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A BOUND FOR RATIOS OF EIGENVALUES OF SCHRÖDINGER OPERATORS WITH SINGLE-WELL POTENTIALS

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Abstract. For Schrödinger operators with nonnegative single-well potentials ratios of eigenvalues are extremal only in the case of zero potential. To prove this, we investigate some monotonicity properties of Prüfer-type variables.

1. Introduction

Consider the Schrödinger operator

$$-y'' + q(x)y = \lambda y \tag{1.1}$$

on the interval $[0, \pi]$ with Dirichlet boundary conditions. If $q \in L_1(0, \pi)$ is real-valued, then the spectrum consists of a growing sequence of infinitely many points, λ_1, λ_2, …; see for example in [3]. Moreover, if $q(x)$ is nonnegative, $\lambda_n \geq n^2$ (as it is seen later, for example, from (2.7) and Lemma 2.1).

Ashbaugh and Benguria in [2] proved the bound

$$\frac{\lambda_n}{\lambda_1} \leq n^2 \tag{1.2}$$

for nonnegative potentials. They also examined the ratio of two arbitrary eigenvalues, and found

$$\frac{\lambda_n}{\lambda_m} \leq \left\lceil \frac{n}{m} \right\rceil^2 \tag{1.3}$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. To show that this estimate is optimal, they constructed multiple-well examples which came arbitrarily near to attain the bound. They formulated the conjecture that if the potential is nonnegative and convex, then

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}, \quad n \geq m, \tag{1.4}$$

holds. In this paper we prove more. Namely, we only need that the potential $q \geq 0$ be single-well. This means that there is a point $a \in [0, \pi]$ such that q is decreasing

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in $[0, a]$ and increasing in $[a, \pi]$ (see in [1]). Our proof relies on some monotonicity properties of the Prüfer-type variables φ and r from (2.3)–(2.4).

2. The main statement

Denote by $y(x, z)$ the unique solution of the initial value problem

$$-y'' + q(x)y = z^2 y, \quad x \in [0, \pi], \quad z > 0, \tag{2.1}$$

$$y(0) = 0, y'(0) = 1, \tag{2.2}$$

and let us introduce Prüfer-type variables:

$$y(x, z) = \frac{r(x, z)}{z} \sin \varphi(x, z), \tag{2.3}$$

$$y'(x, z) = r(x, z) \cos \varphi(x, z), \tag{2.4}$$

$$\varphi(0, z) = 0, \tag{2.5}$$

where $r(x, z) > 0$, and we denote by prime the derivative with respect to x (and by dot the derivative with respect to z). Define further

$$\psi = \frac{\varphi}{z}. \tag{2.6}$$

An easy computation shows that for these variables the following equations hold:

$$\varphi' = z - \frac{q}{z} \sin^2 \varphi, \tag{2.7}$$

$$\frac{r'}{r} = \frac{q}{z} \sin \varphi \cos \varphi. \tag{2.8}$$

Remark. These formulae hold in the usual sense at the continuity points of q and in both half-sided senses at the jumps of q: $\varphi'_{\pm}(x, z) = z - \frac{q(x \pm 0)}{z} \sin^2 \varphi(x, z)$, and analogously for r.

It is obvious that $y = 0$ iff $\sin \varphi = 0$, hence z^2 is an eigenvalue iff $\varphi(\pi, z)$ is a multiple of π. Denote by z_n the square root of λ_n.

Lemma 2.1. *$\varphi(\pi, z_n) = n\pi$.*

Proof. See equation (2.7) in Ashbaugh and Benguria [2]. □

Our idea is to show that (under certain conditions) $\psi(x, z)$ is a monotone increasing function in z, since this will imply (1.4).

Theorem 2.2. *Let $q(x) \geq 0$ be monotone decreasing in $[0, x_0]$. Then $\dot{\psi}(x_0, z) \geq 0$, i.e., $\psi(x_0, z)$ is a monotone increasing function in $z > 0$. If there is a $z > 0$ with $\dot{\psi}(x_0, z) = 0$, then $q = 0$ in $(0, x_0]$.*

This theorem implies various results for different boundary conditions. For example, we mention the following corollary.

Corollary 2.3. *Consider equation* (1.1) *with the Dirichlet-Neumann boundary conditions*

$$y(0) = y'(\pi) = 0. \tag{2.9}$$

If the potential q is nonnegative and decreasing, then for the m-th and n-th eigenvalues with $m \leq n$,

$$\frac{\lambda_n}{\lambda_m} \leq \frac{(2n-1)^2}{(2m-1)^2}, \tag{2.10}$$

and if for two different m and n equality holds, then $q = 0$ in $(0, \pi]$.

The proof will be given in Section 3.

The main statement of this paper reads as follows:

Theorem 2.4. *Consider equation* (1.1) *with the Dirichlet boundary conditions*

$$y(0) = y(\pi) = 0. \tag{2.11}$$

If the potential q is nonnegative and single-well, then for the m-th and n-th eigenvalues with $m \leq n$,

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}, \tag{2.12}$$

and if for two different m and n equality holds, then $q = 0$ in $(0, \pi)$.

The proof will be given in Section 4.

3. The proof of Theorem 2.2

Lemma 3.1. *If $q(x)$ is monotone decreasing in $[0, x_0]$, then (for $z > 0$) $\varphi(x_0, z)$ is a strictly monotone increasing function of x in $[0, x_0]$. Moreover, $\varphi'_\pm(x, z) > 0$ for $z > 0$.*

Proof. Fix z. From (2.7) if $\varphi'_\pm(\tilde{x}, z) \leq 0$, then $q(x) \geq z^2$ for $x < \tilde{x}$. Through $y'' = (q - z^2)y$, y is convex, positive and increasing, so $y' > 0$, and by that, $\varphi(x, z) < \frac{\pi}{2}$ for $x < \tilde{x}$. For small $x > 0$, $\varphi'_\pm(0, z) > 0$ (see (2.7)). The function $\varphi'_\pm(x, z)$ is continuous at the continuity points of $q(x)$, and (since $q(x)$ is monotone decreasing) cannot jump downward. Thus if somewhere $\varphi'_\pm(x, z)$ is negative or zero, there exists a point $x_2 \in (0, \tilde{x}]$ where $\varphi'_-(x_2, z) = 0$ and $\varphi'_\pm(x, z) > 0$ for $x < x_2$. Choose an arbitrary point $x_1 \in (0, x_2)$, then $0 < \varphi(x, z) < \frac{\pi}{2}$ in $[x_1, x_2]$ and

$$(z \cot \varphi(x, z))' = q(x) - z^2 - (z \cot \varphi(x, z))^2 \tag{3.1}$$

in $[x_1, x_2]$. Now we have

$$(\cot \varphi)'_\pm(x, z) = \frac{-1}{\sin^2 \varphi} \varphi'_\pm(x, z),$$

which implies $\sqrt{q(x) - z^2} < z \cot \varphi(x, z)$ for $x \in [x_1, x_2)$ and $\sqrt{q(x_2 - 0) - z^2} = z \cot \varphi(x_2, z)$. We show that this is not possible. Indeed, choose x_3 arbitrarily from

$[x_1, x_2)$. From (3.1) we get

$$\left[\log\left(z\cot\varphi(x,z)-\sqrt{q(x)-z^2}\right)\right]_{x_1}^{x_3}$$

$$=\int_{x_1}^{x_3}\frac{\mathrm{d}(z\cot\varphi(x,z)-\sqrt{q(x)-z^2})}{z\cot\varphi(x,z)-\sqrt{q(x)-z^2}}$$

$$=\int_{x_1}^{x_3}\frac{q(x)-z^2-(z\cot\varphi(x,z))^2}{z\cot\varphi(x,z)-\sqrt{q(x)-z^2}}\ \mathrm{d}x$$

$$-\int_{x_1}^{x_3}\frac{\mathrm{d}q(x)}{2\sqrt{q(x)-z^2}(z\cot\varphi(x,z)-\sqrt{q(x)-z^2})}$$

$$\geq\int_{x_1}^{x_3}-(z\cot\varphi(x,z)+\sqrt{q(x)-z^2})\ \mathrm{d}x$$

by the monotonicity of q. Now $z\cot\varphi(x,z)$ is continuous and bounded in $[x_1, x_2)$, thus $-(z\cot\varphi(x,z)+\sqrt{q(x)-z^2})$ is bounded from below, and hence

$$\left[\log\left(z\cot\varphi(x,z)-\sqrt{q(x)-z^2}\right)\right]_{x_1}^{x_3}\geq K \tag{3.2}$$

with K independent of x_3. If we let x_3 approach x_2, this implies that

$$z\cot\varphi(x_2,z)>\sqrt{q(x_2-0)-z^2}, \tag{3.3}$$

i.e., $\varphi'_-(x_2,z)>0$, a contradiction. ☐

In the following formulae we sometimes write $\varphi(x)$ instead of $\varphi(x,z)$.

Lemma 3.2.

$$\dot\varphi(x)=\int_0^x(1+\frac{q(t)}{z^2}\sin^2\varphi(t))e^{-\int_t^x\frac{q}{z}\sin 2\varphi}\ \mathrm{d}t. \tag{3.4}$$

Proof. Differentiate equation (2.7) with respect to z:

$$\dot\varphi'(x,z)=1+\frac{q(x)}{z^2}\sin^2\varphi(x)-\frac{q(x)}{z}\sin 2\varphi(x,z)\dot\varphi(x,z). \tag{3.5}$$

This is a linear differential equation in $x\to\dot\varphi(x,z)$. Multiplying both sides by $e^{\int_0^x\frac{q}{z}\sin 2\varphi}$, we have

$$(\dot\varphi(x,z)e^{\int_0^x\frac{q}{z}\sin 2\varphi})'=(1+\frac{q(x)}{z^2}\sin^2\varphi(x,z))e^{\int_0^x\frac{q}{z}\sin 2\varphi}. \tag{3.6}$$

Using $\dot\varphi(0)=0$, we get (3.4). ☐

Remark. From (2.8) we can rewrite (3.4):

$$\dot\varphi(x)=\int_0^x(1+\frac{q(t)}{z^2}\sin^2\varphi(t))\frac{r^2(t)}{r^2(x)}\ \mathrm{d}t. \tag{3.7}$$

From equation (3.7) it is obvious that $\varphi(x,z)$ is strictly monotone increasing in z.

Corollary 3.3.

$$\dot{\psi}(x) = \frac{2}{r^2(x)z^2}\int_0^x r^2(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi). \tag{3.8}$$

Proof.

$$\dot{\psi}(x) = \frac{\dot{\varphi}(x)}{z} - \frac{\varphi(x)}{z^2}$$

$$= \frac{1}{r^2(x)z^2}\left\{\int_0^x r^2(t)[2(z-\varphi'(t))+\varphi'(t)]\ \mathrm{d}t - r^2(x)\varphi(x)\right\}$$

$$= \frac{2}{r^2(x)z^2}\int_0^x [r^2(t)(z-\varphi'(t)) - r(t)r'(t)\varphi(t)]\ \mathrm{d}t$$

$$= \frac{2}{r^2(x)z^2}\int_0^x r^2(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi).$$

□

From now on, we define the potential to be zero on (x_0, ∞) and extend the definition of φ, r and ψ accordingly. Then $\varphi(x,z) \to \infty$ if z is fixed and $x \to \infty$.

Lemma 3.4. *If* $0 < |\varphi| < \frac{\pi}{2}$, *then* $\sin^2\varphi - \varphi\sin\varphi\cos\varphi > 0$.

Proof. This is a simple corollary of $\varphi < \tan\varphi$ if $0 < \varphi < \frac{\pi}{2}$. □

Corollary 3.5. *Let* $k \geq 0$ *be an integer,* $k\pi \leq c \leq k\pi + \frac{\pi}{2}$, $k\pi + \frac{\pi}{2} \leq d \leq (k+1)\pi$. *Then for any fixed* z

$$\int_{\varphi^{-1}(k\pi)}^{\varphi^{-1}(c)} r^2(t)(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi) \geq -k\frac{\pi}{2}\left[r^2\right]_{\varphi^{-1}(k\pi)}^{\varphi^{-1}(c)}, \tag{3.9}$$

$$\int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(d)} r^2(t)(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi) \geq -(k+1)\frac{\pi}{2}\left[r^2\right]_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(d)}, \tag{3.10}$$

and equality holds iff $q = 0$ *in the corresponding open interval.*

Proof.

$$\int_{\varphi^{-1}(k\pi)}^{\varphi^{-1}(c)} r^2(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi)$$

$$= \int_{\varphi^{-1}(k\pi)}^{\varphi^{-1}(c)} r^2(\frac{q}{z}\sin^2(\varphi - k\pi) - \frac{q}{z}(\varphi - k\pi)\sin\varphi\cos\varphi)$$

$$-k\pi\int_{\varphi^{-1}(k\pi)}^{\varphi^{-1}(c)} r^2(\frac{q}{z}\sin\varphi\cos\varphi).$$

Now $(\varphi - k\pi)$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so, by the preceding lemma, the first term is positive, except when $q = 0$. The second term is the same as in the right-hand side of (3.9) as we can easily see from (2.8).

The other part of the lemma can be proved in the same way:

$$\int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(d)} r^2(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi)$$

$$= \int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(d)} r^2(\frac{q}{z}\sin^2(\varphi-(k+1)\pi) - \frac{q}{z}(\varphi-(k+1)\pi)\sin\varphi\cos\varphi)$$

$$-(k+1)\pi\int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(d)} r^2(\frac{q}{z}\sin\varphi\cos\varphi)$$

$$\geq -(k+1)\pi\int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(d)} rr' = -(k+1)\frac{\pi}{2}\left[r^2\right]_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(d)}.$$

□

Corollary 3.6. *Let* $0 \leq C \leq \frac{\pi}{2}$, $0 \leq D \leq \pi$. *Then*

$$\int_0^{\varphi^{-1}(C)} r^2(t)(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi) \geq 0, \tag{3.11}$$

$$\int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(k\pi+\frac{\pi}{2}+D)} r^2(t)(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi) \geq -(k+1)\frac{\pi}{2}\left[r^2\right]_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(k\pi+\frac{\pi}{2}+D)}, \tag{3.12}$$

and equality holds iff $q = 0$ *in the corresponding open interval.*

(3.11). follows from (3.9) with $k = 0$. If $D \leq \frac{\pi}{2}$, then (3.12) is the same as (3.10). If not, it is the sum of (3.10) with $d = (k+1)\pi$ and (3.9) with $c = k\pi + \frac{\pi}{2} + D$ (and with k replaced by $k+1$). □

Lemma 3.7. $r(\varphi^{-1}(k\pi + 3\frac{\pi}{2})) \leq r(\varphi^{-1}(k\pi + \frac{\pi}{2}))$, *if* $k = 0, 1, 2, \ldots$. *Moreover, the function* r *is monotone increasing betweeen* $\varphi^{-1}(k\pi)$ *and* $\varphi^{-1}(k\pi + \frac{\pi}{2})$ *and is monotone decreasing between* $\varphi^{-1}(k\pi + \frac{\pi}{2})$ *and* $\varphi^{-1}((k+1)\pi)$.

Proof. Since the logarithmic function is strictly increasing, it is enough to prove

$$\left[\log r^2\right]_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(k\pi+\frac{3}{2}\pi)} \leq 0. \tag{3.13}$$

The monotonicity of $\log r^2$, (hence of r) follows from the sign of its derivative, $\frac{q}{z}\sin 2\varphi$. By substituting $u = \varphi(x)$:

$$\int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}((k+1)\pi)} \frac{2r'}{r} = \int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}((k+1)\pi)} \frac{q}{z}\sin 2\varphi$$

$$= \int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}((k+1)\pi))} \frac{q(x)\sin 2\varphi(x)}{z^2 - q(x)\sin^2\varphi(x)}\varphi'(x)\ \mathrm{d}x$$

$$= \int_{k\pi+\frac{\pi}{2}}^{(k+1)\pi} \frac{q(\varphi^{-1}(u))\sin 2u}{z^2 - q(\varphi^{-1}(u))\sin^2 u}\ \mathrm{d}u.$$

Note that $\sin 2u < 0$ for $k\pi + \frac{\pi}{2} < u < (k+1)\pi$, while the denominator is always positive, as we have seen in Lemma 3.1. Hence if we replace q by its minimum, $q(\varphi^{-1}((k+1)\pi))$, the value of the fraction will increase:

$$\int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}((k+1)\pi)} \frac{2r'}{r} \leq \int_{k\pi+\frac{\pi}{2}}^{(k+1)\pi} \frac{q(\varphi^{-1}((k+1)\pi))\sin 2u}{z^2 - q(\varphi^{-1}((k+1)\pi))\sin^2 u}\ \mathrm{d}u$$

$$= \left[-\ln(z - \frac{q(\varphi^{-1}((k+1)\pi))}{z}\sin^2 u)\right]_{k\pi+\frac{\pi}{2}}^{(k+1)\pi}.$$

The other part of the integral can be handled in an analogous way except that this time $\sin 2u > 0$ on the interval in question and we replace q by its maximum:

$$\int_{\varphi^{-1}((k+1)\pi)}^{\varphi^{-1}(k\pi+3\frac{\pi}{2})} \frac{2r'}{r} \leq \left[-\ln(z - \frac{q(\varphi^{-1}((k+1)\pi))}{z}\sin^2 u)\right]_{(k+1)\pi}^{k\pi+3\frac{\pi}{2}}. \tag{3.14}$$

Summing up,

$$\int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(k\pi+3\frac{\pi}{2})} \frac{2r'}{r} \leq \left[-\ln(z - \frac{q(\varphi^{-1}((k+1)\pi))}{z}\sin^2 u)\right]_{k\pi+\frac{\pi}{2}}^{k\pi+3\frac{\pi}{2}} = 0. \tag{3.15}$$

□

Proof of Theorem 2.2. If $\varphi(x_0, z) \le \frac{\pi}{2}$, then the statement of the theorem immediately follows from (3.8) and (3.11). If not, let $\varphi(x_0, z) = \frac{\pi}{2} + k\pi + D$, with $0 \le D \le \pi$:

$$\int_0^{\varphi^{-1}(\frac{\pi}{2}+k\pi+D)} r^2(t)(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi)\ \mathrm{d}t$$

$$= \int_0^{\varphi^{-1}(\frac{\pi}{2})} r^2(t)(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi)\ \mathrm{d}t$$

$$+ \sum_{i=1}^{k} \int_{\varphi^{-1}(i\pi-\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2})} r^2(t)(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi)\ \mathrm{d}t$$

$$+ \int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(k\pi+\frac{\pi}{2}+D)} r^2(t)(\frac{q}{z}\sin^2\varphi - \frac{q}{z}\varphi\sin\varphi\cos\varphi)\ \mathrm{d}t.$$

Here every term is nonnegative by Corollary 3.6 and Lemma 3.7, and if their sum is zero, then q has to be zero in the whole interval $(0, x_0]$. □

Proof of Corollary 2.3. By the current boundary conditions,

$$z_n\psi(\pi, z_n) = (n - \frac{1}{2})\pi. \tag{3.16}$$

Let m be less than n. Then $\frac{(2m-1)\pi}{2z_m} = \psi(\pi, z_m) \le \psi(\pi, z_n) = \frac{(2n-1)\pi}{2z_n}$, and thus $\frac{z_n}{z_m} \le \frac{2n-1}{2m-1}$ and $\frac{\lambda_n}{\lambda_m} \le \frac{(2n-1)^2}{(2m-1)^2}$. If equality holds, then $\psi(\pi, z_m) = \psi(\pi, z_n)$, and by Theorem 2.2 this implies that $q = 0$ in $(0, \pi]$. □

4. THE PROOF OF THEOREM 2.4

Let the potential $q(x)$ be monotone decreasing in $[0, a]$ and monotone increasing in $[a, \pi]$. Denote by $\tilde{q}(x)$ the reverse of the potential, i.e., $\tilde{q}(x) = q(\pi - x)$. Then $y(\pi - x, z_n)$ is an eigenfunction for the potential $\tilde{q}(x)$. Moreover, define

$$\tilde{y}(x, z_n) = (-1)^{n+1}\frac{y(\pi - x, z_n)}{r(\pi, z_n)}, \tag{4.1}$$

$$\tilde{r}(x, z_n) = \frac{r(\pi - x, z_n)}{r(\pi, z_n)}, \tag{4.2}$$

and

$$\tilde{\varphi}(x, z_n) = n\pi - \varphi(\pi - x, z_n). \tag{4.3}$$

Then

$$\tilde{y}(0, z_n) = 0, \tag{4.4}$$

$$\tilde{y}'(0, z_n) = 1, \tag{4.5}$$

which means that $\tilde{y}(x, z_n)$ is the solution of the initial value problem (2.1)–(2.2) with $\tilde{q}$ instead of q. It is also simple that

$$\tilde{y}(x, z_n) = \frac{\tilde{r}(x, z_n)}{z_n}\sin\tilde{\varphi}(x, z_n), \tag{4.6}$$

$$\tilde{y}'(x, z_n) = \tilde{r}(x, z_n) \cos \tilde{\varphi}(x, z_n), \tag{4.7}$$

$$\tilde{\varphi}(0, z_n) = 0, \tag{4.8}$$

which prove that $\tilde{r}$ and $\tilde{\varphi}$ are the Prüfer-variables for $\tilde{y}$. According to (2.6), let $\tilde{\psi} = \frac{\tilde{\varphi}}{z}$.

Proof of Theorem 2.4. Consider the function $\Psi(z) = \psi(a, z) + \tilde{\psi}(\pi - a, z)$. This is, by Theorem 2.2, the sum of two monotone increasing functions. By (4.3),

$$z_n \Psi(z_n) = n\pi. \tag{4.9}$$

Let m be less than n. Then $\frac{m\pi}{z_m} = \Psi(z_m) \leq \Psi(z_n) = \frac{n\pi}{z_n}$, and thus $\frac{z_n}{z_m} \leq \frac{n}{m}$ and $\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}$. If equality holds, then $\Psi(z_m) = \Psi(z_n)$, hence $\psi(a, z_m) = \psi(a, z_n)$ and $\tilde{\psi}(\pi - a, z_m) = \tilde{\psi}(\pi - a, z_n)$. By Theorem 2.2 this implies that $q = 0$ in $(0, a]$ and $\tilde{q} = 0$ in $(0, \pi - a]$, i.e., $q = 0$ in $(0, \pi)$. □

5. Remarks

Remark 1. If the potential is not monotone decreasing, then ψ might not increase in z at some point. For example, let q be zero in $[0, \frac{2}{3}\pi]$ and 1 otherwise. Pick $z = 3/2$. We can easily see that

$$y(x, \frac{3}{2}) = \begin{cases} \frac{2}{3} \sin \frac{3}{2} x & \text{if } x \leq \frac{2}{3}\pi, \\ -\frac{2}{\sqrt{5}} \sin \frac{\sqrt{5}}{2}(x - \frac{2}{3}\pi) & \text{otherwise,} \end{cases} \tag{5.1}$$

$$y'(x, \frac{3}{2}) = \begin{cases} \cos \frac{3}{2} x & \text{if } x \leq \frac{2}{3}\pi, \\ -\cos \frac{\sqrt{5}}{2}(x - \frac{2}{3}\pi) & \text{otherwise.} \end{cases} \tag{5.2}$$

From (2.7)

$$\varphi'(x) \geq \frac{3}{2} - \frac{2}{3} > 0. \tag{5.3}$$

It can be easily checked that $\varphi(\frac{2}{3}\pi, \frac{3}{2}) = \pi$, $\varphi(\pi, \frac{3}{2}) \approx 4.27083 < \frac{3}{2}\pi$. Combining this with (2.8), we get

$$r(x) = \exp(\int_0^x \frac{q}{z} \sin \varphi \cos \varphi) = \begin{cases} 1 & \text{if } x \leq \frac{2}{3}\pi, \\ \exp(\int_\pi^{\varphi(x)} \frac{\frac{2}{3} \sin v \cos v}{\frac{3}{2} - \frac{2}{3} \sin^2 v} \, dv) & \text{otherwise;} \end{cases} \tag{5.4}$$

hence

$$\begin{aligned} \dot{\psi}(\pi) &= \frac{2}{r^2(\pi) z^2} \int_0^\pi r^2 (\frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi) \\ &= \frac{2}{r^2(\pi) z^2} \int_{\frac{2}{3}\pi}^\pi \frac{r^2 (\frac{2}{3} \sin^2 \varphi - \frac{2}{3} \varphi \sin \varphi \cos \varphi)}{\frac{3}{2} - \frac{2}{3} \sin^2 \varphi} \varphi', \end{aligned}$$

and substituting $\varphi(x) = u$ to this integral, we can numerically compute it (using Maple or Mathematica):

$$\int_\pi^{4.27083} \exp(\int_\pi^u \frac{\frac{2}{3} \sin 2v}{\frac{3}{2} - \frac{2}{3} \sin^2 v} \, dv) \frac{(\frac{2}{3} \sin^2 u - \frac{2}{3} u \sin u \cos u)}{\frac{3}{2} - \frac{2}{3} \sin^2 u} \, du \approx -0.811 \tag{5.5}$$

which means that ψ is not growing at $z = \frac{3}{2}$.

Remark 2. We could ask whether the sum of the $\psi(\pi)$'s belonging to the potential $q(x)$ and $q(\pi - x)$ is a monotone increasing function in z. But the last example shows that this is not necessarily the case either. Indeed,

$$\dot{\psi}(\pi) \approx \frac{2}{r^2(\pi)z^2}(-0.811) \approx -0.292. \tag{5.6}$$

In a similar manner we can compute $\dot{\psi}(x, \frac{3}{2})$ belonging to $q(\pi - x)$, which approximately equals 0.0306, so the sum is also negative.

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