

A SIMPLE PROOF FOR FOLDS ON BOTH SIDES IN COMPLEXES OF GRAPH HOMOMORPHISMS

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ABSTRACT. In this paper we study implications of folds in both parameters of Lovász' $\text{Hom}(-, -)$ complexes. There is an important connection between the topological properties of these complexes and lower bounds for chromatic numbers. We give a very short and conceptual proof of the fact that if $G - v$ is a fold of G , then $\text{bdHom}(G, H)$ collapses onto $\text{bdHom}(G - v, H)$, whereas $\text{Hom}(H, G)$ collapses onto $\text{Hom}(H, G - v)$.

We also give an easy inductive proof of the only nonelementary fact which we use for our arguments: if φ is a closure operator on P , then $\Delta(P)$ collapses onto $\Delta(\varphi(P))$.

1. INTRODUCTION

The Hom complexes were defined by Lovász. It has been shown in [1, 2, 3] that the algebro-topological invariants of Hom 's can be used to provide lower bounds for chromatic numbers of graphs, a notoriously difficult problem. Such a complex $\text{Hom}(G, H)$ depends on two parameters, which are both (not necessarily simple) graphs. $\text{Hom}(G, H)$ is usually not simplicial; however, it is a regular CW complex, and its cells are products of simplices. The topological and combinatorial properties of the Hom complexes have been studied extensively in a recent series of papers [1, 2, 3, 5, 6, 7, 8, 11].

The spectral sequence computations in [3] required certain manipulations with the first parameter graph; these specific manipulations are usually called *folds*. It was proved in [2, Proposition 5.1] that folds in the first parameter yield homotopy equivalence. It was noted in [6, Lemma 3.1], [8] that we can fold in the second parameter if the deleted vertex is an identical twin. More recently, it was observed by Csorba and Dochtermann, [5, 8], that a proof of [2, Proposition 5.1] can be modified to cover the folds in the second parameter completely, with further complications arising in certain situations. All of the previously-known proofs follow the same path of using [2, Proposition 3.2], in turn based on Quillen's theorem A, [12, p. 85].

In this paper we simplify and generalize the situation. We prove that folds in the first parameter induce a collapsing (a sequence of elementary collapses) on

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the barycentric subdivision of the involved complexes, whereas folds in the second parameter induce a collapsing on the complexes themselves.

Our proof is simple and conceptual. We also derive in an elementary inductive way the only fact of the topological combinatorics which we use: the existence of the collapsing induced by a closure operator on a poset.

2. CLOSURE OPERATORS

For a poset P we let $\Delta(P)$ denote its *nerve*: the simplicial complex whose simplices are all chains of P . For a regular CW complex X we let $\mathcal{P}(X)$ denote its face poset, in particular, $\Delta(\mathcal{P}(X)) = \text{Bd } X$. By analogy, we denote $\mathcal{P}(\Delta(P)) = \text{Bd } P$. A cellular map $\varphi : X \rightarrow Y$ between regular CW complexes induces an order-preserving map $\mathcal{P}(\varphi) : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$. For a simplicial complex X and its subcomplex Y we say that X *collapses onto* Y if there exists a sequence of elementary collapses leading from X to Y . If X collapses onto Y , then Y is a strong deformation retract of X .

Recall that an order-preserving map φ from a poset P to itself is called a *descending closure operator* if $\varphi^2 = \varphi$ and $\varphi(x) \leq x$, for any $x \in P$; analogously, φ is called an *ascending closure operator* if $\varphi^2 = \varphi$ and $\varphi(x) \geq x$, for any $x \in P$. That ascending and descending closure operators induce strong deformation retraction is well known in topological combinatorics; see, e.g., [4, Corollary 10.12], where it is proved by using Quillen's theorem A, [12, p. 85].

As a sample, it yields an extremely short proof of the fact that the complex of disconnected graphs is a strong deformation retract of the order complex of the partition lattice: the ascending closure operator takes each disconnected graph G to the partition whose blocks are the connected components of G . We remark that the first complex appeared in the work of Vassiliev on knot theory, [13], whereas the second complex encodes the geometry of the braid arrangement by means of the Goresky-MacPherson theorem; see [10].

In this paper we give a short and self-contained inductive proof of the fact that $\Delta(P)$ collapses onto $\Delta(\varphi(P))$.

Theorem 2.1. *Let P be a poset, and let φ be a descending closure operator. Then $\Delta(P)$ collapses onto $\Delta(\varphi(P))$. By symmetry the same is true for an ascending closure operator.*

Proof. We use induction on $|P| - |\varphi(P)|$. If $|P| = |\varphi(P)|$, then φ is the identity map, and the statement is obvious. Assume that $P \setminus \varphi(P) \neq \emptyset$ and let $x \in P$ be one of the minimal elements of $P \setminus \varphi(P)$.

Since φ fixes each element in $P_{<x}$, $\varphi(x) < x$, and φ is order preserving, we see that $P_{<x}$ has $\varphi(x)$ as a maximal element; see Figure 2.1. Thus the link of x in $\Delta(P)$ is $\Delta(P_{>x}) * \Delta(P_{<x}) = \Delta(P_{>x}) * \Delta(P_{<\varphi(x)}) * \varphi(x)$, in particular, it is a cone with apex $\varphi(x)$.

Let $\sigma_1, \dots, \sigma_t$ be the simplices of $\Delta(P_{>x}) * \Delta(P_{<\varphi(x)})$ ordered so that the dimension is weakly decreasing. Then

$$(\sigma_1 \cup \{x\}, \sigma_1 \cup \{x, \varphi(x)\}), \dots, (\sigma_t \cup \{x\}, \sigma_t \cup \{x, \varphi(x)\})$$

is a sequence of elementary collapses leading from $\Delta(P)$ to $\Delta(P \setminus \{x\})$. Since φ restricted to $P \setminus \{x\}$ is again a descending closure operator, $\Delta(P \setminus \{x\})$ collapses onto $\Delta(\varphi(P \setminus \{x\})) = \Delta(\varphi(P))$ by the induction assumption. \square

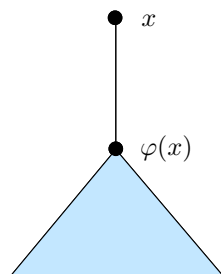


FIGURE 2.1. $P_{<x} = P_{\leq\varphi(x)}$.

Remark 2.2. There is a direct way to describe the elementary collapses in the proof of the Theorem 2.1. For $x \in \text{Bd } P \setminus \text{Bd } \varphi(P)$, $x = (x_1 < \dots < x_k)$, let $1 \leq i \leq k$ be the minimal possible index, such that $x_i \notin \varphi(P)$. Then either $i = 1$ or $x_{i-1} \in \varphi(P)$. If $\varphi(x_i) = x_{i-1}$, then match $(x \setminus \{x_{i-1}\}, x)$, otherwise match $(x, x \cup \varphi(x_i))$; the latter is possible since either $i = 1$, or $x_i > x_{i-1}$ and $\varphi(x_i) \neq x_{i-1}$ imply $\varphi(x_i) > \varphi(x_{i-1}) = x_{i-1}$. It is now easy to see that this is an acyclic matching and thus alternatively derive the result by using discrete Morse theory; see [9].

3. Hom COMPLEXES AND FOLDS

Let G be a graph whose set of vertices is denoted by $V(G)$, and whose set of edges is denoted by $E(G)$. Let v be a vertex of G , and let $G - v$ denote the graph which is obtained from G by deleting the vertex v and all adjacent to v edges. Let $\mathcal{N}(v)$ denote the set of neighbors of v , i.e., $\mathcal{N}(v) = \{x \in V(G) \mid (v, x) \in E(G)\}$.

Definition 3.1. $G - v$ is called a **fold** of G if there exists $u \in V(G)$, $u \neq v$, such that $\mathcal{N}(u) \supseteq \mathcal{N}(v)$.

Let $G - v$ be a fold of G . We let $i : G - v \hookrightarrow G$ denote the inclusion homomorphism, and let $f : G \rightarrow G - v$ denote the folding homomorphism defined by $f(v) = u$ and $f(x) = x$, for $x \neq v$.

The following construction, which is due to Lovász, has been in the center of latest advances in the area of topological obstructions to graph colorings; see [1, 2, 3, 5, 6, 7, 8, 11].

Let $\Delta^{V(H)}$ be a simplex whose set of vertices is $V(H)$. Let $C(G, H)$ denote the direct product $\prod_{x \in V(G)} \Delta^{V(H)}$, i.e., the copies of $\Delta^{V(H)}$ are indexed by vertices of G .

Definition 3.2 ([11, Definition 2.1.5]). $\text{Hom}(G, H)$ is the subcomplex of $C(G, H)$ defined by the following condition: $\sigma = \prod_{x \in V(G)} \sigma_x \in \text{Hom}(G, H)$ if and only if for any $x, y \in V(G)$, if $(x, y) \in E(G)$, then (σ_x, σ_y) is a complete bipartite subgraph of H , i.e., $\sigma_x \times \sigma_y \subseteq E(H)$.

Note that $\text{Hom}(G, H)$ is a polyhedral complex whose cells are indexed by all functions $\eta : V(G) \rightarrow 2^{V(H)} \setminus \{\emptyset\}$, such that if $(x, y) \in E(G)$, then $\eta(x) \times \eta(y) \subseteq E(H)$. The closure of a cell η consists of all cells $\tilde{\eta}$, satisfying $\tilde{\eta}(v) \subseteq \eta(v)$, for all $v \in V(G)$.

It was noted in [2] that $\text{Hom}(G, -)$ is a covariant functor, while $\text{Hom}(-, G)$ is a contravariant functor from the category of graphs and graph homomorphisms,

Graphs to Top. For a graph homomorphism $\varphi : G \rightarrow G'$ the topological maps induced by composition are denoted as $\varphi^H : \text{Hom}(H, G) \rightarrow \text{Hom}(H, G')$ and $\varphi_H : \text{Hom}(G', H) \rightarrow \text{Hom}(G, H)$; this is the notation introduced in [2]. Observe that φ^H is cellular on the mentioned complexes, whereas φ_H is cellular on their first barycentric subdivisions.

Theorem 3.3. *Let $G - v$ be a fold of G and let H be some graph. Then $\text{Bd Hom}(G, H)$ collapses onto $\text{Bd Hom}(G - v, H)$, whereas $\text{Hom}(H, G)$ collapses onto $\text{Hom}(H, G - v)$. The maps i_H and f^H are strong deformation retractions.*

Proof. First we show that $\text{Bd Hom}(G, H)$ collapses onto $\text{Bd Hom}(G - v, H)$. Identify $\mathcal{P}(\text{Hom}(G - v, H))$ with the subposet of $\mathcal{P}(\text{Hom}(G, H))$ consisting of all η , such that $\eta(v) = \eta(u)$. Let X be the subposet consisting of all $\eta \in \mathcal{P}(\text{Hom}(G, H))$ satisfying $\eta(v) \supseteq \eta(u)$. Then $\mathcal{P}(\text{Hom}(G - v, H)) \subseteq X \subseteq \mathcal{P}(\text{Hom}(G, H))$. Consider order-preserving maps

$$\mathcal{P}(\text{Hom}(G, H)) \xrightarrow{\alpha} X \xrightarrow{\beta} \mathcal{P}(\text{Hom}(G - v, H)),$$

defined by

$$\alpha\eta(x) = \begin{cases} \eta(u) \cup \eta(v), & \text{for } x = v; \\ \eta(x), & \text{otherwise;} \end{cases} \quad \beta\eta(x) = \begin{cases} \eta(u), & \text{for } x = v; \\ \eta(x), & \text{otherwise;} \end{cases}$$

for all $x \in V(G)$; see Figure 3.1. Maps α and β are well defined because $G - v$ is a fold of G . Clearly $\beta \circ \alpha = \mathcal{P}(i_H)$, α is an ascending closure operator, and β is a descending closure operator. Since $\text{Im } \mathcal{P}(i_H) = \mathcal{P}(\text{Hom}(G - v, H))$, the statement follows from Theorem 2.1.

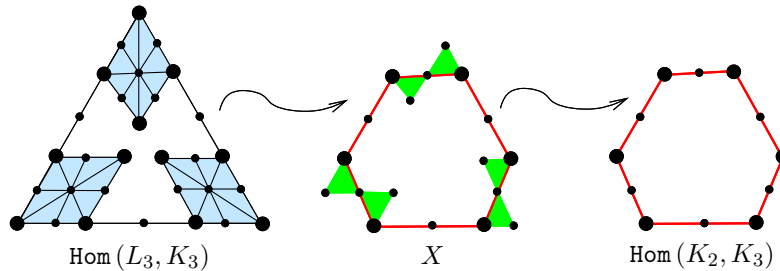


FIGURE 3.1. A two-step folding of the first argument in $\text{Hom}(L_3, K_3)$.

We show that $\text{Hom}(H, G)$ collapses onto $\text{Hom}(H, G - v)$ by presenting a sequence of elementary collapses. Denote $V(H) = \{x_1, \dots, x_t\}$. For $\eta \in \mathcal{P}(\text{Hom}(H, G))$, let $1 \leq i(\eta) \leq t$ be the minimal index such that $v \in \eta(x_{i(\eta)})$. Write $\mathcal{P}(\text{Hom}(H, G))$ as a disjoint union $A \cup B \cup \mathcal{P}(\text{Hom}(H, G - v))$, defined as follows: for $\eta \in A \cup B$ we have $\eta \in A$ if $u \notin \eta(x_{i(\eta)})$, and we have $\eta \in B$ otherwise.

There is a bijection $\varphi : A \rightarrow B$ which adds u to $\eta(x_{i(\eta)})$ without changing other values of η . Adding u to $\eta(x_{i(\eta)})$ yields an element in $\mathcal{P}(\text{Hom}(H, G))$ since $G - v$ is a fold of G . Clearly, $\varphi(\alpha)$ covers α , for all $\alpha \in A$. We take the set $\{(\alpha, \varphi(\alpha)) \mid \alpha \in A\}$ to be our collection of the elementary collapses. These are ordered lexicographically after the pairs of integers $(i(\alpha), -\dim \alpha)$.

Let us see that these collapses can be performed in this lexicographic order. Take $\eta > \alpha$, $\eta \neq \varphi(\alpha)$. Assume $i(\eta) = i(\alpha)$. If $\eta \in B$, then $\eta = \varphi(\tilde{\alpha})$, $i(\tilde{\alpha}) = i(\alpha)$, and

$\dim \tilde{\alpha} > \dim \alpha$. Otherwise $\eta \in A$ and $\dim \eta > \dim \alpha$. The third possibility is that $i(\eta) < i(\alpha)$. In either case η has been removed before α . \square

Instead of verifying that the sequence of collapses is correct in the last paragraph of the proof, we could simply note that the defined matching is acyclic and derive the result by discrete Morse theory, [9].

Remark 3.4. In analogy with the first part of the proof, we can show that $\text{Bd Hom}(H, G)$ collapses onto $\text{Bd Hom}(H, G - v)$ by rewriting $\mathcal{P}(f^H)$ as a composition of two closure operators.

Indeed, let Y be the subposet consisting of all $\eta \in \mathcal{P}(\text{Hom}(H, G))$ such that for all $x \in V(H)$, $\eta(x) \cap \{u, v\} \neq \{v\}$, i.e., for any $x \in V(H)$, we have: if $v \in \eta(x)$, then $u \in \eta(x)$. Then $\mathcal{P}(\text{Hom}(H, G - v)) \subseteq Y \subseteq \mathcal{P}(\text{Hom}(H, G))$. Consider order preserving maps

$$\mathcal{P}(\text{Hom}(H, G)) \xrightarrow{\varphi} Y \xrightarrow{\psi} \mathcal{P}(\text{Hom}(H, G - v))$$

defined by

$$\varphi\eta(x) = \begin{cases} \eta(x) \cup \{u\}, & \text{if } v \in \eta(x); \\ \eta(x), & \text{otherwise;} \end{cases} \quad \psi\eta(x) = \begin{cases} \eta(x) \setminus \{v\}, & \text{if } v \in \eta(x); \\ \eta(x), & \text{otherwise;} \end{cases}$$

for all $x \in V(H)$.

The map ϕ is well defined because $G - v$ is a fold of G , and the map ψ is well defined by the construction of Y . We see that $\psi \circ \varphi = \mathcal{P}(f^H)$, φ is an ascending closure operator, and ψ is a descending closure operator. Since $\text{Im } \mathcal{P}(f_H) = \mathcal{P}(\text{Hom}(H, G - v))$, the statement follows from Theorem 2.1.

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