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HARDY'S THEOREM AND ROTATIONS

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ABSTRACT. We prove an extension of Hardy's classical characterization of real Gaussians of the form $e^{-\pi\alpha x^2}$, $\alpha > 0$, to the case of complex Gaussians in which α is a complex number with positive real part. Such functions represent rotations in the complex plane of real Gaussians. A condition on the rate of decay of analytic extensions of a function f and its Fourier transform $\widehat{f}$ along some pair of lines in the complex plane is shown to imply that f is a complex Gaussian.

1. Hardy's theorem and Fourier uncertainty

A Fourier uncertainty principle is, generally speaking, a statement that limits the rate at which a function and its Fourier transform can decay, or otherwise restricts the "joint localization" of a Fourier pair.

We normalize the Fourier transform $\widehat{f}$ of f by setting $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\, d\xi$ when $f \in L^1(\mathbb{R})$. Hardy's theorem is an uncertainty principle stating that if $|f(x)| \leq Ce^{-\pi\alpha x^2}$ and $|\widehat{f}(\xi)| \leq C'e^{-\pi\beta\xi^2}$, then

(i) if $\alpha\beta > 1$, then $f = 0$, while
(ii) if $\alpha\beta = 1$, then f is a multiple of $e^{-\pi\alpha x^2}$.

Hardy's theorem has been extended in several different directions in recent years, including extensions to Euclidean space (*e.g.*, [SST], [THA]) and, much more generally, to groups of homogeneous type (*e.g.*, [ACDS]) and semisimple Lie groups (*e.g.*, [SS], [CSS]; *cf.* also [S]), and to statements about the decay of time-frequency distributions on phase space (*e.g.*, [GZ], [BDJ], [GR], [HL]). Other important directions include generalizations of Beurling's important variation of Hardy's theorem ([H], [BDJ], [CP], [BR]) and statements about the decay of eigenvalues of Hermite expansions and related operators (*e.g.*, [JAVE], [HL2]). The insightful survey [FS] discusses several of these developments in the context of Fourier uncertainty principles.

Nearly all of the extensions just noted are proved by reduction to Hardy's original theorem (or to Beurling's theorem). In any case, they all reduce to some form of the maximum principle. Given all these directions of generalization, it is interesting to ask whether any new light can be shed on the base case of localization of analytic functions and their Fourier transforms.

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Many years ago, Gelfand and Shilov [GS] proved mapping properties under the Fourier transform of analytic functions satisfying certain growth/decay conditions along $\mathbb{R}$ and $i\mathbb{R}$. As Hörmander [H] pointed out, the Gelfand-Shilov estimates show that Beurling's theorem and other variations and extensions of Hardy's theorem are, in a sense, sharp (*cf.* [BDJ]). This raises a natural question that seems not to have been addressed before: what can be said about joint localization of Fourier pairs f and $\widehat{f}$ when localization is phrased in terms of decay along arbitrary directions in the complex plane?

2. Hardy's theorem and rotations

We answer the question just asked in the form of the following extension of Hardy's theorem. Namely, we extend Hardy's characterization of 'real' Gaussians as optimizers of joint decay of f and $\widehat{f}$ to the case of Gaussians with complex factors, that is, of the form $e^{\pi i\zeta x^2}$, $\zeta = \beta + i\alpha$ in which $\alpha > 0$. Such functions are restrictions to the real axis of rotations in the complex plane of analytic extensions of real Gaussians of the form $e^{-\gamma x^2}$, $\gamma > 0$.

In order to bring rotations of $\mathbb{C}$ to bear on this localization problem, one presumes that f can be defined on $\mathbb{C}$. We do so by setting $\mathcal{F}(f)(\zeta) = \widehat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\zeta}\,dx$ and $f(z) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i z\xi}\,d\xi$, at least for any $\zeta, z \in \mathbb{C}$ for which the corresponding integrals exist. The extension of Hardy's theorem that we have in mind then takes the following form.

Theorem 2.1. *Suppose that $f \in L^1(\mathbb{R})$ and for some $\psi_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the integral*

$$\widehat{f}(se^{i\psi_0}) = \int_{-\infty}^{\infty} f(t)\exp(-2\pi i t s e^{i\psi_0})\,dt$$

converges for all real s and satisfies the bound $|\widehat{f}(se^{i\psi_0})| \le ce^{-\pi s^2/\alpha}$ for some positive constants c and α. Then f has an analytic extension to the complex plane. Suppose, in addition, that for some $\theta_0 \in \mathbb{R}$ the extension of f satisfies the bound $|f(re^{i\theta_0})| \le Ce^{-\pi\alpha r^2}$ for some $C > 0$ and all $r \in \mathbb{R}$, where α is as above. Then f is a rotation of a multiple of $e^{-\pi\alpha x^2}$ through the angle $-\theta_0$ in the plane. That is, $f(z) = Ce^{-\pi\alpha e^{-2i\theta_0}z^2}$. Moreover, in this case we have $\theta_0 \equiv -\psi_0 \mod \pi$ and $|\psi_0| < \pi/4$.

This result generalizes Hardy's theorem in two ways. First, there is no *a priori* relationship between the inverse Fourier transform of $\widehat{f}$ and that of a complex rotation of its analytic extension. Establishing such a relationship under the condition on f is the heart of the proof. It requires a certain regularization argument. This argument invokes the Phragmen-Lindelöf theorem, which is where the condition $\psi_0 \neq \pm\pi/2$ comes into play. Secondly, no prior relationship between θ_0 and ψ_0 is assumed. The relationship $\theta_0 \equiv -\psi_0 \mod \pi$ follows once f is determined to be Gaussian. The conclusion that $|\psi_0| < \pi/4$ is forced by the hypothesis that $f \in L^1$. That is, in order to have $|f(x)| = Ce^{-\pi\alpha\cos(2\theta_0)x^2} \in L^1$ one should take $|\theta_0| < \pi/4$.

There are several possible generalizations of Theorem 2.1, particularly along the lines of conjugate growth conditions on f and $\widehat{f}$ à la consequences of Beurling's theorem (*cf.* [H]) and Gelfand-Shilov spaces. Some investigation along these lines was initiated in [HL2].

3. Hardy's theorem for complex Gaussians: Proof of Theorem 2.1

Being essentially a result in complex variables, Hardy's theorem relies on the fact that the functions in question have analytic extensions. The first step for us is to ensure the same type of behavior here.

Lemma 3.1. *Let $f \in L^1(\mathbb{R})$ and define $\widehat{f}(\zeta) = \int_{-\infty}^{\infty} f(t)e^{-2\pi it\zeta}\,dt$ ($\zeta \in \mathbb{C}$) wherever the integral exists. If $\psi_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is such that $\widehat{f}(re^{i\psi_0})$ is defined for all $r \in \mathbb{R}$ and satisfies $|\widehat{f}(re^{i\psi_0})| \leq Ce^{-\pi r^2/\alpha}$, then f has an analytic extension to the complex plane.*

Proof. The proof of the lemma relies on a certain double regularization of f. We will only argue the case $0 \leq \psi_0 < \pi/2$. The case $-\pi/2 < \psi_0 \leq 0$ is similar. Fix $\phi \in (0, \frac{\pi}{2} - \psi_0)$. For $\epsilon > 0$, define $G_{\epsilon,\phi}(z) = \sqrt{\epsilon}\exp(-\pi i\epsilon e^{-2i\phi}z^2)$ ($z \in \mathbb{C}$). Then

$$|G_{\epsilon,\phi}(re^{i\theta})| = \sqrt{\epsilon}\, e^{-\pi\epsilon\sin 2(\phi-\theta)r^2}$$

and, as a function of r, $G_{\epsilon,\phi}(re^{i\theta})$ decays if and only if $\phi - \frac{\pi}{2} < \theta < \phi$. On the Fourier side we have $\widehat{G_{\epsilon,\phi}}(\zeta) = e^{i\phi}e^{-i\pi/4}\exp(\pi ie^{2i\phi}\zeta^2/\epsilon)$ so that

$$|\widehat{G_{\epsilon,\phi}}(re^{i\theta})| = e^{-\pi r^2\sin 2(\theta+\phi)/\epsilon}, \tag{3.1}$$

which vanishes at infinity if and only if $-\phi < \theta < \frac{\pi}{2} - \phi$. Thus both $G_{\epsilon,\phi}$ and $\widehat{G_{\epsilon,\phi}}$ decay at infinity along $\mathbb{R}$. Now with ϕ fixed as above and $\epsilon > 0$, we define the double regularization $f_{\epsilon,\phi}$ of $f \in L^1(\mathbb{R})$ by

$$f_{\epsilon,\phi}(t) = \frac{1}{\sqrt{\epsilon}}\, e^{-i\phi}e^{\pi i/4}(fG_{\epsilon,\phi}) * G_{1/\epsilon,\phi}(t).$$

Then $f_{\epsilon,\phi} \in L^1(\mathbb{R})$, while

$$\widehat{f_{\epsilon,\phi}}(\zeta) = \frac{1}{\sqrt{\epsilon}}e^{-i\phi}e^{\pi i/4}(fG_{\epsilon,\phi})^\wedge(\zeta)(G_{1/\epsilon,\phi})^\wedge(\zeta) \qquad (\zeta \in \mathbb{C}) \tag{3.2}$$

defines an entire function since $0 < \phi < \pi/2$. From (3.2), the fact that $G_{\epsilon,\phi} \to 1$ locally uniformly on $\mathbb{R}$, and the observation that $\widehat{G_{1/\epsilon,\phi}}(0) = e^{i\phi}e^{-i\pi/4}$, we see that $\widehat{f_{\epsilon,\phi}} \to \widehat{f}$ uniformly on the real line as $\epsilon \to 0$. Consequently, $f_{\epsilon,\phi} \to f$ in the sense of tempered distributions, as $\epsilon \to 0$, whenever $f \in L^1(\mathbb{R})$. For $\zeta = re^{i\theta}$,

$$\begin{aligned}
|\widehat{f_{\epsilon,\phi}}(\zeta)| &\leq \frac{1}{\sqrt{\epsilon}}\left|(fG_{\epsilon,\phi})^\wedge(re^{i\theta})\right|\left|(G_{1/\epsilon,\phi})^\wedge(re^{i\theta})\right| \\
&= \frac{1}{\sqrt{\epsilon}}\left|\int_{-\infty}^{\infty} f(x)G_{\epsilon,\phi}(x)\exp(-2\pi ixre^{i\theta})\,dx\right| e^{-\pi\epsilon r^2\sin 2(\theta+\phi)} \\
&\leq \left(\int_{-\infty}^{\infty} |f(x)|e^{-\pi\epsilon x^2\sin 2\phi}e^{2\pi xr\sin\theta}\,dx\right) e^{-\pi\epsilon r^2\sin 2(\theta+\phi)} \\
&= e^{-\pi\epsilon r^2\sin 2(\theta+\phi)}e^{\pi r^2\sin^2\theta/(\epsilon\sin 2\phi)} \\
&\qquad\qquad \times\int_{-\infty}^{\infty}|f(x)|e^{-\pi\epsilon\sin 2\phi(x-r\sin\theta/(\epsilon\sin 2\phi))^2}\,dx \\
&\leq \|f\|_1\exp\left(\pi r^2\left[\frac{\sin^2\theta}{\epsilon\sin 2\phi} - \epsilon\sin 2(\theta+\phi)\right]\right). \qquad (3.3)
\end{aligned}$$

Restricting to $\zeta = \xi \in \mathbb{R}$ one then has from (3.3) with $r = |\xi|$ and $\theta = 0$,

$$|\widehat{f_{\epsilon,\phi}}(\xi)| \leq \|f\|_1 e^{-\pi\epsilon\xi^2\sin 2\phi}. \tag{3.4}$$

Also, for $\zeta = re^{i\psi_0}$, because of the Gaussian decay of $(G_{\epsilon,\phi})^\wedge$ on the sectors between the lines $\theta = 0$ and $\theta = \psi_0$,

$$\begin{aligned}
(fG_{\epsilon,\phi})^\wedge(re^{i\psi_0}) &= \int_{-\infty}^{\infty} f(x)G_{\epsilon,\phi}(x)\exp(-2\pi ixre^{i\psi_0})\,dx \\
&= e^{i\psi_0}\int_{-\infty}^{\infty} f(x)\exp(-2\pi ixre^{i\psi_0}) \\
&\qquad \times \int_{-\infty}^{\infty} (G_{\epsilon,\phi})^\wedge(se^{i\psi_0})\exp(2\pi ixse^{i\psi_0})\,ds\,dx \\
&= e^{i\psi_0}\int_{-\infty}^{\infty} (G_{\epsilon,\phi})^\wedge(se^{i\psi_0})\int_{-\infty}^{\infty} f(x)\exp(2\pi ix(s-r)e^{i\psi_0})\,dx\,ds \\
&= e^{i\psi_0}\int_{-\infty}^{\infty} (G_{\epsilon,\phi})^\wedge(se^{i\psi_0})\widehat{f}((r-s)e^{i\psi_0})\,ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
|(fG_{\epsilon,\phi})^\wedge(re^{i\psi_0})| &\leq \int_{-\infty}^{\infty} |(G_{\epsilon,\phi})^\wedge(se^{i\psi_0})||\widehat{f}((r-s)e^{i\psi_0})|\,ds \\
&\leq C\int_{-\infty}^{\infty} e^{-\pi s^2\sin 2(\psi_0+\phi)/\epsilon}e^{-\pi(r-s)^2/\alpha}\,ds \\
&= Ce^{-\pi r^2/\alpha}\exp\left(\frac{\pi r^2}{1+\alpha\sin 2(\psi_0+\phi)/\epsilon)}\right) \\
&\qquad \times \int_{-\infty}^{\infty}\exp\left(-\frac{\pi}{\alpha}(1+\alpha\sin 2(\psi_0+\phi)/\epsilon)s^2\right)ds \\
&= C\frac{\sqrt{\alpha\epsilon}}{\sqrt{\epsilon+\alpha\sin 2(\psi_0+\phi)}}\exp\left(\frac{-\pi r^2\sin 2(\psi_0+\phi)}{\epsilon+\alpha\sin 2(\psi_0+\phi)}\right).
\end{aligned}$$

So by (3.2) and (3.1), on the line $\theta = \psi_0$ we have the estimate

$$\begin{aligned}
|(f_{\epsilon,\phi})^\wedge(re^{i\psi_0})| &\leq C\sqrt{\alpha}\exp\left(\frac{-\pi r^2\sin 2(\psi_0+\phi)}{\epsilon+\alpha\sin 2(\psi_0+\phi)}\right)\frac{\exp\left(-\pi r^2\epsilon\sin 2(\psi_0+\phi)\right)}{\sqrt{\epsilon+\alpha\sin 2(\psi_0+\phi)}} \\
&= C\frac{\sqrt{\alpha}\exp\left(-\pi r^2\sin 2(\psi_0+\phi)\left(\epsilon+\frac{1}{\epsilon+\alpha\sin 2(\psi_0+\phi)}\right)\right)}{\sqrt{\epsilon+\alpha\sin 2(\psi_0+\phi)}}. \qquad (3.5)
\end{aligned}$$

This decay of $\widehat{f_{\epsilon,\phi}}$ along the line $\zeta = re^{i\psi_0}$ as in (3.5) is slightly worse than that assumed of f. But this loss is compensated by the gain in decay of $\widehat{f_{\epsilon,\phi}}(\xi)$ in (3.4): the assumption that $f \in L^1(\mathbb{R})$ only gives $|\widehat{f}(\xi)| \leq \|f\|_1 < \infty$ on the real axis.

The decay estimates on $\widehat{f_{\epsilon,\phi}}$ allow us to apply the Phragmen-Lindelöf theorem [SW] to the function

$$h_{\epsilon,\phi}(\zeta) = \widehat{f_{\epsilon,\phi}}(\zeta)\exp(-i\pi\epsilon e^{2i\phi}\zeta^2)$$

on the sectors between the lines $\theta = 0$ and $\theta = \psi_0$. First, from (3.3) we have

$$\begin{aligned} |h_{\epsilon,\phi}(\zeta)| &= |(f_{\epsilon,\phi})^\wedge(\zeta)| e^{\pi\epsilon r^2 \sin 2(\theta+\phi)} \\ &\leq \|f\|_1 \exp\left(\pi r^2 \left(\frac{\sin^2\theta}{\epsilon \sin 2\phi} - \epsilon \sin 2(\theta+\phi)\right)\right) e^{\pi\epsilon r^2 \sin 2(\theta+\phi)} \\ &= \|f\|_1 \exp\left(\frac{\pi r^2 \sin^2\theta}{\epsilon \sin 2\phi}\right). \end{aligned} \tag{3.6}$$

In particular, taking $\theta = 0$ we have from (3.6),

$$|h_{\epsilon,\phi}(\xi)| \leq \|f\|_1 \qquad (\xi \in \mathbb{R}), \tag{3.7}$$

while (3.5) gives

$$\begin{aligned} |h_{\epsilon,\phi}(re^{i\psi_0})| &\leq C\frac{\sqrt{\alpha}}{\sqrt{\epsilon + \alpha \sin 2(\psi_0+\phi)}} \exp\left(\frac{-\pi r^2 \sin 2(\psi_0+\phi)}{\epsilon + \alpha \sin 2(\psi_0+\phi)}\right) \\ &\leq C\frac{\sqrt{\alpha}}{\sqrt{\epsilon + \alpha \sin 2(\psi_0+\phi)}}. \end{aligned} \tag{3.8}$$

Given (3.6), (3.7) and (3.8), the Phragmen-Lindelöf theorem implies that $|h_{\epsilon,\phi}(\zeta)| \leq C$ on the sectors between the lines $\theta = 0$ and $\theta = \psi_0$. It follows that

$$|\widehat{f_{\epsilon,\phi}}(re^{i\theta})| \leq Ce^{-\pi\epsilon r^2 \sin 2(\theta+\phi)} \qquad (0 \leq \theta \leq \psi_0).$$

Consider the integral $I = \int_{-\infty}^{\infty} \widehat{f_{\epsilon,\phi}}(\xi) e^{2\pi i \xi w}\, d\xi$ which defines $f_{\epsilon,\phi}$ at $w \in \mathbb{C}$. Thinking of I as a contour integral over the real line in the complex plane, we wish to show that I equals the contour integral $J = \int_{-\infty}^{\infty} \widehat{f_{\epsilon,\phi}}(re^{i\psi_0}) \exp(2\pi i r e^{i\psi_0} w) e^{i\psi_0}\, dr$ over the line $re^{i\psi_0}$ $(-\infty < r < \infty)$. This may be achieved by applying Cauchy's theorem on the boundaries of the sectors

$$S_1 = \{\zeta = re^{i\theta} \in \mathbb{C};\ 0 \leq r \leq R,\ 0 \leq \theta \leq \psi_0\}, \quad S_2 = -S_1$$

and showing (in the case of S_1) that if Γ_R is the contour parameterized by $\gamma_R(\theta) = R\,e^{i\theta}$ $(0 \leq \theta \leq \psi_0)$, then $K_R = \int_{\Gamma_R} \widehat{f_{\epsilon,\phi}}(\zeta) e^{2\pi i \zeta w}\, d\zeta \to 0$ as $R \to \infty$ for all w. If $w = se^{i\gamma}$, then

$$\begin{aligned} |K_R| &= \left| \int_0^{\psi_0} \widehat{f_{\epsilon,\phi}}(Re^{i\theta}) \exp(2\pi i R e^{i\theta} s e^{i\gamma}) iRe^{i\theta}\, d\theta \right| \\ &\leq CR \int_0^{\psi_0} e^{-\pi\epsilon R^2 \sin 2(\theta+\phi)} e^{-2\pi s R \sin(\gamma+\phi)}\, d\theta. \end{aligned}$$

But on the range $0 \leq \theta \leq \psi_0$ we have

$$\sin 2(\theta+\phi) \geq \min\{\sin 2\phi, \sin 2(\psi_0+\phi)\} = C(\psi_0,\phi) > 0$$

and therefore

$$|K_R| \leq Re^{-\pi\epsilon R^2 C(\psi_0,\phi)} \int_0^{\psi_0} e^{-2\pi s R \sin(\gamma+\theta)}\, d\theta \leq Re^{-\pi\epsilon R^2 C(\psi_0,\phi)} e^{2\pi s R} \psi_0,$$

which approaches 0 as $R \to \infty$ for all $w = se^{i\gamma}$. Similarly, if Γ'_R is parameterized by $\gamma'_R(\theta) = Re^{i\theta}$ $(\pi \leq \theta \leq \pi + \psi_0)$, then $K'_R = \int_{\Gamma'_R} \widehat{f_{\epsilon,\phi}}(\zeta) e^{2\pi i \zeta w}\, d\zeta \to 0$ as $R \to \infty$ for all w. We conclude from Cauchy's theorem that for all $w \in \mathbb{C}$,

$$\int_{-\infty}^{\infty} \widehat{f_{\epsilon,\phi}}(\xi) e^{2\pi i \xi w}\, d\xi = e^{i\psi_0} \int_{-\infty}^{\infty} \widehat{f_{\epsilon,\phi}}(re^{i\psi_0}) e^{2\pi i r e^{i\psi_0} w}\, dr. \tag{3.9}$$

This allows us to define $f_{\epsilon,\phi}$ on the complex plane as a Fourier integral along the line $re^{i\psi_0}$ $(-\infty < r < \infty)$. In view of (3.5), the collection $\{f_{\epsilon,\phi}\}_{\epsilon>0}$ is analytic and uniformly bounded on compact sets. By Montel's theorem (see [P]), $\{f_{\epsilon,\phi}\}_{\epsilon>0}$ has a subsequence converging to an analytic function on all of $\mathbb{C}$. Since $f_{\epsilon,\phi} \to f$ on $\mathbb{R}$ in the sense of tempered distributions as $\epsilon \to 0$, it follows that this analytic function is the analytic extension of f. Applying a similar argument for the case $-\pi/2 < \psi_0 \le 0$ completes the proof of the lemma. $\square$

The proof of Theorem 2.1 can now be completed by arguing along the same lines as in the standard proof of Hardy's theorem. By taking $\epsilon \to 0$ in the decay estimate (3.5) and applying (3.9), one concludes that the analytic extension of f must satisfy

$$|f(w)| = \left|\int_{-\infty}^{\infty} \widehat{f}(re^{i\psi_0}) \exp(2\pi i r e^{i\psi_0} w) e^{i\psi_0}\, dr\right| \le C_\alpha \exp(\pi\alpha(\mathrm{Im}\,(e^{i\psi_0} w))^2). \tag{3.10}$$

Suppose first that f is even, *i.e.*, $f(z) = \sum_n c_n z^{2n}$. Let $h(z) = f(\sqrt{z}) = \sum_n c_n z^n$. Then by (3.10),

$$|h(re^{i\theta})| \le C_\alpha \exp\left(\pi\alpha(\mathrm{Im}\,(e^{i\psi_0}\sqrt{r}e^{i\theta/2}))^2\right) = C_\alpha e^{\pi\alpha r \sin^2(\psi_0+\theta/2)}, \tag{3.11}$$

i.e., h is of exponential type. On the line $\theta = 2\theta_0$ we have by the decay assumption on $f(re^{i\theta_0})$,

$$|h(re^{2i\theta_0})| = |f(\sqrt{r}e^{i\theta_0})| \le Ce^{-\pi\alpha r}. \tag{3.12}$$

Let $0 < \delta < \pi$ and $H_u(\zeta) = \exp\left(\dfrac{i\alpha\pi\zeta e^{i(-2\theta_0-\delta/2)}}{\sin\delta/2}\right) h(\zeta)$. Then by (3.11),

$$\begin{aligned} |H_u(re^{i\theta})| &= \left|\exp\left(\frac{i\alpha\pi r e^{i(\theta-2\theta_0-\delta/2)}}{\sin\delta/2}\right)\right| |h(re^{i\theta})| \\ &= C_\alpha \exp\left(\pi\alpha r\left(\sin^2(\psi_0+\theta/2) - \frac{\sin(\theta-2\theta_0-\delta/2)}{\sin\delta/2}\right)\right). \end{aligned} \tag{3.13}$$

On the line $\theta = 2\theta_0$ we have by (3.12) the estimate

$$|H_u(re^{2i\theta_0})| \le C_\alpha \exp\left(\frac{-\pi\alpha r\sin(-\delta/2)}{\sin\delta/2}\right)\exp(-\pi\alpha r) = C_\alpha.$$

Also, on the line $\theta = 2\theta_0 + \delta$ we have

$$|H_u(re^{i(2\theta_0+\delta)})| \le C_\alpha e^{-\pi\alpha r\cos^2(\psi_0+\theta/2)} \le C_\alpha.$$

Now we apply the Phragmen-Lindelöf theorem on the sector $2\theta_0 \le \theta \le 2\theta_0 + \delta$ to obtain $|H_u(re^{i\theta})| \le C$, *i.e.*,

$$|h(re^{i\theta})| \le C\exp\left(\frac{\pi\alpha r\sin(\theta-2\theta_0-\delta/2)}{\sin\delta/2}\right)$$

for $2\theta_0 \le \theta \le 2\theta_0 + \delta$. Letting $\delta \to \pi^-$ gives

$$|h(re^{i\theta})| \le C\exp\left(\frac{\pi\alpha r\sin(\theta-2\theta_0-\pi/2)}{\sin\pi/2}\right) = Ce^{-\pi\alpha r\cos(\theta-2\theta_0)}$$

on the half-plane $2\theta_0 \le \theta \le 2\theta_0 + \pi$. A similar analysis on the half-plane $2\theta_0 - \pi \le \theta \le 2\theta_0$ using the function $H_l(\zeta) = \exp\left(\frac{-\pi\alpha i\zeta e^{i(-2\theta_0+\delta/2)}}{\sin\delta/2}\right) h(\zeta)$ yields $|h(re^{i\theta})| \le$

$Ce^{-\pi\alpha r\cos(\theta-2\theta_0)}$ on this half-plane. So h satisfies this bound on the plane. Consider now the function

$$\widetilde{H}(z) = \exp\left(\pi\alpha z e^{-2i\theta_0}\right) h(z).$$

$\widetilde{H}$ is entire and the bound on h gives

$$\begin{aligned} |\widetilde{H}(re^{i\theta})| &= \left|\exp\left(\pi\alpha r e^{i(\theta-2\theta_0)}\right)\right| \left|h(re^{i\theta})\right| \\ &\le C\exp\left(\pi\alpha r\cos(\theta-2\theta_0)\right)\exp\left(-\pi\alpha r\cos(\theta-2\theta_0)\right) = C. \end{aligned}$$

By Liouville's theorem, $\widetilde{H}(z) = C$, *i.e.*, $h(z) = C\exp\left(-\pi\alpha z e^{-2i\theta_0}\right)$ and consequently $f(z) = h(z^2) = C\exp\left(-\pi\alpha z^2 e^{-2i\theta_0}\right)$. This completes the proof under the assumption that f is even.

If f is odd, then $z^{-1}f(z)$ is even and analytic (since $f(0) = 0$) and we may apply the above proof to get $f(z) = Cz\exp\left(-\pi\alpha z^2 e^{-2i\theta_0}\right)$. However, we find then that $f(re^{i\theta_0})$ does not satisfy the assumed bound unless $C = 0$. Finally, write $f = f_e + f_o$ with $f_e(z) = (f(z) + f(-z))/2$, $f_o(z) = (f(z) - f(-z))/2$. Since $(f_e)^\wedge = (\widehat{f})_e$ and $(f_o)^\wedge = (\widehat{f})_o$, we see that f_e and f_o satisfy the same bounds as f. Hence $f_o \equiv 0$, $f = f_e$ is even and we have completed the proof of Theorem 2.1. ☐

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