PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 5, Pages 1467–1472
S 0002-9939(05)08159-1
Article electronically published on October 18, 2005

REMARKS ON SPECTRA AND L^1 MULTIPLIERS FOR CONVOLUTION OPERATORS

WŁODZIMIERZ BĄK AND ANDRZEJ HULANICKI

(Communicated by David R. Larson)

Abstract. We prove that the spectrum of a convolution operator on a locally compact group G by a self-adjoint L^1-function f is the same on $L^1(G)$ and $L^2(G)$ and consequently on all L^p spaces, $1 \leq p < \infty$, if and only if a Beurling algebra contains non-analytic functions on $\mathbb{R}$ operating on f into L^1.

1. Introduction

Suppose B_1 and B_2 are two Banach spaces such that $B_0 \subset B_1 \cap B_2$ is dense in B_1 and in B_2. Let

$$T : B_0 \to B_1 \cap B_2$$

be a linear operator such that

$$\|T\|_{B_1 \to B_1} < \infty \text{ and } \|T\|_{B_2 \to B_2} < \infty.$$

Spectral properties of T on B_1 and B_2 have been studied in various context, e.g. [1], [2], [6], [7], [15].

Let G be a locally compact group, and let $L^p(d_r x)$ be the space of p-integrable functions w.r.t. the right invariant Haar measure. Let $f = f^* \in L^1(d_r x)$ and let

$$T_f \xi = \xi * f, \quad \xi \in L^p(G), \quad 1 \leq p < \infty.$$

The spectra of convolution operators $T_f : B_1 \to B_1$ and $T_f : B_2 \to B_2$, $B_1 = L^1(d_r x)$ and $B_2 = L^2(d_r x)$ have been discussed for half a century. [23] is an interesting survey article that lists existing literature (as of 1976). Still, there is a host of recent papers on the subject and a number of questions of some importance that are open and seem to be tractable, e.g. [3], [7], [21], [9].

Let $\mathcal{H}$ be a Hilbert space, and let π be a unitary representation of G on $\mathcal{H}$. Suppose that the representation π is faithful on $L^1(d_r x)$ and

$$\mathrm{Sp}_{L^1(d_r x)} T_f = \mathrm{Sp}_{\mathcal{H}} \pi(f). \tag{1}$$

Equality (1) for *all* $f = f^* \in L^1(d_r x)$, by [25], is equivalent to the definition of what Gelfand called the *symmetry of* $L^1(d_r x)$ over fifty years ago.

Received by the editors September 1, 2004 and, in revised form, December 21, 2004.

2000 *Mathematics Subject Classification.* Primary 43A10, 43A20.

This work was partially done within the project TMR Network "Harmonic Analysis", contract no. ERB FMRX-CT97-0159.

This research was partially financed by the European Commission IHP Network 2002–2006 *Harmonic Analysis and Related Problems* (Contract Number: HPRN-CT-2001-00273-HARP) and by KBN grant 1 P03A 018 26.

Let $g = g^* \in L^1(d_r x)$. Then $\mathrm{Sp}_{\mathcal{H}}\pi(g) \subset [a,b] \subset \mathbb{R}$. By the spectral theorem,

$$\pi(g)\xi = \int_a^b \lambda dE(\lambda)\xi, \quad \xi \in \mathcal{H}. \tag{2}$$

Let $\mathbf{A}$ be the algebra of functions on $[a,b]$ with absolutely convergent Fourier series, and let $\omega : \mathbb{Z} \ni n \to [1,\infty)$ be a function such that

$$\omega(k+l) \leq \omega(k) \cdot \omega(l). \tag{3}$$

Then

$$A(\omega) = \{F \in \mathbf{A} : \sum_{n\in\mathbb{Z}} |\widehat{F}(n)|\omega(n) = \|F\|_{A(\omega)} < \infty\}, \tag{4}$$

where $\widehat{F}(n) = \frac{1}{b-a}\int\limits_a^b F(t)e^{\frac{-2\pi int}{b-a}}\, dt$ is a Banach subalgebra of $\mathbf{A}$. If ω satisfies

$$\lim_{|n|\to\infty} \omega(n)^{\frac{1}{|n|}} = 1, \tag{5}$$

then $A(\omega)$ contains functions that have no holomorphic extension onto a neighborhood of $[a,b]$ in $\mathbb{C}$. We say that a function $F \in \mathbf{A}$ *operates* on f into $L^1(d_r x)$ if

$$F(f)\xi = \int_a^b F(\lambda)dE(\lambda)\xi = \xi * g, \quad \xi \in \mathcal{H}, \tag{6}$$

for some $g \in L^1(d_r x)$. We then say that F is a $L^1(d_r x)$ *spectral multiplier* for f.

The aim of this note is to spell out the following observation.

Theorem 1.1. *Equality* (1) *is equivalent to the existence of a weight ω which satisfies* (5)*, and every F in $A(\omega)$ is a $L^1(d_r x)$ spectral multiplier for f.*

The proof of this theorem uses more or less standard arguments from Banach *-algebras theory that will be presented in Section 2 and yields similar facts for $L^1(G,\omega)$ discussed in [7].

2. BANACH *-ALGEBRAS

Let X be a Banach *-algebra, and let π be a faithful *-representation of X into the algebra $\mathcal{L}(\mathcal{H})$ of linear bounded operators on a Hilbert space $\mathcal{H}$, and let $\widetilde{X}$ be the C^*-algebra that is the closure in the operator norm of $\pi(X)$, and let $f = f^*$ be an element of X. Of course,

$$\mathrm{Sp}_{\mathcal{H}}\pi(f) = \mathrm{Sp}_{\widetilde{X}} f. \tag{7}$$

A semi-simple Banach *-algebra X is called *symmetric* if for every element $g = g^* \in X$, $\mathrm{Sp}_X f \subset \mathbb{R}$, and this is equivalent ([22], [24], [25]), to the property that for all $g \in X$,

$$\mathrm{Sp}_X g = \mathrm{Sp}_{\widetilde{X}} g. \tag{8}$$

Moreover, if X_0 is a symmetric Banach *-algebra, $X_0 \subset X$ and $g = g^* \in X_0$, then (8) holds.

If $f = f^*$, the spectrum $\mathrm{Sp}_{\mathcal{H}}\pi(f)$ of the operator $\pi(f)$ is a closed subset of an interval $I \subset \mathbb{R}$. We say that a bounded measurable function $F : I \to \mathbb{C}$ operates on f into X if there is an element $g \in X$ such that

$$\pi(g)\xi = \int_{\mathrm{Sp}_{\mathcal{H}}\pi(f)} F(\lambda) dE(\lambda)\xi, \quad \xi \in \mathcal{H}.$$

We then write $g = F(f)$. For a fixed element $f = f^* \in X$, the set of functions $F \in \mathbf{A}$ that operate on f forms an algebra under point-wise multiplication, denoted by $\langle f \rangle$.

Thus Theorem 1.1 follows from

Theorem 2.1. *Let X be a semi-simple Banach *-algebra, and let $\widetilde{X}$ be a C^*-algebra such that $X \subset \widetilde{X}$. Let $f = f^*$ be a fixed element in X. Then the algebra $\langle f \rangle$ contains $A(\omega)$, ω satisfying* (5) *if and only if*

$$Sp_X f = Sp_{\widetilde{X}} f. \tag{9}$$

For a weight ω assume that $A(\omega) \subset \langle f \rangle$. Let $\mathcal{A}_f^\omega = \{F(f) : F \in A(\omega)\}$. $\mathcal{A}_f^\omega$ contains A_f the Banach *-algebra generated by f. Let $\mathfrak{M}$ be the Gelfand space of homomorphisms of $\mathcal{A}_f$ into $\mathbb{C}$, and for $g \in \mathcal{A}_f$ let $\hat{g}$ be the Gelfand transform of g.

Now we are going to prove the following

Lemma 2.2. *The following conditions are equivalent:*

i) *for each $F \in A(\omega)$ we have $F(f) \in X$,*
ii) *there exists $C > 0$ such that $\|e^{inf}\|_X \le C\omega(n)$ for $n \in \mathbb{Z}$.*

Proof. i) ⇒ ii) Let $T : A(\omega) \ni F \to F(f) \in X$. We are going to verify that the linear operator T is closed. Let $F_n \to F$ in $A(\omega)$ and let $g_n = F_n(f) \to g$ in X. Obviously F_n converges uniformly to F on $[a, b]$ and, by the spectral theorem, $\pi(g_n) = \int_a^b F_n(\lambda)\, dE(\lambda)$ converges to $\int_a^b F(\lambda)\, dE(\lambda)$ in the operator norm. On the other hand, since $\pi \to \mathcal{L}(\mathcal{H})$ is continuous, $\pi(g_n) \to \pi(g)$ in the operator norm. Hence $\pi(g) = \int_a^b F(\lambda)\, dE(\lambda)$. By the closed graph theorem, it follows that there exists a constant C, such that $\|T(F)\|_X \le C\|F\|_{A(\omega)}$. Now, given ω, for every $n \in \mathbb{Z}$ the function $F(x) = e^{inx}$ belongs to $A(\omega)$, so since

$$e^{inf} = \sum_{k=0}^{\infty} \frac{(inf)^k}{k!} \in \mathcal{A}_f \subset \mathcal{A}_f^\omega,$$

$$\|e^{inf}\|_X \le C\|e^{in\cdot}\|_{A(\omega)} = C\omega(n).$$

ii) ⇒ i) Let $F \in A(\omega)$. Then

$$\begin{aligned} \|T(F)\|_X &= \|F(f)\|_X = \|\sum_{n\in\mathbb{Z}} \widehat{F}(n)e^{inf}\|_X \le \sum_{n\in\mathbb{Z}} |\widehat{F}(n)| \cdot \|e^{inf}\|_X \\ &\le C \cdot \sum_{n\in\mathbb{Z}} |\widehat{F}(n)|\omega(n) < \infty. \end{aligned}$$

Hence $F(f) \in X$. □

Now to prove Theorem 2.1, by virtue of (7) and (8), it suffices to show

(a) *if* $A(\omega) \subset \langle f \rangle$ *and* ω *satisfies* (5), *then* $\mathcal{A}_f$ *is symmetric,*

and

(b) *if* $\mathcal{A}_f$ *is symmetric, then there exists a weight* ω *that satisfies* (5) *and* $A(\omega) \subset \langle f \rangle$.

To prove (a) we need to show that $\hat{f}(\lambda)$ is real for every $\lambda \in \mathfrak{M}$. The lemma says that

$$\max_{\lambda \in \mathfrak{M}} |e^{in\hat{f}(\lambda)}| \leq \|e^{inf}\|_{\mathcal{A}_f} \leq \omega(n)$$

so, if ω satisfies (5), $\hat{f}(\lambda)$ is real.

To prove (b) suppose $\mathcal{A}_f$ is symmetric. Then, by the spectral radius formula,

$$\|e^{inf}\|_{\mathcal{A}_f}^{\frac{1}{n}} \to \sup_{M \in \mathfrak{M}} |e^{i\hat{f}(M)}| = 1.$$

Hence, if we take $\omega(n) = \|e^{inf}\|_{\mathcal{A}_f}$, then ω satisfies (5) and by the lemma, $A(\omega) \subset \langle f \rangle$.

3. REMARKS AND PROBLEMS

Let G be a connected Lie group, let $\mathcal{G}$ be its Lie algebra and let $X_1, ..., X_k \in \mathcal{G}$ be *left* invariant vector fields on G such that the smallest Lie subalgebra of the Lie algebra $\mathcal{G}$ of G that contains $X_1, ..., X_k$ is $\mathcal{G}$. Let $\Delta = X_1^2 + ... + X_k^2$, where Δ is called a sublaplacian. It is the infinitesimal generator of a convolution semigroup

$$p_s * p_t = p_{s+t}, \quad s,t \in \mathbf{R}^+, \quad p_t^* = p_t \in L^1(d_r x).$$

Also

the p_t's are smooth and rapidly decreasing.

The operator Δ is essentially self-adjoint on $C_c^\infty(G) \subset L^2(d_r x)$; cf. e.g [8]. Also for all $1 \leq p < \infty$,

$$\mathrm{Sp}_{L^p(d_r x)} T_{p_t} = \overline{\exp[t\mathrm{Sp}_{L^p(d_r x)}\Delta]}. \tag{10}$$

For a connected Lie group G let $U = U^{-1}$ be an open neighborhood with compact closure. G is said to be of *polynomial growth* if

$$\int_{U^n} d_r x = O(n^k) \quad \text{for some } k \quad \text{as} \quad n \to \infty. \tag{11}$$

If a connected Lie group is not of polynomial growth, then it is of *exponential growth*, i.e.

$$e^{\varepsilon n} \leq \int_{U^n} d_r x \leq e^{\varepsilon' n} \quad \text{for some} \quad \varepsilon, \varepsilon' > 0. \tag{12}$$

For a group of polynomial growth there is a k such that C^k functions with compact support operate on T_{p_t} into $L^1(d_r x)$ for *all* sublaplacians [16], and was known somewhat earlier [15] that

$$\mathrm{Sp}_{L^p(g)}\Delta = \mathrm{Sp}_{L^2(g)}\Delta. \tag{13}$$

On the other hand, for exponentially growing groups it may happen that only functions that have holomorphic extensions in a neighborhood in $\mathbb{C}$ of a non-isolated point of $\mathrm{Sp}_{\mathcal{H}}\Delta\ \mathbf{R}$ operate on T_{p_t} into $L^1(d_r x)$, and one might think that this is the rule [4], [13]. Then also

$$\mathrm{Sp}_{L^1(d_r x)}\Delta \text{ is not contained in } \mathbf{R}. \tag{14}$$

But still for particular solvable Lie groups of exponential growth G and certain sublaplacians Δ, there exist $L^1(d_r x)$ multipliers of finite smoothness [5], [10], [11]. It is an open problem whether for some other sublaplacians Δ' on G (14) still holds.

At the other end, for discrete groups G, there has been some recent interest in spectral properties of convolution operators on $l^2(G)$, [9]. The classical theorem by H. Kesten [20] states that if (and only if) a group G is amenable, then for every symmetric *non-negative* function f on G such that

$$\bigcup_{n=1}^{\infty} (\operatorname{supp} f)^n = G, \tag{15}$$

we have

$$\max\{|\lambda| : \lambda \in \mathrm{Sp}_{l^2(G)} T_h\} = \max\{|\lambda| : \lambda \in \mathrm{Sp}_{l^1(G)} T_h\} \tag{16}$$

for all $h = f$. It is known that if (16) holds for *all* h in the *-algebra generated by $f = f^* \in l^1(G)$, then

$$\mathrm{Sp}_{l^2(G)} T_f = \mathrm{Sp}_{l^1(G)} T_f \tag{17}$$

(see [15] and [7]). It is an open problem whether there exists an amenable group G and *non-negative* $f \in l^1(G)$ which satisfies (15) such that (17) fails. It seems very probable that there are amenable groups G and functions $f, \mu \in l^1(G)$ such that μ is non-negative, both satisfy (15) with (17) holding for μ, and failing for f. It is also likely that there are μ's that satisfy (15) on such a group which admits $l^1(G)$ spectral multipliers of finite smoothness. It seems that an old conjecture formulated in [23] that all groups G for which $L^1(G)$ is symmetric must be amenable is still not resolved. In [14], an incomplete "proof" of this conjecture that has never been made correct is presented.

References

[1] B. A. Barnes, *When is the spectrum of a convolution operator on L^p independent of p?* Proc. Edinburgh Math. Soc. (2) **33** (1990), no. 2, 327–332. MR1057760 (91f:47046)

[2] B. A. Barnes, *A note on invariance of spectrum for symmetric Banach *-algebras*, Proceedings of the AMS **126** (1998), 3545-3547. MR1473655 (99b:46078)

[3] R. Chill and E. Fasangovaá, *Equality of two spectra arising in harmonic analysis and semigroup theory*, Proc. Amer. Math. Soc. 130(3) (2001), 675–681, MR1866019 (2002h:47055)

[4] M. Christ and D. Müller, *On L^p spectral multipliers for a solvable Lie group*, Geom. Funct. Anal. 6 (1996), no. 5, 860–876. MR1415763 (98b:22017)

[5] M. Cowling, S. Giulini, A. Hulanicki, and G. Mauceri, *Spectral multipliers for a distinguished Laplacian on certain groups of exponential growth*, Studia Math. **111**(2) (1994), 103–121. MR1301761 (95j:43002)

[6] E. Fasangá, *Spectral mapping theorems and spectral space-independence*, Proc. Amer. Math. Soc.

[7] G. Fendler, K. Gröchenig, M. Leinert, J. Ludwig, and C. Molitor-Braun, *Weighted Group algebras on Groups of Polynomial Growth*, Preprint, 2004.

[8] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, Princeton University Press, 1982. MR0657581 (84h:43027)

[9] R. I. Grigorchuk and A. Zuk,*The lamplighter group as a group generated by a 2-state automaton, and its spectrum*, Geom. Dedicata **87** (2001), no. 1-3, 209–244. MR1866850 (2002j:60009)

[10] W. Hebisch, *Boundedness of L^1 spectral multipliers for an exponential solvable Lie group,* Coll. Math. **73** (1997), 155–164. MR1436956 (98e:22005)

[11] W. Hebisch, *Spectral multipliers on exponential growth solvable Lie groups,* Math. Z. **229** (1998), 435–441. MR1658573 (2000a:43004)

[12] W. Hebisch, *Spectral multipliers on metabelian groups,* Revista. Math. **16** (2000), 597–604. MR1813330 (2001m:22016)

[13] W. Hebisch, J. Ludwig, and D. Müller, *Sublaplacians of holomorphic L^p-type on exponential solvable groups*, Preprint, 2001.

[14] A. Hulanicki, *On symmetry in group algebras*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 1–2. MR0149323 (26:6813)

[15] A. Hulanicki *On L_p-spectra of the laplacian on a Lie group with polynomial growth,* Proc. Amer. Math. Soc. **44** (1974), 482-484. MR0360931 (50:13378)

[16] A. Hulanicki, *Subalgebras of $L_1(G)$ associated with Laplacian on a Lie group,* Colloq. Math. **31** (1974), 259–287. MR0372536 (51:8743)

[17] A. Hulanicki, *A functional calculus for Rockland operators on nilpotent Lie groups,* Studia Math. **78** (1984), no. 3, 253–266. MR0782662 (86g:22009)

[18] J.-P. Kahane, *Séries de Fourier absolument convergentes*, Springer-Verlag 1970. MR0275043 (43:801)

[19] Y. Katznelson, *An introuction to harmonic analysis*, New York - London, 1968.

[20] H. Kesten, *Full Banach mean values on countable groups*, Math. Scand. **7** (1959), 146–156. MR0112053 (22:2911)

[21] J. Ludwig, A. Müller, and S. Souaifi, *Holomorphic L^p-type for sub-Laplacians on connected Lie groups*, preprint, 2004.

[22] M. A. Naimark, *Normed rings*, Groningen, 1959. MR0110956 (22:1824)

[23] T. W. Palmer, *Classes of nonabelian, noncompact, locally compact groups*, Rocky Mountain J. Math. **8** (1978), no. 4, 683–741. MR0513952 (81j:22003)

[24] C. Rickart, *Banach algebras*, Van Nostrand, 1960. MR0115101 (22:5903)

[25] S. Shirali and W. M. Ford, *Symmetry in complex involutory Banach algebras II*, Duke Math. J. **37** (1970), 275–280. MR0261362 (41:5977)

Instytut Matematyczny, Uniwersytet Wrocławski, Plac Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: `bak@math.uni.wroc.pl`

Instytut Matematyczny, Uniwersytet Wrocławski, Plac Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: `hulanick@math.uni.wroc.pl`