

**ELLIPTIC OPERATORS ON PLANAR GRAPHS:
UNIQUE CONTINUATION FOR EIGENFUNCTIONS
AND NONPOSITIVE CURVATURE**

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ABSTRACT. This paper is concerned with elliptic operators on plane tessellations. We show that such an operator does not admit a compactly supported eigenfunction if the combinatorial curvature of the tessellation is nonpositive. Furthermore, we show that the only geometrically finite, repetitive plane tessellations with nonpositive curvature are the regular $(3, 6)$, $(4, 4)$ and $(6, 3)$ tilings.

1. INTRODUCTION

Since the work of Aronszajn [Aro], unique continuation for elliptic operators on Riemannian manifolds and Schrödinger operators has been investigated in very many papers of which we only mention [Ag, ABG, H, JK]. It was shown under rather general assumptions that a function f on a connected Riemannian manifold vanishes identically whenever it vanishes in one point to infinite order and satisfies

$$(1) \quad Hf = 0$$

where H is an elliptic operator. In particular, f satisfying (1) must vanish identically if it vanishes on the nonempty complement of a compact set.

Despite all analogies between elliptic operators on graphs and those on manifolds, unique continuation does not hold on graphs. In fact, it is rather easy to give examples of elliptic operators on graphs with compactly supported, i.e. finitely supported, eigenfunctions. These examples have recently attracted some attention since they play a role in the investigation of the so-called integrated density of states for random operators [DLMSchY, KLS, Ves].

More precisely, in [DLMSchY], Dodziuk et al. study certain periodic Laplacians on graphs viz Laplacians on infinite graphs, which are coverings of finite graphs by amenable groups. They show that the eigenvalues of the operator on the whole graph are already eigenvalues of suitable restrictions to finite graphs. Moreover, they relate the points of discontinuity of the integrated density of states to existence of compactly supported eigenfunctions.

Independently, existence of compactly supported eigenfunctions was studied by three of the authors in [KLS] for certain aperiodically ordered graphs. This study

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does not only give examples of compactly supported eigenfunctions, but characterizes discontinuities of the integrated density of states by occurrence of compactly supported eigenfunctions.

More recently, there is related work of Veselić [Ves]. This work characterizes discontinuities of the integrated density of states by occurrence of compactly supported eigenfunctions for certain random graphs, including the periodic case of coverings of finite graphs by amenable groups.

While a common framework to operators on these three classes is still missing, the link between discontinuities of the integrated density of states and the occurrence of compactly supported eigenfunctions is by now well established.

The aim of this paper is to investigate combinatorial conditions on the graph which guarantee nonexistence of compactly supported eigenfunctions.

In this context, the only result available so far is due to Delyon/ Souillard [DS]. They show absence of compactly supported eigenfunctions for random Schrödinger operators on the d -dimensional lattice and use this to conclude continuity of the corresponding integrated density of states.

Here, we will restrict our attention to graphs tessellating the plane and establish a connection between the absence of compactly supported eigenfunctions of an elliptic operator and the combinatorial curvature of the graph introduced in [BP1]. Our main result uses the geometric/combinatorial methods developed in [BP1, BP2] and states the following:

Result 1. *If the curvature of the plane tessellation \mathcal{G} is nonpositive, then no elliptic operator on \mathcal{G} admits a compactly supported eigenfunction.*

Note that our result implies Delyon/Souillard's result in the particular case of a two-dimensional lattice.

While the result a priori applies to general plane tessellating graphs, it is of limited use when it comes to application to geometrically more rigid tessellations with certain repetitivity properties (as are the ones encountered in [KLS]). Namely, our second result shows:

Result 2. *If a geometrically finite, repetitive plane tessellation has nonpositive curvature, then it coincides with one of the three regular combinatorial tessellations $(3, 6)$, $(4, 4)$ or $(6, 3)$.*

2. NOTATION AND RESULTS

In this section we introduce the model that we consider and present our results.

A planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ is a graph which is embedded in the plane \mathbb{R}^2 . Here, \mathcal{V} denotes the set of vertices and \mathcal{E} the set of edges of \mathcal{G} . \mathcal{F} denotes the family of connected components of the complement of the image of \mathcal{G} in \mathbb{R}^2 . The elements of \mathcal{F} are called *faces* of \mathcal{G} . We will always assume that our graphs are compactly finite, i.e. every point in \mathbb{R}^2 has a neighbourhood which meets only finitely many faces.

We call a face $f \in \mathcal{F}$ a *polygon* if \bar{f} is homeomorphic to a closed disc and its boundary defines a simply closed path of vertices. In this case, the boundary is called the *boundary cycle* ∂f of f . The edges which belong to ∂f are called the *sides* of f . The number of sides of f is denoted by $E_{\partial f}$. If the polygon f has k sides it is called a k -gon. The number of edges emanating from a vertex $v \in \mathcal{V}$ is

called the *degree of v* , denoted by $|v|$. Two vertices are called *adjacent* if they are connected by an edge. We write $v \sim w$ if v and w are adjacent vertices.

Definition 1. A planar graph \mathcal{G} in \mathbb{R}^2 is called *tessellating* if the following conditions are satisfied:

- i) Any edge is a side of precisely two different faces.
- ii) Any two faces are disjoint or have precisely either a vertex or a side in common.
- iii) Any face $f \in \mathcal{F}$ is a polygon with finitely many sides.
- iv) Every vertex has finite degree.

In order to present our results we have to introduce the notions of combinatorial curvature and of elliptic operators.

A *corner* of a tessellating graph is a pair $(v, f) \in \mathcal{V} \times \mathcal{F}$ such that $v \in \partial f$. The set of all corners of \mathcal{G} is denoted by $\mathcal{C} := \mathcal{C}(\mathcal{G})$.

Definition 2. Let \mathcal{G} be a plane tessellation. Then the function $\kappa : \mathcal{C} \rightarrow \mathbb{R}$ defined by

$$\kappa(v, f) := \frac{1}{|v|} + \frac{1}{E_{\partial f}} - \frac{1}{2}$$

is called the *curvature* (on the graph \mathcal{G}). The graph \mathcal{G} is said to have nonpositive curvature if $\kappa(v, f) \leq 0$ for every $(v, f) \in \mathcal{C}$.

Let $C(\mathcal{V})$ be the vector space of all complex-valued functions on \mathcal{V} . Let $C_c(\mathcal{V})$ be the subspace of $C(\mathcal{V})$ consisting of functions which vanish outside a finite set of vertices.

Definition 3. A linear operator $L : C(\mathcal{V}) \rightarrow C(\mathcal{V})$ is called *elliptic* if its matrix $a : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ satisfies $a(v, w) \neq 0$ whenever $v \sim w$ and $a(v, w) = 0$ whenever $v \not\sim w$ and $v \neq w$.

Thus an elliptic operator L is of the form

$$Lu(v) := \sum_{w \in \mathcal{V}} a(v, w)u(w) = a(v, v)u(v) + \sum_{w \sim v} a(v, w)u(w)$$

for every vertex v in \mathcal{G} . A well-studied example is the nearest neighbour Laplacian where $a(v, v) = |v|$ for all $v \in \mathcal{V}$ and $a(v, w) = 1$ whenever $v \sim w$.

As usual, a nonvanishing function $u \in C(\mathcal{V})$ is called an eigenfunction of the elliptic operator L for the eigenvalue λ if $Lu - \lambda u = 0$, i.e. if

$$\sum_{w \sim v} a(v, w)u(w) = (\lambda - a(v, v))u(v) \quad \text{for all } v \in \mathcal{V}.$$

The precise version of our main result now reads as follows.

Theorem 4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation with nonpositive curvature and L an elliptic operator on $C(\mathcal{V})$. Then L does not admit an eigenfunction in $C_c(\mathcal{V})$.*

Note that the above theorem takes only the combinatorial structure of the tessellation into account and that the precise geometric shape of the faces is of no importance. Our next result, however, is concerned with geometrically more rigid tessellations \mathcal{G} . We assume that \mathcal{G} is built up by copies of finitely many fixed geometric tiles $f_1, \dots, f_N \in \mathcal{F}$ and that every finite configuration of tiles in \mathcal{G} can be found repeatedly in any sufficiently large Euclidean ball.

Definition 5. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation. Then \mathcal{G} is called *geometrically finite*, if there are finitely many faces $\{f_1, \dots, f_N\} \subset \mathcal{F}$, the *generators* of \mathcal{G} , such that every $f \in \mathcal{F}$ is an isometric image of one of the generators, i.e., $f = t + Af_j$, for $j \in \{1, \dots, N\}$, $t \in \mathbb{R}^2$ and $A \in SO(2)$. A geometrically finite tessellation \mathcal{G} is called *repetitive*, if for every finite set of faces $\{g_1, g_2, \dots, g_k\} \subset \mathcal{F}$ there is an $R > 0$ with the following property: In every Euclidean ball $B_R(x) \subset \mathbb{R}^2$ there are k faces $\hat{g}_1, \hat{g}_2, \dots, \hat{g}_k \in \mathcal{F}$ and $t \in \mathbb{R}$, $A \in SO(2)$, such that $\hat{g}_j = t + Ag_j$, for $j = 1, \dots, k$.

Theorem 6. A geometrically finite, repetitive plane tessellation \mathcal{G} of nonpositive curvature coincides with one of the three regular combinatorial tessellations (3, 6), (4, 4) or (6, 3).

The three regular tessellations are illustrated in [GS, Figure 1.2.1].

3. GEOMETRY OF PLANE TESSELLATIONS

In this section we discuss some geometric aspects of plane tessellations following [BP1, BP2]. Moreover, for nonpositively curved plane tessellations we prove the impossibility of a certain vertex labeling of the boundary cycle of distance balls.

Let a plane tessellation $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be given.

Two faces in \mathcal{F} are called *neighbours* if they have an edge in common. A sequence (f_0, f_1, \dots) of faces is a (*connected*) *path* if any two consecutive faces f_j, f_{j+1} are neighbours. If this sequence is finite, i.e. of the form (f_0, f_1, \dots, f_n) , the length of this path is defined to be n . The (*combinatorial*) *distance* $d(f, g)$ of two faces f and g is the smallest number n for which there exists a path (f_0, f_1, \dots, f_n) with $f_0 = f$ and $f_n = g$.

Similarly, for a finite set P in \mathcal{F} and an arbitrary $f \in \mathcal{F}$, we define $d(f, P) := \min\{d(f, g) : g \in P\}$. Thus, for each finite set of faces P and any $k \in \mathbb{N}_0$ we can define the k -neighbourhoods of P by

$$B_k(P) := \{f \in \mathcal{F} \mid d(f, P) \leq k\}.$$

A special example of much relevance in our considerations arises if P consists of only one element: For a given face f_0 and $k \in \mathbb{N}_0$, we define the *distance balls*

$$(2) \quad B_k(f_0) = \{f \in \mathcal{F} \mid d(f, f_0) \leq k\}.$$

Accordingly, we define the *distance spheres* by

$$(3) \quad A_k(f_0) = \{f \in \mathcal{F} \mid d(f, f_0) = k\}.$$

Apparently, each distance ball arises from the previous distance ball by adding a distance sphere.

It turns out that this gives a very well-behaved “onion-like” layered structure on the distance balls, provided \mathcal{G} has nonpositive curvature. This layered structure is the crucial ingredient in the proofs of our main results. It was analyzed in detail in [BP1, BP2].

In order to discuss the relevant results we need some more definitions.

A finite subset P of \mathcal{F} is called a *polygon* if $\bigcup_{f \in P} \bar{f} \subset \mathbb{R}^2$ is homeomorphic to a closed disc. Then, ∂P denotes the cycle of boundary vertices. If P is a polygon and v belongs to ∂P , we define the *inner degree of V with respect to P* by

$$|v|_P^i := \text{number of faces of } f \in P \text{ meeting in } v$$

and the exterior degree of v with respect to P by

$$|v|_P^e := |v| - |v|_P^i.$$

We introduce now a labeling of the boundary vertices of the polygon P with the letters a and b . First, for any vertex v of \mathcal{G} , we define

$$N(v) = \min\{E_{\partial f} \mid f \in \mathcal{F} : v \in \partial f\}.$$

A vertex $v \in \partial P$ obtains label a if $|v|_P^i = 1$, or if both $N(v) = 3$ and $|v|_P^i \leq 3$ hold. All other vertices of ∂P obtain label b . Moreover, if v is an a -vertex satisfying $|v|_P^i = 1$, then we say that v is of type a^+ . One should think of the labeling as a sort of book-keeping of convexity: a^+ -vertices of ∂P are considered to be particularly convex and b -vertices to be particularly concave.

The following proposition gives a simple fact about this labeling.

Proposition 7. *Let \mathcal{G} be a plane tessellation with nonpositive curvature. Let P be a polygon. If $v \in \partial P$ satisfies $|v|_P^e = 1$, then no edge starting in v belongs to $\partial B_1(P)$. Moreover, in this case v has a b -label.*

Proof. The first statement is obvious from the definition of the exterior degree. If v were an a -vertex, then it would satisfy $|v| \leq 4$ and it would be adjacent to a triangle, in contradiction to nonpositive curvature. \square

We say $w = (v_0, v_1, v_2, \dots, v_k)$ is a connected (vertex-)path of length $|w| = k$ in \mathcal{G} if all consecutive vertices of w are connected by an edge. A connected path $w \subset \partial P$ is called *admissible* if for every vertex $v \in w$ with label b its neighbours in w carry the label a^+ . We call the polygon P *admissible* iff ∂P is admissible. Thus, admissible polygons have the property that particularly concave vertices are compensated by particularly convex vertices in their neighbourhood.

It turns out that admissibility is preserved under taking k -neighbourhoods if the curvature is nonpositive. More precisely, the following is proven in Proposition 2.5 and Proposition 2.6 of [BP2]:

Lemma 8. *Let \mathcal{G} be a plane tessellation with nonpositive curvature. Let P be an admissible polygon. Then the set $B_1(P)$ is an admissible polygon. Moreover, for every face $f \in B_1(P) - P$,*

- (a) $\partial f \cap \partial P$ is a connected path of edges of length ≤ 2 and
- (b) $\partial f \cap \partial B_1(P)$ is a connected path of length ≥ 1 .

Apparently, every face $f \in \mathcal{F}$ is admissible. Thus, the lemma immediately implies (see Corollary 2.7 in [BP2] as well):

Lemma 9. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation with nonpositive curvature. Let $f \in \mathcal{F}$ be an arbitrary face. Then, every distance ball $B_k(f)$ is an admissible polygon and every face of the distance sphere $A_k(f)$ contributes at least one edge to the boundary $\partial B_k(f)$.*

Now, the main result of [BP2] is the following combinatorial analogue of the Hadamard-Cartan theorem in differential geometry. Note, that it can be seen as describing a nice layered structure of \mathcal{G} .

Theorem 10. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation. For a given face $f \in \mathcal{F}$, we define the cut locus $C(f) \subset \mathcal{F}$ of f in the metric space (\mathcal{F}, d) to be the set*

$$C(f) := \{g \in \mathcal{F} \mid d(f', f) \leq d(g, f) \text{ for all neighbours } f' \text{ of } g\},$$

i.e., the set of all faces on which the distance function $d_f(g) = d(f, g)$ attains a local maximum. If \mathcal{G} has nonpositive curvature, then $C(f) = \emptyset$ for all $f \in \mathcal{F}$.

This theorem has the following consequence.

Lemma 11. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation with nonpositive curvature. Let $f \in \mathcal{F}$ be an arbitrary face. Then the boundary $\partial B_k(f)$ imposes a cyclic enumeration (f_1, \dots, f_n) of $A_{k+1}(f)$ such that faces intersect if and only if they are consecutive and in this case each intersection contains a vertex $v \in \partial B_k(f)$.*

Proof. This follows from Theorem 3.2 of [BP1] and the fact that \mathcal{G} has no cut locus by the previous theorem. \square

We are heading towards the result on the vertex labeling mentioned in the beginning of this section.

We will use the following lemma from [BP2]. The lemma is rather technical but very important for the proof of Theorem 4.

Lemma 12 (Lemma 2.8 in [BP2]). *Let \mathcal{G} be a plane tessellation with nonpositive curvature and let $A_k(f) = \{f_1, f_2, \dots, f_n\}$ be a distance sphere in \mathcal{G} with the cyclic enumeration of Lemma 11. Then there is at least one face $f_j \in A_k(f)$ with one of the following properties: either $|\partial f_j \cap \partial B_{k-1}(f)| = 1$ or f_j does not share edges with both f_{j-1} and $f_{j+1} \pmod n$.*

The following proposition is a key step in our proof. It may be of independent interest.

Proposition 13. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation with nonpositive curvature and let B_k be a distance ball for a given $f_0 \in \mathcal{F}$ with the closed simple sequence of vertices $v_0, v_1, v_2, \dots, v_{l-1}, v_l = v_0$ describing ∂B_k .*

If l is even and every b -vertex on ∂B_k satisfies $|v|_{B_k}^e = 1$, then the closed simple label-sequence of vertices describing ∂B_k is not of the form $a^+, b, a^+, b, a^+, b, \dots, a^+, b$.

Proof. Note that $\partial B_0 = \partial f_0$ is of the form a^+, a^+, \dots, a^+ . Let $k \geq 1$ and assume that the label sequence of B_k is of the form a^+, b, \dots, a^+, b . Then, for every $f \in A_k := A_k(f_0)$, $\partial f \cap \partial B_k$ is a connected path of length ≤ 2 (for otherwise there would be at least two consecutive vertices of label a^+). By Lemma 11, the faces of A_k can be enumerated so that only faces with consecutive indices intersect. We call faces in A_k with such a nonempty intersection “neighbours in A_k ”.

Case 1: There is a b -vertex v of ∂B_k satisfying $|v|_{B_k}^i = 2$.

As $|v|_{B_k}^e = 1$, we have $|v| = 3$. Thus, nonpositivity of the curvature implies that all faces adjacent to v have to be at least 6-gons. Let $f \in B_k$ be a face with $v \in \bar{f}$. Note that f must belong to A_k , as $|v| = 3$. Therefore, the edges of f fall into three different types: those which belong to ∂B_{k-1} , those which belong to ∂B_k and those which are a common edge with one of the two neighbours of f in A_k .

As all distance balls are admissible, Lemma 8 (a) gives $|\partial f \cap \partial B_{k-1}| \leq 2$. Together with the fact $|\partial f \cap \partial B_k| \leq 2$ we conclude that f has to be a 6-gon, sharing an edge with both its neighbours. Moreover, both b -vertices of ∂B_k belonging to \bar{f} satisfy $|v|_{B_k}^i = 2$.

By repetition of the above arguments this implies that, consecutively, every face of A_k has to be a 6-gon, sharing an edge with both its neighbours. This situation is not possible, by Lemma 12.

Case 2 : Every b -vertex of ∂B_k satisfies $|v|_{B_k}^i \geq 3$.

This means that none of the faces f in A_k shares a common edge with any of its neighbours in A_k , and since $\partial f \cap \partial B_{k-1}$ is a connected path of at most 2 edges (see Lemma 8 (a)), all faces $f \in A_k$ are at most 4-gons.

Case 2.1 : A_k contains a 4-gon f .

The face f contributes an edge to the boundary of B_k and therefore at least one of its edges carries an a^+ label. Thus f contributes precisely two edges to the boundary of B_k and there exists a unique vertex v of ∂f which does not belong to ∂B_k .

Assume that v is an a -vertex with respect to the labeling of ∂B_{k-1} . Then v is adjacent to a triangle and we have $|v|_{B_{k-1}}^i \geq 5$, for curvature reasons, which contradicts the fact that v has label a .

Hence, v is a b -vertex with respect to B_{k-1} . The neighbours of v along ∂B_{k-1} are then a^+ -vertices (since B_{k-1} is admissible). Therefore, the two b -vertices v', v'' with respect to B_k which belong to f are also a^+ -vertices with respect to B_{k-1} , i.e., we have $|v'|_{B_k}^i = 3$. Thus we have $|v'| = 4$ because of $|v'|_{B_k}^e = 1$, and the neighbour f' of f in A_k with $v' \in \partial f'$ cannot be a triangle, for curvature reasons.

This shows that all faces in A_k are 4-gons and that we have, again, for B_{k-1} the situation that the vertices of ∂B_{k-1} are labeled as $a^+, b, a^+, b, \dots, a^+, b$ and all b -vertices satisfy $|v|_{B_{k-1}}^e = 1$ and $|v|_{B_{k-1}}^i \geq 4$, for curvature reasons. Again, by our curvature restriction, A_{k-1} must contain a 4-gon, since there are vertices $v' \in \partial B_k \cap \partial B_{k-1}$ with $|v'| = 4$. Thus we may apply induction and conclude that $B_0 = \{f_0\}$ is a 4-gon with label-sequence a^+, b, a^+, b , which is a contradiction.

Case 2.2 : A_k consists only of triangles.

The label sequence forces each triangle to contribute two edges to the boundary of B_k . Let f be a triangle in A_k . For curvature reasons, the two b -vertices (with respect to B_k) of f satisfy $|v|_{B_{k-1}}^i \geq 3$ (note that $|v|_{B_k}^e = 1$). This means that the face f' of B_{k-1} which shares an edge with f does not share a common edge with both of its neighbours in A_{k-1} and that $|\partial f \cap \partial B_{k-1}| = 1$. Since $\partial f' \cap \partial B_{k-2}$ is a connected path of at most 2 edges, by Lemma 8 (a), f' has to be, again, a triangle.

Note that, by curvature reasons, the unique vertex of f' which is not also a vertex of f has to be a b -vertex with respect to B_{k-2} . Moreover, by admissibility of B_{k-2} , its label sequence is, again, given by $a^+, b, a^+, b, \dots, a^+, b$, and all b -vertices of ∂B_{k-2} satisfy $|v|_{B_{k-2}}^e = 0$. Now, B_{k-2} satisfies, again, the conditions of the proposition. Since neither Case 1 nor Case 2.1 can be given, we conclude that, again, Case 2.2 is given for A_{k-2} , namely, A_{k-2} consists only of triangles. We can repeat the same arguments inductively.

In the case that k was even, induction leads to $B_0 = \{f_0\}$ being a single triangle. Its three vertices would have to be labeled as a^+, b, \dots, a^+, b , which is not possible for parity reasons. In the case that k was odd, we end up with B_1 consisting of the center face f_0 and triangles attached to each of the edges of f_0 . The property $|v|_{B_1}^e = 1$ of each b -vertex of ∂B_1 then implies $|v| = 4$, which yields a contradiction to nonpositive curvature. □

4. PROOF OF THEOREM 4

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation with nonpositive curvature. We choose $f_0 \in \mathcal{F}$ and define $B_k := B_k(f_0)$ and $A_k = B_k \setminus B_{k-1}$. Note that $B_0 = \{f_0\}$. Let $u \in C_c(\mathcal{V})$ be a compactly supported eigenfunction.

By Lemma 9, B_k is a polygon for every $k \in \mathbb{N}_0$, i.e., the boundary of B_k defines a simple closed path of vertices. In particular, the boundary is a Jordan curve and thus divides the plane into an interior and an exterior part. Therefore, we can define the set of vertices \mathcal{V}_k by

$$\mathcal{V}_k := \{\text{vertices on } \partial B_k\} \cup \{\text{vertices outside of } B_k\}.$$

Theorem 4 follows if we prove the following two steps:

- i) There exists an $n_0 \in \mathbb{N}$ such that u vanishes on \mathcal{V}_{n_0} .
- ii) If u vanishes on \mathcal{V}_k for some $k \in \mathbb{N}$, then u vanishes on \mathcal{V}_{k-1} as well.

Here, i) is immediate from the local finiteness of the graph. Thus, the main point is to prove ii).

To prove ii), we need the following lemma.

Lemma 14. *Let $u \in C_c(\mathcal{V})$ be an eigenfunction of the elliptic operator L on $C(\mathcal{V})$ with $u|_{\mathcal{V}_{k+1}} \equiv 0$. Then, $u(v) = 0$ for all vertices $v \in \partial B_k$ with $|v|_{B_k}^e > 1$.*

Proof. Let $v \in \partial B_k$. If $|v|_{B_k}^e > 2$, then, obviously, v is also a vertex of ∂B_{k+1} and we have $u(v) = 0$. Thus, we only have to consider the case $|v|_{B_k}^e = 2$:

Choose $f \in A_k$ with $v \in \partial f$. Let $f' \in A_{k+1}$ be a neighbour of f . $|v|_{B_k}^e = 2$ implies that f' shares a common edge e , emanating from v , with a neighbour f'' in A_{k+1} . Assume that e connects the vertices v and v' . Since two faces have at most one edge in common, we conclude that v' belongs to ∂B_{k+1} . Now, v' cannot belong to ∂B_k , for otherwise f' or f'' would be in the cut locus $C(f_0)$ (which is empty by Theorem 10). Now, we inspect the other vertices adjacent to v' .

Case 1: All vertices adjacent to v' and different from v belong to \mathcal{V}_{k+1} .

Then, none of these vertices belongs to ∂B_k . Thus, we conclude from the fact $u(v') = 0$ that

$$0 = (\lambda - a(v', v')) u(v') = a(v', v) u(v) + \underbrace{\sum_{v'' \sim v', v'' \neq v} a(v', v'') u(v'')}_{=0},$$

and thus $u(v) = 0$ as $a(v', v) \neq 0$ by ellipticity.

Case 2: There is a vertex $w \neq v$ adjacent to v' not belonging to \mathcal{V}_{k+1} .

As w is different from v , we infer that $|w|_{B_{k+1}}^i > 2$. Thus, there is at least one face $\hat{f} \in \{f', f''\} \subset A_{k+1}$ which satisfies $v' \in \partial \hat{f}$ and $\partial \hat{f} \cap \partial B_{k+1} = v'$. Therefore, the boundary cycle ∂B_{k+1} does not share an edge with the face $\hat{f} \in A_{k+1}$, and we have $\hat{f} \in C(f_0)$, contradicting Theorem 10. \square

In what follows we let

$$(4) \quad v_0, v_1, v_2, \dots, v_{l-1}, v_l = v_0$$

be the closed simple sequence of vertices describing ∂B_k .

Lemma 15. *Let $u \in C_c(\mathcal{V})$ be a compactly supported eigenfunction of an elliptic operator on L with $u|_{\mathcal{V}_{k+1}} \equiv 0$. If u vanishes for two consecutive vertices of the sequence (4), then u vanishes on all of ∂B_k .*

Proof. Assume that $u(v_j) = u(v_{j+1}) = 0$. The vertex v_{j+2} satisfies either $|v_{j+2}|_{B_k}^e > 1$, in which case we have $u(v_{j+2}) = 0$, by Lemma 14, and we can continue by considering the consecutive vertices v_{j+1}, v_{j+2} . Otherwise, we have $|v_{j+2}|_{B_k}^e = 1$, in which case v_{j+2} carries label b with respect to the polygon B_k by Proposition 7.

Since the distance ball B_k is admissible, it follows that v_{j+1} is an a^+ -vertex, i.e., we have $|v_{j+1}|_{B_k}^i = 1$. This implies that v_j and v_{j+2} are the only vertices in $\overline{B_k}$ which are adjacent to v_{j+1} . Thus, all other vertices adjacent to v_{j+1} belong to \mathcal{V}_{k+1} and u vanishes on them.

Defining $\mathcal{V}'_{k+1} := \mathcal{V}_{k+1} \setminus \{v_j, v_{j+2}\}$, we can therefore calculate

$$\begin{aligned} 0 &= (\lambda - a(v_{j+1}, v_{j+1}))u(v_{j+1}) \\ &= a(v_{j+1}, v_j) \underbrace{u(v_j)}_{=0} + a(v_{j+1}, v_{j+2})u(v_{j+2}) + \underbrace{\sum_{v \sim v_{j+1}, v \in \mathcal{V}'_{k+1}} a(v_{j+1}, v)u(v)}_{=0} \\ &= a(v_{j+1}, v_{j+2})u(v_{j+2}). \end{aligned}$$

Thus, by ellipticity, we conclude that $u(v_{j+2}) = 0$.

Hence, we can also continue with the consecutive vertices v_{j+1}, v_{j+2} in this case. The lemma follows now by iteration. \square

Proof of Theorem 4. We follow our strategy and show that $u|_{\mathcal{V}_{k+1}} \equiv 0$ yields $u|_{\partial B_k} \equiv 0$.

By Lemma 14, u vanishes on all vertices with $|v|_{B_k}^e > 1$. If v is an edge which does not satisfy $|v|_{B_k}^e > 1$, it must satisfy $|v|_{B_k}^e = 1$. Then v carries a b -label by Proposition 7. As B_k is admissible it is then enclosed by two a^+ -vertices v_{j-1} and $v_{j+1} \pmod{l}$. By Proposition 7 again, these vertices have exterior degree at least 2 and then, by Lemma 14, u vanishes on them.

These considerations show that u vanishes at least for every second vertex of ∂B_k . Now, by Lemma 15, there remains only one case for $u|_{\partial B_k} \not\equiv 0$: l in (4) is even and, by the admissibility of distance balls, the corresponding label-sequence is $a^+, b, a^+, b, a^+, b, \dots, a^+, b$, where every a^+ -vertex satisfies $|v|_{B_k}^i = 1$ (by definition) and every b -vertex satisfies $|v|_{B_k}^e = 1$. But this case is impossible by Proposition 13. \square

5. PROOF OF THEOREM 6

The basic idea in the proof of Theorem 6 is that the existence of a face with a negatively curved corner together with repetitivity implies exponential growth of the number of faces in a combinatorial distance ball. On the other hand, geometrical finiteness implies that combinatorial and Euclidean balls are comparable and that the number of faces inside a Euclidean ball can only grow quadratically with the radius. Obviously, the two growth properties are contradictory and \mathcal{G} is forced to have zero curvature.

Assume that $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ is geometrically finite (with generators $f_1, \dots, f_N \in \mathcal{F}$) and repetitive. Nonpositive curvature implies that we have no cut-locus. We prove that $\kappa(v, f) = 0$ for all corners. Assume there is $(v, f) \in \mathcal{C}$ with $\kappa(v, f) < 0$. By repetitivity, there is a constant $C > 0$ and a radius $R > 0$ such that every Euclidean ball $B_R(x) \subset \mathbb{R}^2$ contains a face f with

$$\chi(f) := \sum_{v \in \bar{f}} \kappa(v, f) \leq -C.$$

We choose $f_0 \in \mathcal{F}$ arbitrarily and write B_k for $B_k(f_0)$. By geometrical finiteness, there are constants $0 < d < D$ and a point $x_0 \in \mathbb{R}^2$ such that we have, for large

enough k :

$$B_{kd}(x_0) \subset \bigcup_{f \in B_k} \bar{f} \subset B_{kD}(x_0).$$

By volume comparison, we immediately obtain the following two facts:

- i) There is a constant $c_1 > 0$ such that, for k large enough, $B_{kd}(x_0)$ contains at least $c_1 k^2$ disjoint Euclidean balls of radius R .
- ii) There is a constant $c_2 > 0$ such that, for k large enough, $B_{kD}(x_0)$ contains at most $c_2 k^2$ faces of \mathcal{F} .

Therefore, the mean Euler-characteristic $\bar{\chi}(B_k) := (\sum_{f \in B_k} \chi(f))/|B_k|$ of distance balls B_k satisfies, for all k large enough,

$$\bar{\chi}(B_k) = \frac{1}{|B_k|} \sum_{f \in B_k} \chi(f) \leq -\frac{c_1}{c_2} C,$$

where $|B_k|$ denotes the number of faces in B_k . By the remark on page 156 of [BP1], this implies that $|B_k(f_0)|$ grows exponentially in k , which contradicts ii).

Consequently, the plane tessellation has zero curvature in all corners, and this immediately yields for each corner $(v, f) \in \mathcal{C}$: $(|v|, E_{\partial f}) \in \{(3, 6), (4, 4), (6, 3)\}$. Finally, face to face extension forces \mathcal{G} to be a regular tiling of type $(3, 6), (4, 4)$ or $(6, 3)$.

6. FURTHER REMARKS

In the previous sections we have undertaken initial steps into investigating the geometric situation leading to compactly supported eigenfunctions. We showed that their existence is connected to curvature properties of the underlying graph in the two-dimensional situation. This raises various questions:

- Do similar results hold in arbitrary dimension?
- What are sufficient conditions for existence of compactly supported eigenfunctions?
- Can one develop a general framework of random operators covering the connection between compactly supported eigenfunctions and the discontinuities of the integrated density of states?

We plan to attack these questions in the future.

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REFERENCES

- [Ag] S. Agmon, *Lower bounds for solutions of Schrödinger equations*, J. Analyse Math. **23** (1970), 1–25. MR0276624 (43:2367)
- [ABG] W. O. Amrein, A.-M. Berthier, V. Georgescu, *L^p -inequalities for the Laplacian and unique continuation*, Ann. Inst. Fourier (Grenoble) **31** (1981), 153–168. MR0638622 (83g:35011)
- [Aro] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pure Appl. **36** (1957), 235–249. MR0092067 (19:1056c)
- [BP1] O. Baues, N. Peyerimhoff, *Curvature and geometry of tessellating plane graphs*, Discrete Comput. Geom. **25** (2001), 141–159. MR1797301 (2001k:57004)
- [BP2] O. Baues, N. Peyerimhoff, *Geodesics in Non-Positively Curved Plane Tessellations*, preprint, see <http://www.math.ethz.ch/~oliver/>

- [deV] Y. Colin de Verdiere, *Spectres de Graphes*, Soc. Math. de France, No. 4, 1998. MR1652692 (99k:05108)
- [DS] F. Delyon, B. Souillard, *Remark on the continuity of the density of states of ergodic finite difference operators*, Comm. Math. Phys. **94** (1984), 289–291. MR0761798 (86c:47005)
- [DLMSchY] J. Dodziuk, P. Linnell, V. Mathai, T. Schick, S. Yates, *Approximating L^2 -invariants, and the Atiyah conjecture*, Commun. Pure Appl. Math. **56** (2003), 839–873. MR1990479 (2004g:58040)
- [GS] B. Grünbaum, G. C. Shepard, *Tilings and patterns*, W. H. Freeman and Company, 1987. MR0857454 (88k:52018)
- [H] L. Hörmander, *Uniqueness theorems for second order elliptic partial differential equations*, Comm. Part. Diff. Equations **8** (1983), 21–64. MR0686819 (85c:35018)
- [JK] D. Jerison, C. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math. **121** (1985), 463–488. MR0794370 (87a:35058)
- [KLS] S. Klassert, D. Lenz, P. Stollmann, *Discontinuities of the integrated density of states for random operators on Delone sets*, Comm. Math. Phys. **241** (2003), 235–243. MR2013799 (2004h:82048)
- [Ves] I. Veselić, *Spectral analysis of percolation Hamiltonians*, to appear in: Mathematische Annalen.

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