

## REMARKS ON A PAPER BY CHAO-PING CHEN AND FENG QI

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(Communicated by Carmen C. Chicone)

ABSTRACT. In a recent paper, Chao-Ping Chen and Feng Qi (2005) established sharp upper and lower bounds for the sequence  $P_n := \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}$ . We show that their result follows easily from a theorem of G. N. Watson published in 1959. We also show that the main result of Chen and Qi's paper is a special case of a more general inequality which admits a very short proof.

### 1. INTRODUCTION AND RESULTS

Let

$$P_n := \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}.$$

This sequence appears in Wallis's well-known product formula of approximation of  $\pi$  and in various other topics of analysis and number theory. In the recent paper [2], Chao-Ping Chen and Feng Qi proved the following inequality:

$$(1.1) \quad \frac{1}{\sqrt{\pi(n + \frac{4}{\pi} - 1)}} \leq P_n < \frac{1}{\sqrt{\pi(n + \frac{1}{4})}}, \quad n = 1, 2, \dots$$

The constants  $\frac{4}{\pi} - 1$  and  $\frac{1}{4}$  are the best possible. Inequality (1.1) improves some earlier results dealing with estimates of the sequence  $P_n$ .

Here we observe that (1.1) follows easily from a known result due to G. N. Watson [15]. Indeed, it is shown in [15] that the function

$$\theta(x) = \left( \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right)^2 - x$$

is strictly decreasing on  $(-1/2, \infty)$ . Applying this result, together with the observation that  $\theta(1) = 4/\pi - 1$ ,  $\lim_{x \rightarrow \infty} \theta(x) = 1/4$  and  $P_n = \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}$ , we obtain the double inequality (1.1).

The purpose of this note is to show that (1.1) is a special case of a more general inequality, which in fact, admits a very short proof. Let  $0 < \alpha < 1$  and

$$d_n(\alpha) := \frac{(1-\alpha)_n}{n!}.$$

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Received by the editors September 15, 2004 and, in revised form, November 30, 2004.

2000 *Mathematics Subject Classification*. Primary 33B15; Secondary 26D20.

*Key words and phrases*. Wallis' inequality, Gamma function, monotonicity, best bounds.

As usual,  $(a)_k$  denotes the Pochhammer symbol, defined by

$$(a)_0 = 1, \text{ and } (a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(k+a)}{\Gamma(a)}, \text{ for } k = 1, 2, \dots$$

Clearly,  $d_n(1/2) = P_n$ . It is also well known and easy to see that

$$(1.2) \quad \frac{1}{\Gamma(1-\alpha)(n+1)^\alpha} < d_n(\alpha) < \frac{1}{\Gamma(1-\alpha)n^\alpha}.$$

Here we show that this inequality can be improved as follows.

**Theorem 1.1.** *For all natural numbers  $n$ , we have*

$$(1.3) \quad \frac{1}{\Gamma(1-\alpha)(n+c_2)^\alpha} \leq d_n(\alpha) < \frac{1}{\Gamma(1-\alpha)(n+c_1)^\alpha},$$

where the constants

$$c_1 = c_1(\alpha) = \frac{1-\alpha}{2} \quad \text{and} \quad c_2 = c_2(\alpha) = \frac{1}{[\Gamma(2-\alpha)]^{1/\alpha}} - 1$$

are the best possible.

It is clear that for  $\alpha = 1/2$ , inequality (1.3) coincides with (1.1).

Inequalities such as (1.2) and (1.3) are of particular importance in certain problems on positive trigonometric sums and positive sums of Gegenbauer polynomials, having  $d_k(\alpha)$  as a sequence of coefficients. See the recent papers [9] and [10]. We also refer to the paper [11] for some interesting estimates of ratios of gamma functions. Some other proofs of (1.1) and related results can be found in the recent articles [3], [4], [5], [6] and [7]. Compare also the papers [12], [13] and [14] for different proofs of (1.1) and various inequalities for the sequence  $P_n$ .

## 2. PROOF OF (1.3)

For the proof of (1.3) we define

$$Q_n(\alpha) := \left\{ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} \right\}^{1/\alpha} - n.$$

Using the asymptotic formula

$$x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right), \quad x \rightarrow \infty$$

(see [1], p. 615, for the complete form of this formula), we easily verify that

$$\lim_{n \rightarrow \infty} Q_n(\alpha) = \frac{1-\alpha}{2} = c_1.$$

Obviously  $c_2 = Q(1, \alpha)$ .

The required inequality (1.3) follows from the fact that the sequence  $Q_n(\alpha)$  is strictly decreasing. This, in turn, is an immediate consequence of a result of N. Elezović, C. Giordano and J. Pečarić, [8] who showed that the function

$$x \mapsto \left( \frac{\Gamma(x+t)}{\Gamma(x+s)} \right)^{1/(t-s)} - x$$

is convex and decreasing on  $(-r, \infty)$  if  $|t-s| < 1$ , where  $s, t$  are given positive numbers and  $r = \min(s, t)$ .

This completes the proof of (1.3). □

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