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MAPS PRESERVING NUMERICAL RANGES OF OPERATOR PRODUCTS

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ABSTRACT. Let H be a complex Hilbert space, $B(H)$ the algebra of all bounded linear operators on H and $S^a(H)$ the real linear space of all self-adjoint operators on H. We characterize the surjective maps on $B(H)$ or $S^a(H)$ that preserve the numerical ranges of products or Jordan triple-products of operators.

1. Introduction

Denote by $\mathbb{C}$ the field of complex numbers and by $\mathbb{R}$ the field of real numbers. For a Hilbert space H, $\langle\cdot,\cdot\rangle$ stands for its inner product, $\mathcal{B}(H)$ the algebra of all bounded linear operators on H and $\mathcal{S}^a(H)$ the real linear space of all self-adjoint operators in $\mathcal{B}(H)$. For every $A \in \mathcal{B}(H)$, the numerical range of A is the set $W(A) = \{\langle Ax, x\rangle \mid x \in H,\ \|x\| = 1\}$ and the numerical radius of A is defined as $w(A) = \sup\{|\lambda| \mid \lambda \in W(A)\}$. A map U on H is called a conjugate unitary operator if U is conjugate linear and $U^*U = UU^* = I$.

Numerical range of operators is a very important concept and is extensively studied in both theory and applications. Particularly, many authors have studied numerical range preserving maps on various operator algebras; see [1]-[6], [9], [11], [12], [13, Chapter 5]. In this paper, we characterize surjective maps $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$ such that

$$W(\phi(A)\phi(B)) = W(AB) \quad \text{for all } A, B \in \mathcal{B}(H). \tag{1.1}$$

Here H, K are two Hilbert spaces. This work is motivated by the result of L. Molnár [10], who characterized surjective maps ϕ on $\mathcal{B}(H)$ such that

$$\sigma(\phi(A)\phi(B)) = \sigma(AB) \quad \text{for all } A, B \in \mathcal{B}(H).$$

Here $\sigma(T)$ is the spectrum of $T \in \mathcal{B}(H)$.

In Section 2, we show that a surjective map $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$ satisfying (1.1) has the form $\phi(A) = \pm UAU^*$ for all $A \in \mathcal{B}(H)$, where U is unitary. Also we show that a surjective map $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$ satisfying

$$W(\phi(B)\phi(A)\phi(B)) = W(BAB) \quad \text{for all } A, B \in \mathcal{B}(H) \tag{1.2}$$

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must be a multiple of a C*-isomorphism (by a cubic root of unity). In Section 3, we treat the problems for maps $\phi : \mathcal{S}^a(H) \to \mathcal{S}^a(K)$. In Section 4, we characterize the maps $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$ satisfying

$$W(\phi(A)^*\phi(B)) = W(A^*B) \quad \text{for all } A, B \in \mathcal{B}(H),$$

and

$$W(\phi(B)\phi(A)^*\phi(B)) = W(BA^*B) \quad \text{for all } A, B \in \mathcal{B}(H).$$

We obtain more general results covering these in the indefinite inner product space context. Some remarks and questions are given in Section 5.

2. Maps on $\mathcal{B}(H)$

In this section we discuss the question of characterizing maps which preserve numerical ranges of operator products or numerical ranges of operator Jordan triple-products. The following are our main results.

Theorem 2.1. *Let H and K be complex Hilbert spaces and let $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a surjective map. Then ϕ satisfies Eq.* (1.1) *if and only if there is a unitary operator $U : H \to K$ such that ϕ is of the form*

$$\phi(A) = \epsilon UAU^*$$

for all $A \in \mathcal{B}(H)$, where $\epsilon = \pm 1$.

Theorem 2.2. *Let H and K be complex Hilbert spaces and let $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a surjective map. Then ϕ satisfies Eq.* (1.2) *if and only if there is a scalar λ with $\lambda^3 = 1$ and a unitary operator $U : H \to K$ such that either*

(1) *$\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}(H)$; or*

(2) *$\phi(A) = \lambda UA^tU^*$ for all $A \in \mathcal{B}(H)$, where A^t is the transpose of A with respect to an arbitrarily fixed orthonormal basis of H.*

The next lemma is crucial for our proofs of Theorem 2.1 and 2.2 as well as other results of this paper, which gives new characterizations of rank-one operators by numerical range of operator products.

Lemma 2.3. *Let $A \in \mathcal{B}(H)$. The following conditions are equivalent:*

(i) *A is a rank-one operator.*

(ii) *For every $B \in \mathcal{B}(H)$ with $AB \neq 0$, $W(AB)$ is either an ellipse which has* 0 *as a focus or a line segment which has* 0 *as an end point.*

(iii) *For every $B \in \mathcal{B}(H)$, $BAB \neq 0$ implies that $W(BAB)$ is either an ellipse which has* 0 *as a focus or a line segment which has* 0 *as an end point.*

Proof. (i)⇒(ii) and (i)⇒(iii) are obvious since, under the assumptions, AB and BAB are of rank one and the numerical range of every rank-one operator has the form stated in (ii).

(ii)⇒(i). Assume that rank$A \geq 2$. Then there exist linearly independent vectors $x_1, x_2 \in H$ such that $Ax_1 \perp Ax_2$ and $\|Ax_1\| = \|Ax_2\| = 1$. Let $B = \alpha x_1 \otimes Ax_1 + \beta x_2 \otimes Ax_2 + \gamma x_1 \otimes Ax_2$, $\alpha, \beta, \gamma \in \mathbb{C}$. It follows that $AB = \alpha Ax_1 \otimes Ax_1 + \beta Ax_2 \otimes Ax_2 + \gamma Ax_1 \otimes Ax_2$. If α, β and γ are all nonzero, then $W(AB)$ is an ellipse which has α, β as focuses, contradicting to the conditions of (ii).

(iii)⇒(i). Assume, on the contrary, that A satisfies (iii) but rank$A \geq 2$. We have to show that there exists a $B \in \mathcal{B}(H)$ such that $W(BAB)$ is neither an ellipse with 0 as a focus, nor a line segment with 0 as an end point.

If $\dim H \geq 4$, then there exists a rank-four projection P such that $\operatorname{rank}(PAP) \geq 2$. In fact, there exist vectors x_1, x_2 with $x_1 \perp x_2$ such that Ax_1 and Ax_2 are linearly independent. Let $X = [x_1, x_2, Ax_1, Ax_2]$, the linear subspace spanned by $\{x_1, x_2, Ax_1, Ax_2\}$. Then $\dim X \leq 4$ and $P_X A P_X$ has rank > 1. Here we denote by P_L the projection with closed subspace L as its range. Take any 4-dimensional subspace H_4 containing X and let $P = P_{H_4}$. It is obvious that $\operatorname{rank}(PAP) \geq 2$.

Denote $A_1 = PA|_{H_4} \in \mathcal{B}(H_4)$. If we have shown that there is an operator $B_1 \in \mathcal{B}(H_4)$ such that $B_1 A_1 B_1 \neq 0$, $0 \in W(B_1 A_1 B_1)$ but $W(B_1 A_1 B_1)$ is neither an ellipse with 0 as a focus, nor a line segment with 0 as an end point, then, let $B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$. It is clear that $W(BAB)$ is neither an ellipse with 0 as a focus, nor a line segment with 0 as an end point since $W(BAB) = \operatorname{conv}\{W(B_1 A_1 B_1) \cup \{0\}\} = W(B_1 A_1 B_1)$ as $0 \in W(B_1 A_1 B_1)$. Where $\operatorname{conv}(\Lambda)$ denotes the convex hull of the set Λ. Thus we get a contradiction and then the proof of (iii)$\Rightarrow$(i) for the case $\dim H \geq 4$ is completed. So, the task of proving (iii)$\Rightarrow$(i) is reduced to the four dimensional case.

Identify $\mathcal{B}(H_4)$ with $M_4(\mathbb{C})$ and assume $A \in M_4(\mathbb{C})$ has rank greater than 1. Then there exists a transformation $S : \mathbb{C}^2 \to \mathbb{C}^4$ with $S^*S = I_2$ such that $S^*AS \in M_2(\mathbb{C})$ is invertible. Thus there is a 2×2 unitary matrix $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ and there are positive numbers s_1, s_2 such that $S^*AS = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} U$. Take $T : \mathbb{C}^2 \to \mathbb{C}^4$ so that $W = \begin{pmatrix} S & T \end{pmatrix} \in M_4(\mathbb{C})$ is unitary. Then

$$W^*AW = \begin{pmatrix} S^*AS & S^*AT \\ T^*AS & T^*AT \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} U & S^*AT \\ T^*AS & T^*AT \end{pmatrix}.$$

Pick nonzero complex numbers b_1 and b_2 so that $\bar{u}_{11} s_1 b_1^2 + \bar{u}_{22} s_2 b_2^2 = 0$. Let $B = W(U^* \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \oplus 0) W^* \in M_4(\mathbb{C})$. Then

$$W^*BABW = U^* \begin{pmatrix} s_1 b_1^2 & 0 \\ 0 & s_2 b_2^2 \end{pmatrix} \oplus 0 = \begin{pmatrix} \bar{u}_{11} s_1 b_1^2 & \bar{u}_{21} s_2 b_2^2 \\ \bar{u}_{12} s_1 b_1^2 & \bar{u}_{22} s_2 b_2^2 \end{pmatrix} \oplus 0.$$

It is easily checked that the matrix $\begin{pmatrix} \bar{u}_{11} s_1 b_1^2 & \bar{u}_{21} s_2 b_2^2 \\ \bar{u}_{12} s_1 b_1^2 & \bar{u}_{22} s_2 b_2^2 \end{pmatrix}$ has two nonzero eigenvalues λ_1 and λ_2 with $\lambda_1 + \lambda_2 = 0$, and hence its numerical range, as well as the numerical range of BAB, contains 0 but is neither an ellipse with 0 as a focus, nor a line segment with 0 as an end point. This completes the proof. □

The rest of this section is devoted to proving Theorem 2.2. The proof of Theorem 2.1 is similar and we omit it here. To do this, we need two more lemmas which are also useful in Section 4.

For finite rank operators in $\mathcal{B}(H)$ one can define a trace functional tr by $\operatorname{tr}(A) = \sum_{k=1}^{n} \langle x_k, f_k \rangle$ when $A = \sum_{k=1}^{n} x_k \otimes f_k$.

Lemma 2.4. *Let $A, C \in \mathcal{B}(H)$. If $\operatorname{tr}(BAB) = \operatorname{tr}(BCB)$ for every rank-one projection $B \in \mathcal{B}(H)$, then $A = C$.*

Proof. Let $B = x \otimes x$, where x is a unit vector. Then B is a rank-1 projection and every rank-1 projection takes this form. By the assumption, we have $\langle Ax, x \rangle =$

$\text{tr}(Ax\otimes x) = \text{tr}(BAB) = \text{tr}(BCB) = \text{tr}(Cx\otimes x) = \langle Cx, x\rangle$. Thus $\langle Ax, x\rangle = \langle Cx, x\rangle$ holds for every unit vector $x \in H$, which entails $A = C$ since H is complex. □

Lemma 2.5. *Let $T \in \mathcal{B}(H)$ be invertible. Then $\langle T^{-1}x, f\rangle\langle Tx, f\rangle = \langle x, f\rangle^2$ for every $x, f \in H$ implies that there exists a $\lambda \in \mathbb{C}$ such that $T = \lambda I$.*

Proof. Fix a nonzero $x \in H$. Then, for every $f \in [x]^\perp \subset H$, we have $\langle T^{-1}x, f\rangle = 0$ or $\langle Tx, f\rangle = 0$ since $\langle T^{-1}x, f\rangle\langle Tx, f\rangle = \langle x, f\rangle^2 = 0$.

Let $M_x = \{f \in [x]^\perp \mid \langle Tx, f\rangle = 0\}$ and $N_x = \{f \in [x]^\perp \mid \langle T^{-1}x, f\rangle = 0\}$. Then $M_x \cup N_x = [x]^\perp$. Because $[x]^\perp$, M_x and N_x are all closed linear subspaces, we must have $M_x \subseteq N_x = [x]^\perp$ or $N_x \subseteq M_x = [x]^\perp$.

If $N_x = [x]^\perp$, then $T^{-1}x \in [x]$. So there exists a $\lambda_x \in \mathbb{C}$ such that $T^{-1}x = \lambda_x x \neq 0$, that is, $Tx = {\lambda_x}^{-1}x$.

If $M_x = [x]^\perp$, then $Tx \in [x]$, that is, $Tx = \lambda_x x$ for some scalar λ_x.

Since x is arbitrary, we see that, for every $x \in H$, there is a scalar λ_x such that $Tx = \lambda_x x$. This implies that there exists a $\lambda \in \mathbb{C}$ such that $T = \lambda I$. □

Now we are in a position to prove Theorem 2.2. Note that, if two rank-one operators have the same numerical ranges, then they have the same nonzero eigenvalues, and hence have the same traces. This simple observation will be used frequently in this paper.

Proof of Theorem 2.2. It is clear that we need only to check the necessity. Suppose that ϕ satisfies Eq. (1.2). For the sake of simplicity we assume $K = H$.

First we check that ϕ preserves rank-one operators in both directions. Let $A \in \mathcal{B}(H)$ be a rank-one operator. For every $T \in \mathcal{B}(H)$, there exists a $B \in \mathcal{B}(H)$ such that $T = \phi(B)$. It follows from $W(T\phi(A)T) = W(BAB)$ and Lemma 2.3 that $\phi(A)$ is a rank-one operator. Similarly, $\phi(A)$ is a rank-one operator will imply that A is a rank-one operator, too.

Second we show that ϕ is linear. Let $A, A' \in \mathcal{B}(H)$ be arbitrarily given and let $B \in \mathcal{B}(H)$ be a rank-one operator. Notice that, for rank-one operators T and S, $W(T) = W(S)$ will imply $\text{tr}(T) = \text{tr}(S)$. Then Eq. (1.2) implies that

$$\begin{aligned} & \text{tr}(\phi(B)(\phi(A + A')\phi(B)) = \text{tr}(B(A + A')B) = \text{tr}(BAB) + \text{tr}(BA'B) \\ = \;& \text{tr}(\phi(B)\phi(A)\phi(B)) + \text{tr}(\phi(B)\phi(A')\phi(B)) = \text{tr}(\phi(B)(\phi(A) + \phi(A'))\phi(B)). \end{aligned}$$

Since $\phi(B)$ runs over all rank-one operators when B runs over all rank-one operators, Lemma 2.4 ensures that $\phi(A + A') = \phi(A) + \phi(A')$, i.e., ϕ is additive. Similarly, we can check that ϕ is homogeneous.

So, ϕ is a linear bijection on $\mathcal{B}(H)$ preserving rank-one operators in both directions. It follows from [8, Lemma 1.2], either

(i) there exist bijective linear operators U and V on H such that $\phi(x \otimes f) = Ux \otimes Vf$ $(\forall x, f \in H)$; or

(ii) there exist bijective conjugate linear operators U and V on H such that $\phi(x \otimes f) = Uf \otimes Vx$ $(\forall x, f \in H)$.

Suppose the case (i) occurs, we will show that ϕ has the form (1) stated in Theorem 2.2. By Eq. (1.2), we have $W(x \otimes f) = W(\phi(I)\phi(x \otimes f)\phi(I))$. So, by taking trace, $\langle x, f\rangle = \langle \phi(I)^2Ux, Vf\rangle$ holds for every $x, f \in H$. It follows that U, V are bounded and $V^*\phi(I)^2U = I$. Eq. (1.2) also yields that $\phi(I)^2 = \phi(I)^{-1}$. Hence $V^*\phi(I)^{-1}U = I$, i.e., $\phi(I) = UV^*$. As $V^*\phi(I)U = V^*UV^*U$ and

$V^*UV^*UV^*U = I$, we have $V^*UV^*U = (V^*U)^{-1}$. Since $W((x\otimes f)I(x\otimes f)) = W((Ux\otimes Vf)\phi(I)(Ux\otimes Vf))$, we see that $\langle x,f\rangle^2 = \langle V^*\phi(I)Ux,f\rangle\langle V^*Ux,f\rangle$, that is, $\langle x,f\rangle^2 = \langle (V^*U)^{-1}x,f\rangle\langle V^*Ux,f\rangle$ holds for all $x,f\in H$. By Lemma 2.5, there exists a $\lambda\in\mathbb{C}$ such that $V^*U=\lambda I$. Notice that $I=(V^*U)^3=\lambda^3 I$, so $\lambda^3=1$ and $V^*=\lambda U^{-1}$. Thus we have $\phi(x\otimes f)=\lambda U(x\otimes f)U^{-1}$ for all rank-one operators $x\otimes f$, where $\lambda^3=1$.

Now, for every A, we have

$$W(\lambda^2 U(x\otimes f)U^{-1}\phi(A)U(x\otimes f)U^{-1}) = W((x\otimes f)A(x\otimes f)).$$

So $\operatorname{tr}(\lambda^2(x\otimes f)U^{-1}\phi(A)U(x\otimes f)) = \operatorname{tr}((x\otimes f)A(x\otimes f))$. It follows from Lemma 2.4 again that $\lambda^2U^{-1}\phi(A)U = A$. Hence $\phi(A)=\lambda UAU^{-1}$ holds for every $A\in\mathcal{B}(H)$.

Let $x\in H$ be a unit vector; then we have

$$\begin{aligned}[0,1] &= W((x\otimes x)(x\otimes x)(x\otimes x)) = W(\phi(x\otimes x)\phi(x\otimes x)\phi(x\otimes x))\\ &= W(U(x\otimes x)U^{-1}U(x\otimes x)U^{-1}U(x\otimes x)U^{-1}) = W(Ux\otimes xU^{-1}).\end{aligned}$$

This implies that $Ux\otimes(U^{-1})^*x$ is a rank-one projection and hence $(U^{-1})^*x$ is linearly dependent of Ux for every unit vector x. It follows that $(U^{-1})^*\in[U]$ and there exists a $\mu>0$ such that $UU^*=\mu I$. Let $U_1=(\sqrt{\mu})^{-1}U$, then $U_1{U_1}^*=I$. So $\phi(A)=\lambda U_1A{U_1}^*$ for all $A\in\mathcal{B}(H)$, where $\lambda^3=1$ and U_1 is unitary. Hence ϕ has the form (1) stated in Theorem 2.2.

Assume the case (ii) occurs, let us show that the form (2) in Theorem 2.2 holds true. Taking $A=x\otimes f$ and $B=I$ in the equation (1.2), we get

$$W(\phi(I)(Uf\otimes Vx)\phi(I)) = W(\phi(I)\phi(x\otimes f)\phi(I)) = W(x\otimes f).$$

Note that both U and V are conjugate linear. So by taking trace we have $\langle x,f\rangle = \langle\phi(I)Uf,\phi(I)^*Vx\rangle = \langle x,V^*\phi(I)^2Uf\rangle$ for every $x,f\in H$. Now it is easily checked that both U and V are bounded, and $V^*\phi(I)^2U=I$. Thus, similar to the corresponding part in the proof of case (i) above, one gets $\phi(I)^2=\phi(I)^{-1}$, $\phi(I)=UV^*$, and $V^*UV^*U=(V^*U)^{-1}$. The equation

$$W((x\otimes f)I(x\otimes f)) = W((Uf\otimes Vx)\phi(I)(Uf\otimes Vx))$$

yields that

$$\begin{aligned}\langle x,f\rangle^2 = &\ \langle\phi(I)Uf,Vx\rangle\langle Uf,Vx\rangle = \langle x,V^*\Phi(I)Uf\rangle\langle x,V^*Uf\rangle\\ = &\ \langle x,(V^*U)^{-1}f\rangle\langle x,V^*Uf\rangle\end{aligned}$$

for every x,f. By Lemma 2.5 we see that $V^*U=\lambda I$ with $\lambda^3=1$. Now for every A, it follows from $W((Uf\otimes Vx)\phi(A)(Uf\otimes Vx)) = W((x\otimes f)A(x\otimes f))$ that

$$\begin{aligned}&\operatorname{tr}((x\otimes f)A(x\otimes f))\\ &\quad= \operatorname{tr}((Uf\otimes Vx)\phi(A)(Uf\otimes Vx)) = \langle\phi(A)Uf,Vx\rangle\langle Uf,Vx\rangle\\ &\quad= \langle U^*\phi(A)^*Vx,f\rangle\langle x,V^*Uf\rangle = \langle U^*\phi(A)^*(\bar\lambda(U^*)^{-1})x,f\rangle\langle x,\lambda f\rangle\\ &\quad= \langle\lambda^2U^*\phi(A)^*(U^*)^{-1}x,f\rangle\langle x,f\rangle\\ &\quad= \operatorname{tr}((x\otimes f)(\lambda^2U^*\phi(A)^*(U^*)^{-1})(x\otimes f))\end{aligned}$$

for every rank-one operator $x\otimes f$. By Lemma 2.4, we obtain that

$$\lambda^2U^*\phi(A)^*(U^*)^{-1} = A,$$

that is,

$$\phi(A) = \bar\lambda UA^*U^{-1}$$

for every $A \in \mathcal{B}(H)$. Now it is trivial to check that U can be taken as a conjugate unitary operator. Pick an orthonormal basis $\{e_j \mid j \in \mathcal{J}\}$ and define a conjugate unitary operator $J : H \to H$ by $Jx = \sum_{j\in\mathcal{J}} \overline{\xi_j} e_j$ if $x = \sum_{j\in\mathcal{J}} \xi_j e_j$. It is clear that $J^2 = I$, $J^* = J$ and $A^* = JA^tJ$, where A^t is the transpose of A with respect to the basis $\{e_j \mid j \in \mathcal{J}\}$. Let $U_1 = UJ$ and $\alpha = \bar{\lambda}$. Then, U_1 is a unitary operator, $\alpha^3 = 1$ and $\phi(A) = \alpha U_1 A^t U_1^*$ for all $A \in \mathcal{B}(H)$, finishing the proof. $\square$

3. Maps on the space of self-adjoint operators

In this section we characterize the maps on the real linear subspace $\mathcal{S}^a(H)$ of all self-adjoint operators on a complex Hilbert space which preserve the numerical ranges of products of operators.

Theorem 3.1. *Let H, K be complex Hilbert spaces and let ϕ: $\mathcal{S}^a(H) \to \mathcal{S}^a(K)$ be a surjective map. Then*

$$W(\phi(A)\phi(B)) = W(AB) \tag{3.1}$$

for all $A, B \in \mathcal{S}^a(H)$ if and only if there is a unitary operator $U : H \to K$ such that ϕ is of the form

$$\phi(A) = \epsilon UAU^*$$

for all $A \in \mathcal{S}^a(H)$, where $\epsilon = \pm 1$.

Similar to Section 2, our proofs are based on the following characterizations of rank-one self-adjoint operators.

Lemma 3.2. *Let $A \in \mathcal{S}^a(H)$. The following conditions are equivalent:*

(i) *A is of rank one.*

(ii) *For every $B \in \mathcal{S}^a(H)$, $AB \neq 0$ implies that $W(AB)$ is either an ellipse with 0 as a focus or a line segment with 0 as an end point.*

Proof of Theorem 3.1. Suppose that Φ satisfies Eq. (3.1). Applying Lemma 3.2 we can prove that ϕ preserves rank-one operators in both directions and ϕ is real linear. Thus ϕ preserves adjacency in both directions with $\phi(0) = 0$. By [7], there exists a bijective linear or conjugate linear operator V on H and a real scalar $c \in \mathbb{R} \setminus \{0\}$ such that $\phi(x \otimes x) = cVx \otimes Vx$ for all $x \in H$.

For every $x \in H$ with $\|x\| = 1$, we have $x \otimes x = (x \otimes x)(x \otimes x)$. Hence

$$[0, 1] = W(x \otimes x) = W(\phi(x \otimes x)^2) = c^2\|Vx\|^2[0,\ \|Vx\|^2].$$

It follows that $c^2\ \|Vx\|^4 = 1$, $\|Vx\| = \frac{1}{\sqrt{|c|}}\|x\|$ for every $x \in H$. Let $U = \sqrt{|c|}V$; then U is a unitary or conjugate unitary operator such that

$$\phi(x \otimes x) = cVx \otimes Vx = c\frac{1}{\sqrt{|c|}}Ux \otimes \frac{1}{\sqrt{|c|}}Ux = c|c|^{-1}Ux \otimes Ux = \epsilon Ux \otimes Ux$$

with $\epsilon = \pm 1$.

There is no loss of generality in assuming that $\epsilon = 1$.

If U is unitary, then for every rank-one operator $T \in \mathcal{S}^a(H)$, we have $\phi(T) = UTU^*$. Thus, for each $A \in \mathcal{S}^a(H)$,

$$\begin{aligned} W(Ax \otimes x) &= W(\phi(A)\phi(x \otimes x)) = W(\phi(A)U(x \otimes x)U^*) \\ &= W(U^*\phi(A)Ux \otimes x) \end{aligned}$$

and hence

$$\langle U^*\phi(A)Ux, x\rangle = \langle Ax, x\rangle$$

for all $x \in H$. This ensures that

$$\phi(A) = UAU^* \qquad \text{for all } A \in \mathcal{S}^a(H),$$

that is, ϕ has the form stated in the theorem.

We assert the case that U is a conjugate unitary operator cannot occur. Assume on the contrary that U is conjugate unitary such that $\phi(x \otimes x) = Ux \otimes Ux$ for all $x \in H$. It follows that, for every $A \in \mathcal{S}^a(H)$ and every $x \in H$,

$$W(Ax \otimes x) = W(\phi(A)\phi(x \otimes x)) = W(\phi(A)Ux \otimes Ux)$$

and consequently,

$$\langle x, Ax\rangle = \langle Ax, x\rangle = \langle \phi(A)Ux, Ux\rangle = \langle x, U^*\phi(A)Ux\rangle.$$

Thus we still have $\phi(A) = UAU^*$ for every $A \in \mathcal{S}^a(H)$. On the other hand, for $T \in \mathcal{B}(H)$,

$$\langle UTU^*x, x\rangle = \langle U^*x, TU^*x\rangle = \langle T^*U^*x, U^*x\rangle,$$

so $W(UTU^*) = W(T^*) = W(T)^*$. Thus we get

$$W(AB) = W(\phi(A)\phi(B)) = W(UABU^*) = W(AB)^* = W(BA)$$

for all $A, B \in \mathcal{S}^a(H)$, which is impossible. The proof is completed. □

4. Maps preserving numerical ranges of skew products

The purpose of this section is to classify the maps which preserve numerical ranges of skew products or Jordan skew triple-products of operators on Hilbert spaces, i.e., the maps ϕ which satisfy

$$W(\phi(A)^*\phi(B)) = W(A^*B) \quad \text{or} \quad W(\phi(B)\phi(A)^*\phi(B)) = W(BA^*B).$$

Taking indefinite inner product structures into consideration, we discuss it here in a more general situation. In fact, we show that

Theorem 4.1. *Let H_i be complex Hilbert spaces and $S_i \in \mathcal{B}(H_i)$ invertible self-adjoint operators, $i = 1, 2$. Let $\phi : \mathcal{B}(H_1) \to \mathcal{B}(H_2)$ be a surjective map. Then*

$$W(S_2^{-1}\phi(A)^*S_2\phi(B)) = W(S_1^{-1}A^*S_1B) \tag{4.1}$$

holds for all $A, B \in \mathcal{B}(H_1)$ if and only if there exist a nonzero real number $c \in \mathbb{R} \setminus \{0\}$, a linear invertible bounded operator $U \in \mathcal{B}(H_1, H_2)$ and a unitary operator $V \in \mathcal{B}(H_1, H_2)$ satisfying $U^*S_2U = cUS_1$ and $S_2V = cVS_1$, respectively, such that

$$\phi(A) = UAV^*$$

for all $A \in \mathcal{B}(H_1)$.

Theorem 4.2. *Let H_i be complex Hilbert spaces and $S_i \in \mathcal{B}(H_i)$ invertible self-adjoint operators, $i = 1, 2$. Let $\phi : \mathcal{B}(H_1) \to \mathcal{B}(H_2)$ be a surjective map. Then*

$$W(\phi(B){S_2}^{-1}\phi(A)^*S_2\phi(B)) = W(B{S_1}^{-1}A^*S_1B) \tag{4.2}$$

for all $A, B \in \mathcal{B}(H_1)$ if and only if there exist a number $c \in \mathbb{R} \setminus \{0\}$ and a unitary operator U such that either

(1) $U^*S_2^{-1}US_1 = cI$ and $\phi(A) = UAU^*$ for all $A \in \mathcal{B}(H_1)$; or

(2) $S_1^tU^*S_2U = cI$ and $\phi(A) = UA^tU^*$ for all $A \in \mathcal{B}(H_1)$. Where A^t is the transpose of A with respect to an arbitrarily fixed basis.

In particular, if both S_1 and S_2 are the identity, we have

Corollary 4.3. *Let H, K be complex Hilbert spaces and let $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a surjective map. Then*

$$W(\phi(A)^*\phi(B)) = W(A^*B) \tag{4.3}$$

for all $A, B \in \mathcal{B}(H)$ if and only if there exist unitary operators U and V in $\mathcal{B}(H_1, H_2)$ such that ϕ is of the form

$$\phi(A) = UAV^*$$

for all $A \in \mathcal{B}(H)$.

Corollary 4.4. *Let H, K be complex Hilbert spaces and let ϕ be a surjective map from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$. Then*

$$W(\phi(B)\phi(A)^*\phi(B)) = W(BA^*B) \tag{4.4}$$

for all $A, B \in \mathcal{B}(H)$ if and only if there exists a unitary operator U such that either $\phi(A) = UAU^*$ for all A, or $\phi(A) = UA^tU^*$ for all A.

So, Eq. (4.3) and Eq. (4.4) give a characterization of $*$-isomorphisms multiplied by a unitary and a characterization of C*-isomorphisms between $\mathcal{B}(H)$ and $\mathcal{B}(K)$, respectively.

We give a proof of Theorem 4.2 here. The proof of Theorem 4.1 is similar.

Proof of Theorem 4.2. It is clear that we need only check the "only if" part. Assume that ϕ satisfies the Eq. (4.2). For $A \in \mathcal{B}(H_i)$, $T \in \mathcal{B}(H_1, H_2)$ and $S \in \mathcal{B}(H_2, H_1)$, we denote $A^\dagger = S_i^{-1}A^*S_i$, $T^\dagger = S_1^{-1}T^*S_2$ and $S^\dagger = S_2^{-1}S^*S_1$, respectively. It is clear that $\dagger$ is an involution, i.e., $(B^\dagger)^\dagger = B$, $(BC)^\dagger = C^\dagger B^\dagger$, $(\alpha B + C)^\dagger = \bar{\alpha}B^\dagger + C^\dagger$ and $(B^{-1})^\dagger = (B^\dagger)^{-1}$. Thus the equation (4.2) becomes

$$W(\phi(B)\phi(A)^\dagger\phi(B)) = W(BA^\dagger B) \tag{4.5}$$

for every $A, B \in \mathcal{B}(H_1)$ and ϕ satisfies Eq. (4.5).

Similar to the proof of Theorem 2.2 in Section 2, we can show that ϕ is a linear bijection preserving rank-one operators in both directions. Thus, either

(i) there exist bijective linear operators $U : H_1 \to H_2$ and $V : H_2 \to H_1$ such that $\phi(x \otimes f) = Ux \otimes Vf$ $(\forall x, f \in H_1)$; or

(ii) there exist bijective conjugate linear operators $U : H_1 \to H_2$ and $V : H_2 \to H_1$ such that $\phi(x \otimes f) = Uf \otimes Vx$ $(\forall x, f \in H_1)$.

Suppose first that ϕ takes the form (i). Let $A = B = I$ in (4.5), we get $\phi(I)\phi(I)^\dagger\phi(I) = I$. So $\phi(I)$ is invertible and $\phi(I)^\dagger = \phi(I)^{-2}$.

Let $B = I$ and $A = x \otimes f$ in (4.5), we get

$$\begin{aligned} & W(\phi(I){S_2}^{-1}(Vf \otimes Ux)S_2\phi(I)) = W(\phi(I)\phi(x \otimes f)^\dagger\phi(I)) \\ = \quad & W((x \otimes f)^\dagger) = W({S_1}^{-1}(f \otimes x)S_1). \end{aligned}$$

So $\langle \phi(I){S_2}^{-1}Vf, \phi(I)^*S_2Ux\rangle = \langle {S_1}^{-1}f, S_1x\rangle = \langle f, x\rangle$. It follows that U and V are bounded, and $(\phi(I)^*S_2U)^*\phi(I){S_2}^{-1}V = I$, i.e.,

$$U^*S_2\phi(I)^2{S_2}^{-1}V = I.$$

As $\phi(I)^\dagger = \phi(I)^{-2}$, we have $\phi(I) = UV^*$, and consequently,

$$\phi(I)^\dagger = \phi(I)^{-2} = (V^*)^{-1}U^{-1}(V^*)^{-1}U^{-1}.$$

Taking $A = I$ and $B = x \otimes f$ $(x, f \in H_1)$ in Eq. (4.5), we get

$$W((Ux \otimes Vf)\phi(I)^\dagger(Ux \otimes Vf)) = W(\phi(x \otimes f)\phi(I)^\dagger\phi(x \otimes f)) = W((x \otimes f)(x \otimes f)).$$

This yields

$$\langle \phi(I)^\dagger Ux, Vf\rangle\langle Ux, Vf\rangle = \langle x, f\rangle^2$$

for every $x, f \in H_1$. So $\langle V^*\phi(I)^\dagger Ux, f\rangle\langle V^*Ux, f\rangle = \langle x, f\rangle^2$, that is,

$$\langle U^{-1}(V^*)^{-1}x, f\rangle\langle V^*Ux, f\rangle = \langle x, f\rangle^2$$

for every $x, f \in H_1$. By Lemma 2.5 in Section 2 we see that there exists a $\lambda \in \mathbb{C}$ such that $V^*U = \lambda I$. Hence $V^* = \lambda U^{-1}$, $\phi(I) = UV^* = \lambda I$ and $\overline{\lambda} I = \phi(I)^\dagger = \phi(I)^{-2} = \lambda^{-2}I$. Since $|\lambda|^2\lambda = 1$, we get $\lambda = 1$. It follows that

$$\phi(x \otimes f) = U(x \otimes f)U^{-1} \tag{4.6}$$

for every $x \otimes f \in \mathcal{B}(H_1)$.

Now, let $A \in \mathcal{B}(H_1)$. For any rank-one operator $B = x \otimes f$, Eq. (4.5) gives that

$$\begin{aligned} & W(U(x \otimes f)U^{-1}\phi(A)^\dagger U(x \otimes f)U^{-1}) \\ = \ & W(\phi(x \otimes f)\phi(A)^\dagger\phi(x \otimes f)) = W((x \otimes f)A^\dagger(x \otimes f)). \end{aligned}$$

Thus we have $\mathrm{tr}((x \otimes f)U^{-1}\phi(A)^\dagger U(x \otimes f)) = \mathrm{tr}((x \otimes f)A^\dagger(x \otimes f))$ for every $x \otimes f$, and by Lemma 2.4 in Section 2, we get $U^{-1}\phi(A)^\dagger U = A^\dagger$. So we have shown that

$$\phi(A) = (U^{-1})^\dagger A U^\dagger = (U^\dagger)^{-1} A U^\dagger \tag{4.7}$$

holds for every $A \in \mathcal{B}(H_1)$. Eqs. (4.6) and (4.7) together yield that

$$\phi(x \otimes f) = (U^{-1})^\dagger(x \otimes f)U^\dagger = U(x \otimes f)U^{-1}$$

for every x,$f \in H_1$. So $(U^{-1})^\dagger x$ and Ux are linearly dependent for every $x \in H_1$, and then, there exists a $\mu \in \mathbb{R} \setminus \{0\}$ such that $UU^\dagger = \mu I$.

In Eq. (4.5), let $A = I$ and $B = x \otimes x$, where $x \in H_1$ with $\|x\| = 1$. Then $[0, 1] = W(x \otimes x) = W((U^\dagger)^{-1}(x \otimes x)U^\dagger)$, this entails that $(U^\dagger)^{-1}x$ and $(U^\dagger)^*x$ are linearly dependent. So there exists a scalar $\alpha > 0$ such that $(U^\dagger)^*U^\dagger = \alpha I$. This implies that there is a unitary operator U_1 and an $\alpha_0 \in \mathbb{R}$ such that ${\alpha_0}^2 = \alpha$ and $(U^\dagger)^* = \alpha_0 U_1$. Thus we have

$$\phi(A) = U_1 A {U_1}^*$$

for every $A \in \mathcal{B}(H_1)$. Note that $\mu I = UU^\dagger = {\alpha_0}^2({U_1}^*)^\dagger {U_1}^* = \alpha({U_1}^*)^\dagger {U_1}^*$. Let $c = \alpha\mu^{-1}$; then ${U_1}^\dagger U_1 = cI$, i.e.,

$$S_1^{-1}U_1^*S_2U_1 = cI.$$

So ϕ has the form stated in Theorem 4.2 (1).

Assume that the case (ii) occurs. We will show that ϕ has the form stated in Theorem 4.2(2). Note that U and V are conjugate linear bijections, a similar argument as that in the beginning of case (i) shows that both U and V are bounded, $\phi(I)^\dagger = \phi(I)^{-2}$ and $\phi(I) = UV^*$. Let $A = I$ and $B = x \otimes f$ in Eq. (4.5), then we get

$$W((Uf \otimes Vx)\phi(I)^\dagger(Uf \otimes Vx)) = W(\phi(x \otimes f)\phi(I)^\dagger\phi(x \otimes f)) = W((x \otimes f)(x \otimes f)).$$

This implies that

$$\langle \phi(I)^\dagger Uf, Vx\rangle\langle Uf, Vx\rangle = \langle x, f\rangle^2$$

for every $x, f \in H_1$. So $\langle x, V^*\phi(I)^\dagger Uf\rangle\langle x, V^*Uf\rangle = \langle x, f\rangle^2$, that is,

$$\langle x, U^{-1}(V^*)^{-1}f\rangle\langle x, V^*Uf\rangle = \langle x, f\rangle^2$$

for every $x, f \in H_1$. By Lemma 2.5 we see that there exists a $\lambda \in \mathbb{C}$ such that $V^*U = \lambda I$. Hence $V^* = \lambda U^{-1}$, $\phi(I) = UV^* = \lambda I$ and $\overline{\lambda} I = \phi(I)^\dagger = \phi(I)^{-2} = \lambda^{-2} I$. It follows that $\lambda = 1$ and

$$\phi(x \otimes f) = Uf \otimes (U^*)^{-1} x \tag{4.8}$$

for every $x \otimes f \in \mathcal{B}(H_1)$. Let $A \in \mathcal{B}(H_1)$. For any rank-one operator $x \otimes f$, Eqs. (4.5) and (4.8) together imply that

$$W((Uf \otimes (U^*)^{-1}x)\phi(A)^\dagger(Uf \otimes (U^*)^{-1}x)) = W((x \otimes f)A^\dagger(x \otimes f)),$$

and hence

$$\mathrm{tr}((x \otimes f)U^*(\phi(A)^\dagger)^*(U^*)^{-1}(x \otimes f)) = \mathrm{tr}((x \otimes f)A^\dagger(x \otimes f)).$$

Applying Lemma 2.4, we see that $U^*(\phi(A)^\dagger)^*(U^*)^{-1} = A^\dagger$, i.e.,

$$U^* S_2 \phi(A) S_2^{-1} (U^*)^{-1} = S_1^{-1} A^* S_1.$$

Thus we have proved that

$$\phi(A) = U_1 A^* U_1^{-1} \tag{4.9}$$

for every $A \in \mathcal{B}(H_1)$, where $U_1 = (S_1 U^* S_2)^{-1}$ is a conjugate linear operator. Then, for any $x \otimes f$ we have

$$(U_1 f \otimes x)U_1^{-1} = \phi(x \otimes f) = Uf \otimes (U^*)^{-1}x,$$

this implies that $U_1 f$ is linearly dependent of Uf for every $f \in H_1$. Thus we must have $U_1 = \alpha U$ and $\phi(x \otimes f) = U_1 f \otimes (U_1^*)^{-1} x$ for every $x \otimes f$.

Now, for every unit vector $x \in H_1$, since

$$\begin{aligned} [0,1] = \quad & W(x \otimes x) = W(\phi(x \otimes x)\phi(I)^\dagger \phi(x \otimes x)) \\ = \quad & W(U_1(x \otimes x)U_1^{-1}U_1(x \otimes x)U_1^{-1}) = W(U_1 x \otimes (U_1^*)^{-1}x), \end{aligned}$$

we see that $(U_1^*)^{-1} = \beta U_1$ for some $\beta > 0$. Let $V_1 = \sqrt{\beta} U_1$. Then V_1 is conjugate unitary and

$$\phi(A) = V_1 A^* V_1^*$$

for all $A \in \mathcal{B}(H_1)$. Equivalently, there exists a unitary operator $U_2 : H_1 \to H_2$ such that

$$\phi(A) = U_2 A^t U_2^* \tag{4.10}$$

for all A, where A^t is the transpose of A with respect to an arbitrarily fixed orthonormal basis. By substituting Eq. (4.10) in Eq. (4.2) with B being rank one, and noting that $W(T^t) = W(T)$, it is easily checked that

$$\mathrm{tr}(B^t U_2^* S_2^{-1} U_2 (A^t)^* U_2^* S_2 U_2 B^t) = \mathrm{tr}(B^t (S_1^{-1} A^* S_1)^t B^t).$$

By Lemma 2.4 again, we get $U_2^* S_2^{-1} U_2 (A^t)^* U_2^* S_2 U_2 = (S_1^{-1} A^* S_1)^t$. This implies that there is a scalar c such that $U_2^* S_2 U_2 = c(S_1^t)^{-1}$. So $S_1^t U_2^* S_2 U_2 = cI$ as desired. The proof is finished. □

5. Remarks and questions

Before concluding this paper we give some remarks and propose a question.

Remark 5.1. The results in this paper are still valid if we replace the numerical range by the closure of numerical range. Moreover, Theorems 2.1 and 2.2 are still true if we replace $\mathcal{B}(H)$ and $\mathcal{B}(K)$ by standard operator algebras $\mathcal{A}$ on H and $\mathcal{B}$ on K, respectively. Recall that a standard operator algebra on H is a subalgebra of $\mathcal{B}(H)$ which contains the identity I and all finite rank operators. Similarly, Theorems 4.1 and 4.2 still hold if we replace $\mathcal{B}(H_i)$ ($\mathcal{B}(H)$ and $\mathcal{B}(K)$) by standard †-operator algebras $\mathcal{A}_i$ on H_i ($\mathcal{A}$ on H and $\mathcal{B}$ on K, resp.).

Remark 5.2. Based on the results and the methods in this paper, it is not difficult to classify the surjective maps that preserve numerical ranges of k-products (or skew k-products) of operators with $k > 2$ of the form $A_1^{\tau_1} A_2^{\tau_2} \cdots A_k^{\tau_k}$, where $A_i^{\tau_i} = A_i$ or A_i^*. For example, let $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a surjection such that

$$W(\phi(A_1)\phi(A_2)\ldots\phi(A_k)) = W(A_1 A_2 \ldots A_k)$$

for all $A_1, A_2, \ldots, A_k \in \mathcal{B}(H)$. Then particularly we have $W(\phi(I)^{k-2}\phi(A)\phi(B)) = W(\phi(A)\phi(B)\phi(I)^{k-2}) = W(AB)$ for all A, B. Similar to the proof of Theorem 2.2, it is easily checked that $\phi(I)$ is invertible, ϕ is linear and preserves rank-one operators in both directions. It follows that $\phi(I)$ is a multiple of the identity I, and there exists a unitary operator U and a kth root λ of 1 such that $\phi(A) = \lambda U A U^*$ for all $A \in \mathcal{B}(H)$.

It is natural and interesting to ask similar questions as in this paper for a more general case of the numerical radius.

Question 5.3. How do we classify the maps which preserve the numerical radius of operator products or operator Jordan triple-products?

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