_2} - Q_{11}) D_1 = D_1 Q_{22} D_1^* D_1.$$

Equating these two expressions for $(I_{\mathcal{H}_2} - Q_{11}) D_1$ and using the injectivity of $D_1 Q_{22}$ then leaves us with

$$I_{\mathcal{H}_3} = D_1^* D_1.$$

In a similar vein, from the second of equations (3) we have $D_1(I_{\mathcal{H}_3} - Q_{22}) = Q_{11} D_1$, while from the last of equations (3) we have $D_1(I_{\mathcal{H}_3} - Q_{22}) = D_1 D_1^* Q_{11} D_1$. Equating these two expressions for $D_1(I_{\mathcal{H}_3} - Q_{22})$ and using the injectivity of $Q_{11} D_1$ then gives

$$I_{\mathcal{H}_2} = D_1 D_1^*.$$

With this identity, the first of equations (3) can then be rewritten as

$$Q_{11} = D_1 (I_{\mathcal{H}_3} - Q_{22}) D_1^*.$$

We have thus arrived at the system of equations

$$\begin{cases} D_1 D_1^* = I_{\mathcal{H}_2}, \\ D_1^* D_1 = I_{\mathcal{H}_3}, \\ Q_{11} = D_1 (I_{\mathcal{H}_3} - Q_{22}) D_1^*. \end{cases}$$

Denote $\hat{Q} = Q_{11}$. Then

$$Q_0 = \begin{pmatrix} \hat{Q} & \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{\frac{1}{2}} D_1 \\ D_1^* \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{\frac{1}{2}} & D_1^* (I - \hat{Q}) D_1 \end{pmatrix}.$$

Now $\lambda_1 P + \lambda_2 Q$ has the following operator matrix form:

$$\lambda_1 P + \lambda_2 Q = \lambda_1 \begin{pmatrix} I & 0 & 0 & P_{11} & P_{12} \\ 0 & I & 0 & P_{21} & P_{22} \\ 0 & 0 & I & P_{31} & P_{32} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$+ \lambda_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} & 0 \\ 0 & 0 & Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1 I & 0 & 0 & \lambda_1 P_{11} & \lambda_1 P_{12} \\ 0 & (\lambda_1+\lambda_2) I & 0 & \lambda_1 P_{21} & \lambda_1 P_{22} \\ 0 & 0 & \lambda_1 I + \lambda_2 \hat{Q} & \lambda_1 P_{31} + \lambda_2 \hat{Q}^{\frac{1}{2}} (I-\hat{Q})^{\frac{1}{2}} D_1 & \lambda_1 P_{32} \\ 0 & 0 & \lambda_2 D_1^* \hat{Q}^{\frac{1}{2}} (I-\hat{Q})^{\frac{1}{2}} & \lambda_2 D_1^* (I-\hat{Q})^{\frac{1}{2}} D_1 & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = \bigoplus_{i=0}^{4} \mathcal{H}_i$.

Denote

$$\bar{Q}_0 = \begin{pmatrix} \lambda_1 I + \lambda_2 \hat{Q} & \lambda_1 P_{31} + \lambda_2 \hat{Q}^{\frac{1}{2}} (I-\hat{Q})^{\frac{1}{2}} D_1 \\ \lambda_2 D_1^* \hat{Q}^{\frac{1}{2}} (I-\hat{Q})^{\frac{1}{2}} & \lambda_2 D_1^* (I-\hat{Q}) D_1 \end{pmatrix}.$$

Obviously, if $\lambda_1 + \lambda_2 \neq 0$, the invertibility of the operator $\lambda_1 P + \lambda_2 Q$ on $\mathcal{H}$ is equivalent to the invertibility of $\bar{Q}_0$ on $\mathcal{H}_2 \oplus \mathcal{H}_3$. Moreover, by Lemma 3, that $\bar{Q}_0$ is invertible if and only if $I - \hat{Q}$ and

$$\lambda_1 I + \lambda_2 \hat{Q} - (\lambda_1 P_{31} + \lambda_2 \hat{Q}^{\frac{1}{2}} (I-\hat{Q})^{\frac{1}{2}} D_1) \lambda_2^{-1} D_1^* (I-\hat{Q})^{-1} D_1 \lambda_2 D_1^* \hat{Q}^{\frac{1}{2}} (I-\hat{Q})^{\frac{1}{2}}$$

are invertible. Noting that $D_1 D_1^* = I_{\mathcal{H}_2}$ and $\lambda_1 \lambda_2 \neq 0$, we have

$$\begin{aligned} &\lambda_1 I + \lambda_2 \hat{Q} - (\lambda_1 P_{31} + \lambda_2 \hat{Q}^{\frac{1}{2}} (I-\hat{Q})^{\frac{1}{2}} D_1) \lambda_2^{-1} D_1^* (I-\hat{Q})^{-1} D_1 \lambda_2 D_1^* \hat{Q}^{\frac{1}{2}} (I-\hat{Q})^{\frac{1}{2}} \\ &= \lambda_1 (I - P_{31} D_1^* \hat{Q}^{\frac{1}{2}} (I-\hat{Q})^{-\frac{1}{2}}). \end{aligned}$$

This shows that the invertibility of $\bar{Q}_0$ is only dependent on the invertibility of the operator $I - P_{31} D_1^* \hat{Q}^{\frac{1}{2}} (I-\hat{Q})^{-\frac{1}{2}}$ if both λ_1 and λ_2 are not zero; that is, the invertibility of $\bar{Q}_0$ is independent of the choice of λ_1 and λ_2 if both λ_1 and λ_2 are not zero.

In other words, the invertibility of $\lambda_1 P + \lambda_2 Q$ is independent of the choice of λ_1 and λ_2 if both λ_1 and λ_2 are not zero and $\lambda_1 + \lambda_2 \neq 0$. □

Remark. By the proof of Theorem 1, if $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$, the invertibility of $\lambda_1 P + \lambda_2 Q$ is independent of the choice of λ_1 and λ_2 if both λ_1 and λ_2 are not zero.

The following consequence is immediate.

Corollary 5. *Let P and Q be two idempotents and $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$. Then the following statements hold:*

(1) *$P + Q$ is invertible if and only if $P - Q$ is invertible;*

(2) *In addition, $\mathcal{R}(\tilde{P}) \cap \mathcal{R}(\tilde{Q}) = \{0\}$, and $P + Q$ is invertible if and only if $\tilde{P} + \tilde{Q}$ is invertible, where $\tilde{K} = I - K$.*

Proof. (1) It is clear from the Remark above.

(2) Obviously, $\tilde{P}$ and $\tilde{Q}$ are idempotents. Since $\mathcal{R}(\tilde{P}) \cap \mathcal{R}(\tilde{Q}) = \{0\}$, $\tilde{P} + \tilde{Q}$ is invertible if and only if $\tilde{P} - \tilde{Q}$ is invertible by (1). But

$$\tilde{P} - \tilde{Q} = I - P - (I - Q) = -(P - Q),$$

so $\tilde{P} - \tilde{Q}$ is invertible if and only if $P - Q$ is invertible. By (1), $P - Q$ is invertible if and only if $P + Q$ is invertible. □

Corollary 6. *Let P and Q be two orthogonal projections such that $P+Q$ is invertible and $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$. Then $P - Q, 1 - PQ, P + Q - PQ$ are all invertible.*

Proof. If $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$, then $\mathcal{H}_1 = \{0\}$. Observe that in the proof of Theorem 1, $P + Q$ has the following operator matrix:

$$\begin{aligned} P + Q &= \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} & 0 \\ 0 & Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I + Q_{11} & Q_{11}^{\frac{1}{2}} (I - Q_{11})^{\frac{1}{2}} D_1 & 0 \\ 0 & D_1^* Q_{11}^{\frac{1}{2}} (I - Q_{11})^{\frac{1}{2}} & D_1^* (I - Q_{11})^{\frac{1}{2}} D_1 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \end{aligned}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{H}_0 \oplus (\bigoplus_{i=2}^4 \mathcal{H}_i)$. Then the invertibility of $P+Q$ implies that $I - Q_{11}$ is invertible. A direct calculation can show that $1 - PQ$ and $P + Q - PQ$ are all invertible. □

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