

## THE AMENABILITY CONSTANT OF THE FOURIER ALGEBRA

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ABSTRACT. For a locally compact group  $G$ , let  $A(G)$  denote its Fourier algebra and  $\hat{G}$  its dual object, i.e., the collection of equivalence classes of unitary representations of  $G$ . We show that the amenability constant of  $A(G)$  is less than or equal to  $\sup\{\deg(\pi) : \pi \in \hat{G}\}$  and that it is equal to one if and only if  $G$  is abelian.

### INTRODUCTION

The theory of amenable Banach algebras begins with B. E. Johnson's memoir [Joh1]. The choice of terminology is motivated by [Joh1, Theorem 2.5]: a locally compact group is amenable (in the usual sense; see [Pie], for example) if and only if its group algebra  $L^1(G)$  is an amenable Banach algebra. For a modern account of the theory of amenable Banach algebras, see [Run].

The Fourier algebra  $A(G)$  of an arbitrary locally compact group  $G$  was introduced by P. Eymard in [Eym]. If  $G$  is abelian, then the Fourier transform yields an isometric isomorphism of  $A(G)$  and  $L^1(\hat{G})$ , where  $\hat{G}$  is the dual group of  $G$ . (In the framework of Kac algebras, this extends to a duality between  $L^1(G)$  and  $A(G)$  for arbitrary  $G$ ; see [E–S].) Since amenable Banach algebras have bounded approximate identities, Leptin's theorem ([Lep]) yields immediately that  $A(G)$  can be amenable only if  $G$  is amenable.

Nevertheless, the tempting conjecture that a locally compact group  $G$  is amenable if and *only if*  $A(G)$  is amenable turned out to be wrong, as Johnson showed in [Joh3]. For any locally compact group  $G$ , let  $\hat{G}$  denote its *dual object*, i.e., the collection of all equivalence classes of (continuous) irreducible unitary representations of  $G$ . For  $\pi \in \hat{G}$ , let  $\deg(\pi)$  denote its *degree*, i.e., the dimension of the corresponding Hilbert space. For compact  $G$ , Johnson showed: If  $G$  is infinite such that  $\{\pi \in \hat{G} : \deg(\pi) = n\}$  is finite for each  $n \in \mathbb{N}$ , the Fourier algebra cannot be amenable. Hence, for example,  $A(\mathrm{SO}(3))$  is not amenable.

This leaves the problem of characterizing those locally compact groups  $G$  for which  $A(G)$  is amenable ([Run, Problem 14]). On the positive side,  $A(G) \cong L^1(\hat{G})$  is amenable whenever  $G$  is abelian, and  $A(G)$  is trivially amenable if  $G$  is finite.

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With a little more effort, one can show that, if  $G$  is *almost abelian*, i.e., has an abelian subgroup of finite index, then  $A(G)$  is still amenable ([L–L–W, Theorem 4.1]). Eventually, the locally compact groups  $G$  with an amenable Fourier algebra were characterized by B. E. Forrest and the author:  $A(G)$  is amenable if and *only if*  $G$  is almost abelian ([F–R, Theorem 2.3]).

In the present note, we will pick up another line of investigation begun in [Joh3] (and continued in [L–L–W]). Suppose that  $A(G)$  is amenable, so that it makes sense to speak of its amenability constant, which we denote by  $\text{AM}_{A(G)}$ . For finite  $G$ , Johnson, in [Joh3], derived a remarkable formula that allows one to compute  $\text{AM}_{A(G)}$  in terms of the degrees of the irreducible unitary representations of  $G$ , namely

$$(1) \quad \text{AM}_{A(G)} = \frac{\sum_{\pi \in \hat{G}} \deg(\pi)^3}{\sum_{\pi \in \hat{G}} \deg(\pi)^2}.$$

From (1), it is immediate that the following are true for finite  $G$ :

- $\text{AM}_{A(G)} \leq \deg(G) := \sup\{\deg(\pi) : \pi \in \hat{G}\}$ ;
- $\text{AM}_{A(G)} = 1$  if and only if  $G$  is abelian.

It is the purpose of this note to show that these two statements on  $\text{AM}_{A(G)}$  are true not only if  $G$  is finite, but for all locally compact groups  $G$ . As a by-product, we obtain an alternative approach to [F–R, Theorem 2.3].

## 1. AMENABILITY PRELIMINARIES

Johnson's original definition of an amenable Banach algebra was in terms of cohomology groups ([Joh1]). We prefer to give another approach, which is based on a characterization of amenable Banach algebras from [Joh2].

Following [E–R], we denote the (completed) Banach space tensor product by  $\otimes^\gamma$ . If  $\mathfrak{A}$  is a Banach algebra, then  $\mathfrak{A} \otimes^\gamma \mathfrak{A}$  becomes a Banach  $\mathfrak{A}$ -bimodule via

$$a \cdot (x \otimes y) := ax \otimes y \quad \text{and} \quad (x \otimes y) \cdot a := x \otimes ya \quad (a, x, y \in \mathfrak{A}).$$

The product of  $\mathfrak{A}$  induces a homomorphism  $\Delta_{\mathfrak{A}} : \mathfrak{A} \otimes^\gamma \mathfrak{A} \rightarrow \mathfrak{A}$  of Banach  $\mathfrak{A}$ -bimodules.

**Definition 1.1.** Let  $\mathfrak{A}$  be a Banach algebra. An *approximate diagonal* for  $\mathfrak{A}$  is a bounded net  $(d_\alpha)_\alpha$  in  $\mathfrak{A} \otimes^\gamma \mathfrak{A}$  such that

$$(2) \quad a \cdot d_\alpha - d_\alpha \cdot a \rightarrow 0 \quad (a \in \mathfrak{A})$$

and

$$(3) \quad a \Delta_{\mathfrak{A}} d_\alpha \rightarrow a \quad (a \in \mathfrak{A}).$$

By [Joh2], a Banach algebra is amenable if and only if it has an approximate diagonal. The advantage of using approximate diagonals to define amenable Banach algebras is that approximate diagonals allow the introduction of a quantitative aspect into the notion of amenability:

**Definition 1.2.** A Banach algebra  $\mathfrak{A}$  is called  $C$ -amenable with  $C \geq 0$  if there is an approximate diagonal for  $\mathfrak{A}$  bounded by  $C$ .

*Remarks.* (1) In view of [Joh2], a Banach algebra is amenable if and only if it is  $C$ -amenable for some  $C \geq 0$ .

(2) By (3), it is impossible for any Banach algebra to be  $C$ -amenable with  $C < 1$ .

**Definition 1.3.** Let  $\mathfrak{A}$  be a Banach algebra. The *amenability constant* of  $\mathfrak{A}$  is defined as

$$AM_{\mathfrak{A}} := \inf\{C \geq 0 : \mathfrak{A} \text{ is } C\text{-amenable}\}.$$

*Remarks.* (1) In terms of Definition 1.3,  $\mathfrak{A}$  is amenable if and only if  $AM_{\mathfrak{A}} < \infty$ .  
 (2) The infimum in Definition 1.3 is easily seen to be a minimum.

**Examples.** (1) Let  $G$  be a locally compact group. Then  $G$  is amenable, if and only if  $L^1(G)$  is 1-amenable ([Sto, Corollary 1.11]). Hence, we either have  $AM_{L^1(G)} = \infty$  or  $AM_{L^1(G)} = 1$  depending on whether  $G$  is amenable or not.

(2) Let  $\mathfrak{A}$  be a  $C^*$ -algebra. Then  $\mathfrak{A}$  is amenable if and only if it is nuclear (see [Run, Chapter 6] for a self-contained exposition). By [Haa, Theorem 3.1], if  $\mathfrak{A}$  is nuclear, then it is already 1-amenable. We thus have again a dichotomy that either  $AM_{\mathfrak{A}} = \infty$  or  $AM_{\mathfrak{A}} = 1$ .

(3) Let  $G$  be a finite group. Then  $AM_{A(G)}$  can be explicitly computed through (1). From (1), it follows immediately that  $AM_{A(G)} = 1$  if and only if  $G$  is abelian, but more is true: if  $G$  is not abelian, then  $AM_{A(G)} \geq \frac{3}{2}$  must hold ([Joh3, Proposition 4.3]). Another consequence of (1) is that, if  $H$  is another finite group, we have

$$AM_{A(G \times H)} = AM_{A(G)} AM_{A(H)}.$$

Consequently, if  $G$  is not abelian,  $AM_{A(G^n)} \geq (\frac{3}{2})^n$  can be arbitrarily large.

## 2. AN ESTIMATE FROM ABOVE FOR $AM_{A(G)}$

In this section, we shall extend (1) to general locally compact groups in the sense that we shall show that, for any locally compact group  $G$ , the inequality  $AM_{A(G)} \leq \deg(G)$  holds.

We require some background from the theory of operator spaces, for which we refer to [E–R], whose notation we adopt. In particular, for a linear space  $E$  and  $n \in \mathbb{N}$ , the symbol  $M_n(E)$  stands for the  $n \times n$ -matrices with entries from  $E$ , and if  $F$  is another linear space, and  $T: E \rightarrow F$  is linear, then the  $n$ -th amplification of  $T$ —from  $M_n(E)$  to  $M_n(F)$ —is denoted by  $T_n$ .

Our first lemma is a minor generalization of [E–R, Proposition 2.2.6] and has an almost identical proof:

**Lemma 2.1.** *Let  $E$  be an operator space, let  $\mathfrak{A}$  be a commutative  $C^*$ -algebra, and let  $n \in \mathbb{N}$ . Then every bounded linear map  $T: E \rightarrow M_n(\mathfrak{A})$  is completely bounded such that  $\|T\|_{cb} = \|T_n\|$ .*

*Proof.* Let  $\Omega$  be a locally compact Hausdorff space such that  $\mathfrak{A} \cong C_0(\Omega)$ . We may identify  $M_n(\mathfrak{A})$  with  $C_0(\Omega, M_n)$ . For  $\omega \in \Omega$ , let

$$T^\omega: E \rightarrow M_n, \quad x \mapsto (Tx)(\omega).$$

By Smith’s lemma ([E–R, Proposition 2.2.2]), each map  $T^\omega$  is completely bounded with  $\|T^\omega\|_{cb} = \|T_n^\omega\|$ , so that

$$(4) \quad \|T^\omega\|_{cb} = \|T_m^\omega\| = \|T_n^\omega\| \quad (m \in \mathbb{N}, m \geq n).$$

Let  $m \in \mathbb{N}$  with  $m \geq n$ . Then we have

$$\begin{aligned} \|T_m\| &= \sup \left\{ \|T_m x\|_{C_0(\Omega, M_{mn})} : x \in M_m(E), \|x\|_{M_m(E)} \leq 1 \right\} \\ &= \sup \left\{ \|T_m^\omega x\|_{M_{mn}} : \omega \in \Omega, x \in M_m(E), \|x\|_{M_m(E)} \leq 1 \right\} \\ &= \sup \{ \|T_m^\omega\| : \omega \in \Omega \} \\ &= \sup \{ \|T_n^\omega\| : \omega \in \Omega \}, \quad \text{by (4),} \\ &= \sup \left\{ \|T_n^\omega x\|_{M_{n^2}} : \omega \in \Omega, x \in M_n(E), \|x\|_{M_n(E)} \leq 1 \right\} \\ &= \sup \left\{ \|T_n x\|_{C_0(\Omega, M_{n^2})} : x \in M_n(E), \|x\|_{M_n(E)} \leq 1 \right\} \\ &= \|T_n\|. \end{aligned}$$

Since  $m \geq n$  was arbitrary, this means that  $\|T\|_{cb} = \|T_n\|$ . □

Our next lemma is related to [Los, Lemma] (following [E–R],  $\otimes^\lambda$  stands for the injective tensor product of Banach spaces):

**Lemma 2.2.** *Let  $\mathfrak{A}$  be a commutative  $C^*$ -algebra, and let  $n \in \mathbb{N}$ . Then the canonical map from  $M_n \otimes^\lambda M_n(\mathfrak{A})$  to  $M_{n^2}(\mathfrak{A})$  has norm at most  $n$ .*

*Proof.* Again, suppose that  $\mathfrak{A} \cong C_0(\Omega)$  for some locally compact Hausdorff space  $\Omega$ .

We may identify  $M_{n^2}(\mathfrak{A})$  with  $C_0(\Omega, M_{n^2})$ . It is sufficient to show that, for each  $\omega \in \Omega$ , the map

$$(5) \quad M_n \otimes^\lambda M_n(\mathfrak{A}) \rightarrow M_{n^2}, \quad \alpha \otimes f \mapsto \alpha \otimes f(\omega)$$

has norm at most  $n$ .

Let  $\omega \in \Omega$ , and note that (5) is the composition of the contraction

$$M_n \otimes^\lambda M_n(\mathfrak{A}) \rightarrow M_n \otimes^\lambda M_n, \quad \alpha \otimes f \mapsto \alpha \otimes f(\omega)$$

with the canonical map from  $M_n \otimes^\lambda M_n \rightarrow M_{n^2}$ , which has norm  $n$  by [Los, Lemma]. Hence, (5) has norm  $n$ . □

**Lemma 2.3.** *Let  $E$  be an operator space, let  $\mathfrak{A}$  be a commutative  $C^*$ -algebra, and let  $n \in \mathbb{N}$ . Then every bounded linear map  $T: E \rightarrow M_n(\mathfrak{A})$  is completely bounded such that  $\|T\|_{cb} \leq n\|T\|$ .*

*Proof.* We can suppose without loss of generality that  $E$  is a minimal operator space, so that, in particular,  $M_m(E) = M_m \otimes^\lambda E$  for all  $m \in \mathbb{N}$ .

By Lemma 2.1, it is enough to show that  $\|T_n\| \leq n\|T\|$ . The map  $T_n: M_n \otimes^\lambda E \rightarrow M_{n^2}(\mathfrak{A})$ , however, is the composition of  $\text{id}_{M_n} \otimes T: M_n \otimes^\lambda E \rightarrow M_n \otimes^\lambda M_n(\mathfrak{A})$ , which has the same norm as  $T$ , and the canonical map from  $M_n \otimes^\lambda M_n(\mathfrak{A})$  to  $M_{n^2}(\mathfrak{A})$ , which has norm at most  $n$  by Lemma 2.2. Hence,  $\|T_n\|$  has norm at most  $n\|T\|$ . □

**Corollary 2.4.** *Let  $E$  be an operator space, let  $\mathfrak{A}_1, \dots, \mathfrak{A}_k$  be commutative  $C^*$ -algebras, let  $n_1, \dots, n_k \in \mathbb{N}$ , and let*

$$\mathfrak{A} = M_{n_1}(\mathfrak{A}_1) \oplus_\infty \dots \oplus_\infty M_{n_k}(\mathfrak{A}_k).$$

*Then every bounded linear map  $T: E \rightarrow \mathfrak{A}$  is completely bounded such that  $\|T\|_{cb} \leq \max\{n_1, \dots, n_k\}\|T\|$ .*

*Proof.* For  $j = 1, \dots, n$ , let  $T_j : E \rightarrow M_{n_j}(\mathfrak{A}_j)$  be the composition of  $T$  with the projection from  $\mathfrak{A}$  onto  $M_{n_j}(\mathfrak{A}_j)$ . It follows that

$$\begin{aligned} \|T\|_{\text{cb}} &= \max\{\|T_1\|_{\text{cb}}, \dots, \|T_k\|_{\text{cb}}\} \\ &\leq \max\{n_1\|T_1\|, \dots, n_k\|T_k\|\}, \quad \text{by Lemma 2.3,} \\ &\leq \max\{n_1, \dots, n_k\} \max\{\|T_1\|, \dots, \|T_k\|\} \\ &= \max\{n_1, \dots, n_k\} \|T\|, \end{aligned}$$

which proves the claim. □

As in [E–R], we write  $\hat{\otimes}$  for the projective tensor product of operator spaces (as opposed to  $\otimes^\gamma$ ). Given two operator spaces  $E$  and  $F$ , we have a canonical contraction from  $E \otimes^\gamma F$  to  $E \hat{\otimes} F$ , and, generally, this is all that can be said about the relation between  $E \otimes^\gamma F$  and  $E \hat{\otimes} F$ .

In special situations, however, stronger statements can be made:

**Proposition 2.5.** *Let  $G$  be a locally compact group such that  $\text{deg}(G) < \infty$ , and let  $E$  be an operator space. Then the canonical contraction from  $A(G) \otimes^\gamma E$  into  $A(G) \hat{\otimes} E$  is a topological isomorphism whose inverse has norm at most  $\text{deg}(G)$ .*

*Proof.* Let  $\text{VN}(G)$  denote the group von Neumann algebra of  $G$ , and recall that  $A(G)^* = \text{VN}(G)$ .

We approach the problem from a dual point of view, and show that every bounded linear map  $T : E \rightarrow \text{VN}(G)$  is completely bounded with  $\|T\|_{\text{cb}} \leq \text{deg}(G)\|T\|$ .

Since  $\text{deg}(G) < \infty$ , the basic structure theory for von Neumann algebras yields that there are commutative von Neumann algebras  $\mathfrak{M}_1, \dots, \mathfrak{M}_k$  as well as  $n_1, \dots, n_k \in \mathbb{N}$ —with  $\max\{n_1, \dots, n_k\} \leq \text{deg}(G)$ —such that

$$\text{VN}(G) \cong M_{n_1}(\mathfrak{M}_1) \oplus_\infty \dots \oplus_\infty M_{n_k}(\mathfrak{M}_k).$$

By Corollary 2.4, we have a canonical—obviously  $w^*$ - $w^*$ -continuous—bijection from  $\mathcal{B}(E, \text{VN}(G))$  to  $\mathcal{CB}(E, \text{VN}(G))$  with norm not exceeding  $\max\{n_1, \dots, n_k\} \leq \text{deg}(G)$ . It follows that the preadjoint from  $A(G) \hat{\otimes} E$  to  $A(G) \otimes^\gamma E$  of this map, which is the identity on  $A(G) \otimes E$ , also has norm at most  $\text{deg}(G)$ . □

*Remark.* By [Tho] or [Moo],  $\text{deg}(G) < \infty$  holds if and only if  $G$  is almost abelian. Hence, what we actually show in the proof of Proposition 2.5, is that  $\mathcal{B}(A(G), E) = \mathcal{CB}(A(G), E)$ —not necessarily with identical norms—for every almost abelian, locally compact group  $G$  and every operator space  $E$ : this result was already proven by Forrest and P. J. Wood ([F–W, Theorem 4.5]). Our approach, however, yields better norm estimates. If  $G$  is a locally compact group,  $H$  is a (closed) abelian subgroup of  $G$  with finite index,  $E$  is any operator space, and  $T : A(G) \rightarrow E$  is a bounded, linear operator, then an inspection of the proof of [F–W, Theorem 4.5] shows that  $\|T\|_{\text{cb}} \leq [G : H]\|T\|$ . Proposition 2.5, on the other hand, yields the estimate  $\|T\|_{\text{cb}} \leq \text{deg}(G)\|T\|$ . In view of Proposition 2.8 below and the example following it, this latter estimate is better.

**Corollary 2.6.** *Let  $G$  and  $H$  be locally compact groups such that  $\text{deg}(G) < \infty$ . Then the canonical contraction from  $A(G) \otimes^\gamma A(H)$  into  $A(G) \hat{\otimes} A(H)$  is a topological isomorphism whose inverse has norm at most  $\text{deg}(G)$ .*

We can now prove the main result of this section:

**Theorem 2.7.** *Let  $G$  be a locally compact group. Then  $\text{AM}_{A(G)} \leq \text{deg}(G)$  holds.*

*Proof.* Since the claim is trivial if  $\text{deg}(G) = \infty$ , we can suppose that  $\text{deg}(G) < \infty$ . Then  $G$  is, in particular, amenable. By [Rua, Theorem 3.6], this means that  $A(G)$  is operator amenable, i.e., there is a bounded net  $(d_\alpha)_\alpha$  in  $A(G) \hat{\otimes} A(G)$  such that (2) and (3) hold (with  $\hat{\otimes}$  instead of  $\otimes^\gamma$ ). An inspection of the proof of [Rua, Theorem 3.6] shows that  $(d_\alpha)_\alpha$  can be chosen to have bound one. By Corollary 2.6,  $(d_\alpha)_\alpha$  can be viewed as a net in  $A(G) \otimes^\gamma A(G)$ , bounded by  $\text{deg}(G)$ . Hence,  $A(G)$  is  $\text{deg}(G)$ -amenable.  $\square$

Let  $G$  be a locally compact group, and let  $H$  be a closed, abelian subgroup of  $G$  with finite index. Then  $A(G)$  is amenable by [L-L-W, Theorem 4.1], and an inspection of the proof of [L-L-W, Theorem 4.1] shows that  $A(G)$  is, in fact,  $[G : H]$ -amenable.

We shall devote the remainder of this section to showing that Theorem 2.7 provides a better estimate.

The following was proved in [Tho] for the case of a normal subgroup ([Tho, Satz 5]):

**Proposition 2.8.** *Let  $G$  be a group, and let  $H$  be an abelian subgroup of  $G$  of finite index. Then  $\text{deg}(G) \leq [G : H]$  holds.*

*Proof.* Set  $n := [G : H]$ , and let  $\pi \in \hat{G}$ . It is well known ([Tho] or [Moo]) that  $m := \text{deg}(\pi) < \infty$ .

We may view  $\pi$  as a  $*$ -representation of the Banach  $*$ -algebra  $\ell^1(G)$ . Then  $\pi(\ell^1(G))$  is isomorphic to the  $C^*$ -algebra  $M_m$  of all complex  $m \times m$ -matrices and  $\pi(\ell^1(H))$  is a commutative  $C^*$ -subalgebra of  $M_m$ . This commutative  $C^*$ -algebra is contained in a maximal commutative  $C^*$ -subalgebra of  $M_m$ , and since—up to unitary equivalence—there is only one such  $C^*$ -subalgebra of  $M_m$ , namely the diagonal matrices, we conclude that  $\dim \pi(\ell^1(H)) \leq m$ .

Let  $x_1, \dots, x_n \in G$  be representatives of the left cosets of  $H$ , and note that

$$\dim \pi(\ell^1(x_j H)) = \dim \pi(x_j) \pi(\ell^1(H)) = \dim \pi(\ell^1(H)) \leq m \quad (j = 1, \dots, n).$$

Since  $\ell^1(G) = \ell^1(x_1 H) \oplus \dots \oplus \ell^1(x_n H)$ , we conclude that

$$m^2 = \dim M_m = \dim \pi(\ell^1(G)) \leq \sum_{j=1}^n \dim \pi(\ell^1(x_j H)) \leq nm,$$

so that  $m \leq n$ .  $\square$

The inequality in Proposition 2.8 can be strict as the following example shows:

**Example.** Let  $A_5$  be the alternating group in five symbols, i.e. the group of all even permutations of  $\{1, \dots, 5\}$ . According to [Con et al.],  $\hat{A}_5$  consists of five elements,  $\pi_1, \dots, \pi_5$  say, with

$$\text{deg}(\pi_1) = 1, \text{deg}(\pi_2) = \text{deg}(\pi_3) = 3, \text{deg}(\pi_4) = 4, \text{and } \text{deg}(\pi_5) = 5,$$

so that

$$\text{AM}_{A(A_5)} = \frac{61}{15} = 4.0666\dots \leq 5 = \text{deg}(A_5).$$

Let  $H$  be an abelian subgroup of  $A_5$ . Assume that  $[A_5 : H] = 5$ . Then  $H$  is contained in a maximal subgroup,  $M$  say, of  $A_5$  whose index is necessarily at most

5. Again according to [Con et al.],  $A_5$  has—up to conjugacy—only three maximal subgroups whose indices are 5, 6, and 10, respectively, so that  $[A_5 : M] = 5$  and thus  $M = H$ . The (up to conjugacy) unique subgroup of  $A_5$  with index 5, however, is isomorphic to  $A_4$ , the alternating group in four symbols, and is therefore not abelian. It follows that  $[A_5 : H] > 5$ .

3. THE CASE  $AM_{A(G)} = 1$

Let  $G$  be a locally compact, almost abelian group. Then Theorem 2.7 provides us with an estimate for  $AM_{A(G)}$  from above. In view of (1) it is clear that it would be naive to expect a similarly simple estimate from below.

Nevertheless, some sort of estimate from below is possible.

By  $B(G)$ , we denote the *Fourier–Stieltjes algebra* of a locally compact group  $G$  (see [Eym]). For any locally compact group  $G$ , we use  $G_d$  to denote the same group equipped with the discrete topology. Finally, the *anti-diagonal* of a group  $G$  is the subset

$$G_\Gamma := \{(x, x^{-1}) : x \in G\}$$

of  $G \times G$ , whose indicator function we denote  $\chi_\Gamma$ .

**Lemma 3.1.** *Let  $G$  be a locally compact group such that  $A(G)$  is amenable. Then  $\chi_\Gamma$  lies in  $B(G_d \times G_d)$  and satisfies  $\|\chi_\Gamma\|_{B(G_d \times G_d)} \leq AM_{A(G)}$ .*

*Proof.* For any function  $f: G \rightarrow \mathbb{C}$ , define

$$\check{f}: G \rightarrow G, \quad x \mapsto f(x^{-1}).$$

The map

$$\check{\vee}: A(G) \rightarrow A(G), \quad f \mapsto \check{f}$$

is an isometry (see [Eym]).

Let  $(d_\alpha)_\alpha$  be an approximate diagonal for  $A(G)$  bounded by  $AM_{A(G)}$ . Since  $\check{\vee}: A(G) \rightarrow A(G)$  is an isometry,  $((\text{id} \otimes \check{\vee})d_\alpha)_\alpha$  is a net in  $A(G) \otimes^\gamma A(G)$  that is also bounded by  $AM_{A(G)}$ . We have a canonical contraction from  $A(G) \otimes^\gamma A(G)$  into  $B(G_d \otimes G_d)$  and may thus view  $((\text{id} \otimes \check{\vee})d_\alpha)_\alpha$  as a net in  $B(G_d \times G_d)$  bounded by  $AM_{A(G)}$ . From (2) and (3), it follows that  $((\text{id} \otimes \check{\vee})d_\alpha)_\alpha$  converges to  $\chi_\Gamma$  pointwise on  $G \times G$ . By [Eym, (2.25) Corollaire], this means that  $\chi_\Gamma \in B(G_d \times G_d)$  with  $\|\chi_\Gamma\|_{B(G_d \times G_d)} \leq AM_{A(G)}$ . □

Lemma 3.1 can be used to give a more direct proof of [F–R, Theorem 2.3].

Recall that the *coset ring*  $\Omega(G)$  of a group  $G$  is the ring of subsets of  $G$  generated by all left cosets of subgroups of  $G$ .

**Proposition 3.2.** *The following are equivalent for a group  $G$ :*

- (i)  $G$  is almost abelian;
- (ii)  $G_\Gamma \in \Omega(G \times G)$ ;
- (iii)  $\chi_\Gamma \in B(G \times G)$ .

*Proof.* (i)  $\iff$  (ii) is [F–R, Proposition 2.2] and (ii)  $\iff$  (iii) follows from Host’s idempotent theorem ([Hos]). □

Combining Lemma 3.1 and Proposition 3.2, we immediately recover [F–R, Theorem 2.3] the following corollary.

**Corollary 3.3.** *The following are equivalent for a locally compact group  $G$ :*

- (i)  $G$  is almost abelian;
- (ii)  $A(G)$  is amenable.

*Remark.* The present proof for Corollary 3.3 is more direct than the one give in [F–R] because it invokes Host’s idempotent theorem directly instead of making the detour over [For et al.].

It remains to be seen whether or not Lemma 3.1 will eventually lead to a more satisfactory bound from below for the amenability constant of a Fourier algebra: very little seems to be known on the norms of idempotents in Fourier–Stieltjes algebras (see the remark below, following Theorem 3.5).

Let  $G$  be an abelian locally compact group, so that  $A(G) \cong L^1(\hat{G})$ . In view of [Sto, Corollary 1.11], this means that  $\text{AM}_{A(G)} = 1$ . Concluding this note, we shall now see that the locally compact groups  $G$  for which  $\text{AM}_{A(G)} = 1$  are precisely the abelian ones. The two ingredients of the proof are Lemma 3.1 and the following proposition that parallels Proposition 3.2:

**Proposition 3.4.** *The following are equivalent for a group  $G$ :*

- (i)  $G$  is abelian;
- (ii)  $G_\Gamma$  is a subgroup of  $G \times G$ ;
- (iii)  $\chi_\Gamma$  lies in  $B(G \times G)$  and has norm one.

*Proof.* (i)  $\iff$  (ii) is straightforward.

(ii)  $\implies$  (iii): If  $G_\Gamma$  is a subgroup of  $G \times G$ , its indicator function is positive definite so that  $\|\chi_\Gamma\|_{B(G \times G)} = \chi_\Gamma(e, e) = 1$ .

(iii)  $\implies$  (ii): By [I–S, Theorem 2.1],  $G_\Gamma$  must be a left coset of some subgroup of  $G \times G$ . Since  $(e, e) \in G_\Gamma$ , this means that  $G_\Gamma$  is, in fact, a subgroup of  $G \times G$ .  $\square$

**Theorem 3.5.** *The following are equivalent for a locally compact group  $G$ :*

- (i)  $G$  is abelian;
- (ii)  $\text{AM}_{A(G)} = 1$ .

*Proof.* We have already observed that (i)  $\implies$  (ii) holds. The converse is an immediate consequence of Lemma 3.1 and Proposition 3.4.  $\square$

*Remark.* If  $G$  is a finite, nonabelian group,  $\text{AM}_{A(G)} \geq \frac{3}{2}$  holds by [Joh3, Proposition 4.3]. In view of Theorem 3.5, one wonders if this estimate from below still holds for arbitrary locally compact groups (with  $\frac{3}{2}$  possibly replaced by another universal constant strictly greater than one). In view of Lemma 3.1, one way of obtaining such a constant would be to find an estimate for  $\|\chi_\Gamma\|$  from below. In [Sae], it is proved for *abelian*  $G$  that the norm of an idempotent in  $B(G)$  is either one or at least  $\frac{1}{2}(1 + \sqrt{2})$ . A similar dichotomy result for general, locally compact groups  $G$  would immediately yield a universal bound (strictly greater than one) from below for  $\text{AM}_{A(G)}$  for nonabelian  $G$ .

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