ON THE OPEN SET CONDITION
FOR SELF-SIMILAR FRACTALS

CHRISTOPH BANDT, NGUYEN VIET HUNG, AND HUI RAO

Abstract. For self-similar sets, the existence of a feasible open set is a natural separation condition which expresses geometric as well as measure-theoretic properties. We give a constructive approach by defining a central open set and characterizing those points which do not belong to feasible open sets.

1. Introduction

1.1. Self-similar sets. Let \( f_1, \ldots, f_m \) be contracting similarity maps on \( \mathbb{R}^n \), that is,
\[
|f_i(x) - f_i(y)| = r_i \cdot |x - y| \quad \text{for all } x, y \in \mathbb{R}^n,
\]
where the \( r_i \in (0, 1) \) are the contraction factors and \( |\cdot| \) denotes the Euclidean norm. There is a unique compact set \( A \) which satisfies the set equation
\[
A = f_1(A) \cup \ldots \cup f_m(A)
\]
and is called the self-similar set generated by the maps \( f_i \). The set \( A \) consists of similar copies \( A_i = f_i(A) \) of itself, where each \( A_i \) consists of smaller copies \( A_{ij} = f_j(f_i(A)) \), and so on. For any integer \( n \), we can consider the set \( \mathcal{S}^n \) of words \( i = i_1 \ldots i_n \) from the alphabet \( \mathcal{S} = \{1, \ldots, m\} \). Writing \( f_i = f_{i_1} \ldots f_{i_n} \) and \( A_i = f_i(A) \), we can express \( A \) as
\[
A = \bigcup \{ A_i \mid i \in \mathcal{S}^n \}.
\]
When \( n \) tends to infinity, this induces a continuous map \( \pi : \mathcal{S}^\infty \to A \) from the set \( \mathcal{S}^\infty \) of sequences \( i_1 i_2 i_3 \ldots \) onto the self-similar set, the so-called address map.

The open set condition (OSC). The \( f_i \) are said to satisfy the OSC if there exists a nonempty open set \( V \subset \mathbb{R}^n \) such that
\[
\bigcup_{i=1}^m f_i(V) \subseteq V \quad \text{and} \quad f_i(V) \cap f_j(V) = \emptyset \quad \text{for } i \neq j.
\]
We call \( V \) a feasible open set of the \( f_i \), or of \( A \). The OSC controls the overlap of the \( A_i \). It was introduced by P.A.P. Moran \cite{Moran1946} in 1946 in order to show that the
canonical Hausdorff measure is positive on $A$. More recently, Schief [6] proved the converse: positive Hausdorff measure implies OSC. They also proved that OSC is equivalent to a combinatorial condition: there exists an integer $N$ such that at most $N$ incomparable pieces $A_j$ of size $\geq \varepsilon$ can intersect the $\varepsilon$-neighborhood of a piece $A_i$ of diameter $\varepsilon$.

The neighbor map condition. An algebraic equivalent for OSC was given by Bandt and Graf [1]. The inverse map $f_i^{-1}$ is used to transform the small pieces $f_i(A)$ and $f_j(A)$ into $A$ and $h(A) = f_i^{-1} f_j(A)$, respectively. Here $h(A)$ is “the potential neighbor set” of $A$ if we imagine that the self-similar structure is extended outward from $A$. The second requirement of the OSC can be written as $f_i(V) \cap f_j(V) = \emptyset$ for $i_1 \neq j_1$, and this is equivalent to $V \cap f_i^{-1} f_j(V) = \emptyset$. If such an open set $V$ exists, then the map $h = f_i^{-1} f_j$ cannot be near the identity map $id$. Let $S^* = \bigcup_{n \geq 1} S^n$. The maps in

$$N = \{h = f_i^{-1} f_j \mid i, j \in S^*, \ i_1 \neq j_1\}$$

will be called neighbor maps. The algebraic formulation of OSC reads as follows.

There is a constant $\kappa > 0$ such that $\|h - id\| > \kappa$ for all neighbor maps $h$.

The norm of an affine mapping $g$ on $\mathbb{R}^n$ is $\|g\| = \sup_{\|x\| \leq 1} |g(x)|$, as usual. Geometrically, the condition says that compared to their size, two pieces $A_i$ and $A_j$ cannot be arbitrarily close to each other.

2. The central open set

So far, no algorithm is known to construct feasible open sets. Here we give a constructive approach. Let us say that $x \in \mathbb{R}^n$ is a forbidden point for $A$ if there is no feasible open set $V$ containing $x$.

All points of a “neighbor set” $h(A) = f_i^{-1} f_j(A)$ are forbidden points for $A$. This will follow from Proposition 5 and can easily be proved directly. Thus an open set $V$ cannot contain points of the set $H = \bigcup \{h(A) \mid h \in N\}$. So let us define the central open set for $f_1, \ldots, f_m$ as

$$V_c = \{x \mid d(x, A) < d(x, H)\},$$
where \( d(x, Y) = \inf \{|x-y| \, | \, y \in Y \} \) denotes the distance from a point to a set. Thus \( d(x, H) = \inf \{d(x, h(A)) \, | \, h \in \mathcal{N} \} \). The definition of \( V_c \) resembles the construction of a fundamental domain for a transformation group, where the neighbor maps play the part of the transformations.

**Theorem 1.** If OSC holds, the central open set \( V_c \) is a feasible open set. If OSC does not hold, then \( V_c \) is empty.

**Proof.** We show that \( V_c \) fulfills condition (1) whenever it is nonempty. It follows that if OSC does not hold, \( V_c \) must be empty. To verify \( f_i(V_c) \subseteq V_c \), consider a point \( x \in V_c \) and \( y = f_i(x) \). Since \( f_i \) is a similarity map with factor \( r_i \),

\[
d(y, A) \leq d(y, f_i(A)) = r_i \cdot d(x, A) < r_i \cdot d(x, H) = d(y, f_i(H)) \leq d(y, H).
\]

In the last step, we used \( H \subseteq f_i(H) \) which follows from the fact that each \( f^{-1}_i f_j(a) \) can be written as \( f_i(f_j^{-1} f_i^{-1} f_j(a)) \).

Next, let \( i \neq j \) and \( x \in V_c \). Then \( d(x, A) < d(x, f_i^{-1} f_j(A)) \) by definition of \( V_c \). Applying \( f_i \) on both sides of the inequality and dividing by \( r_i \), we obtain

\[
d(f_i(x), f_i(A)) < d(f_i(x), f_j(A)).
\]

The points of \( f_i(V_c) \) are nearer to \( A_i \) than to any other piece of \( A \). A similar statement holds for the points of \( f_j(V_c) \). This proves \( f_i(V_c) \cap f_j(V_c) = \emptyset \).

If OSC holds, then by Schief [6] there is an open set \( V \) with \( A \cap V \neq \emptyset \). Since \( V \) does not intersect \( \overline{H} \) (cf. Proposition 3), each point in \( V \cap A \) belongs to \( V_c \), and so \( V_c \) is nonempty. \( \Box \)

**Corollary 2.** OSC holds if and only if \( A \) is not contained in \( \overline{H} \).

M. Moran [4] even claimed that OSC holds if and only if \( H \) contains no dense subset of \( A \), that is, \( A \neq \overline{A} \cap H \). But since his proof contains a gap, his assertion remains open as long as we do not know more about \( H \). So let us study the structure of forbidden points of \( A \) in greater detail.

![Figure 2. a) Neighbor sets and central open set for the Koch curve. b) A feasible open set which contains the central one.](image-url)
3. The fixed points of neighbor maps

Hutchinson [3] characterized the self-similar set $A$ as the closure of the set of fixed points of the $f_i, i \in S^*$. It turns out that the set of forbidden points has a similar structure. Let $J$ denote the set of fixed points of neighbor maps,

$$J = \{ x \in \mathbb{R}^n \mid h(x) = x \text{ for some } h \in \mathcal{N}\}.$$ 

Proposition 3. All points of the closure $\overline{J}$ are forbidden points for $A$, and $H \subseteq \overline{J}$.

Proof. Assume $x \in \overline{J}$ belongs to an open set $V$. Then $V$ contains the fixed point $y$ of a neighbor map $f_i^{-1} f_j$. Thus $f_i(V) \cap f_j(V)$ contains $f_i(y)$, so $V$ is not feasible.

To show that $H \subseteq \overline{J}$ we fix a point $b \in H$, say $b \in f_i^{-1} f_j(A)$. We assume that $|i| < |j|$ so that $h = f_i^{-1} f_j$ is contractive. Actually we may enlarge $j$ as far as we want, for the only requirement is that $f_i(b) \in f_j(A)$.

Let $c$ be the fixed point of the map $h = f_i^{-1} f_j$. For $a = h^{-1}(b)$ we have

$$|c - b| = |h(c) - h(a)| = r_h \cdot |c - a|.$$ 

We fix $i$ and extend $j$ to obtain a sequence of $h_n$ such that $b \in h_n(A)$. In this procedure, one can show that $c$ is bounded, $r_h$ tends to 0, and $a$ is bounded since it is in $A$. Therefore, the fixed points of $h_n$ tend to $b$. □

Our problem here is whether there are forbidden points outside $\overline{J}$. For self-similar sets on the line, we will prove that there are no such points. When OSC holds, our statement is a bit more general. We start with a simple observation.

Lemma 4 (cf. [1], Prop. 1(i)). Given a forbidden point $x$ of $A$ and an $\varepsilon > 0$, there is a neighbor map $h$ with $|h(x) - x| < \varepsilon$.

Proof. Let $B$ denote the open ball with center $x$ and radius $\varepsilon/2$, and consider $V = \bigcup_{i \in S^*} f_i(B)$. Since $V$ is not a feasible open set, $f_i(V) \cap f_j(V) \neq \emptyset$ for some $i \neq j$. So there are $i, j \in S^*$ with $f_if_i(B) \cap f_jf_j(B) \neq \emptyset$. The map $h = f_i^{-1} f_j$ fulfills the condition if $h$ is contractive; otherwise $h^{-1}$ fulfills the condition. □

Theorem 5. Let the mappings $f_i(x) = u_i x + v_i$ on $\mathbb{C}^n$ with $u_i \in \mathbb{C}$ and $v_i \in \mathbb{C}^n$ satisfy OSC. Then any forbidden point belongs to $\overline{J}$.

Proof. Any neighbor map can be written as $h(x) = r x + (1 - r)c$ or $h(x) = x + b$ with $r \in \mathbb{C}$ and $c, b \in \mathbb{C}^n$.

Let $x$ be a forbidden point. OSC means $\|h - id\| > \kappa$ for all neighbor maps and some $\kappa > 0$. Let $\varepsilon < \kappa/2$ be given, and take $h$ from Lemma 4. $h$ cannot be a translation since then $\|h - id\| = |h(x) - x|$. Thus

$$\varepsilon > |h(x) - x| = |1 - r| \cdot |c - x|.$$ 

By the definition of norm, there is a $y$ with $|y| \leq 1$ such that

$$\kappa \leq |h(y) - y| = |1 - r| \cdot |c - y|.$$ 

Subtracting the two inequalities and dividing by $|1 - r|$, we obtain

$$(\kappa - \varepsilon)/|1 - r| < |c - y| - |c - x| \leq |x - y| \leq |x| + 1.$$
Now we use the first inequality again:
\[ |c - x| < \frac{\varepsilon}{|1 - r|} < \frac{\varepsilon}{\kappa - \varepsilon} \cdot (|x| + 1) < \frac{2\varepsilon}{\kappa} \cdot (|x| + 1). \]

For \( \varepsilon \to 0 \) this shows that \( x \) is in \( \mathcal{J} \).

\[ \square \]

**Theorem 6.** An IFS on \( \mathbb{R} \) does not satisfy OSC if and only if \( \mathcal{J} = \mathbb{R} \).

**Proof.** We need only show that no OSC implies that \( \mathcal{J} = \mathbb{R} \). Let \( f_i(x) = r_i x + d_i \), \( 1 \leq i \leq m \). Without loss of generality, we may assume that \( d_1 = 0 \). It follows that \( 0 \in A \). Define \( D = \{ h(0) \mid h \in \mathcal{N} \} \).

First \( D \subseteq \mathcal{J} \), for \( h(0) \in h(A) \subset \mathcal{J} \) by Proposition 3. In the following, we will show that \( \overline{D} = \mathbb{R} \) and hence \( \mathcal{J} = \mathbb{R} \).

We can assume that \( r_1 > 0 \). Otherwise we consider the IFS formed by the \( f_{ij} \) with \( i, j \in S \) for which \( J \) is not larger and \( f_{11}(x) = r_1^2 x \).

Take any neighbor map \( h = f_{i1}^{-1} f_{j1}(x) \), where \( f_{i1}^{-1}(x) = a_1 x + b_1 \) and \( f_{j1}(x) = a_2 x + b_2 \). Then \( h(x) = a_1 a_2 x + a_1 b_2 + b_1 \). Pick any \( \delta > 0 \); we will show that there exists a neighbor map \( h^* \in \mathcal{N} \) such that
\[ \frac{\delta}{2} \leq h^*(0) - h(0) \leq (1/r_1 + 1/2)\delta. \]

Let \( g(x) = ax + b \) be a neighbor map in \( \mathcal{N} \). Denote \( g^{-1}(x) = x/a - b/a = a' x + b' \).

Since there is no OSC, we can choose \( g \) arbitrarily near to the identity map. Without loss of generality, we may assume that \( a_1 b > 0 \); otherwise we exchange \( g \) and \( g^{-1} \).

We choose \( g \) so near to the identity map that
\[ |(a - 1)a_1 b_2| < \frac{\delta}{2}, \quad |a_1 b| < \delta \quad \text{and} \quad |(a' - 1)a_1 b_2| < \frac{\delta}{2}, \quad |a_1 b'| < \delta. \]

Let \( g_1(x) = (f_{i1}^{-1})^k \cdot g \cdot f_{j1}(x) \); then \( g_1(x) \in \mathcal{N} \) and \( g_1(x) = ax + r_1^{-k} b \). Choose the integer \( k \) such that
\[ \delta \leq r_1^{-k} a_1 b < \frac{\delta}{r_1}. \]

Let \( h^*(x) = f_{i1}^{-1} \cdot g_1 \cdot f_{j1}(x) \). Then
\[ \frac{\delta}{2} < h^*(0) - h(0) = (1 - a_1) a_1 b_2 + r_1^{-k} a_1 b < (1/r_1 + 1/2)\delta. \]

Likewise let \( g_2(x) = (f_{i1}^{-1})^k \cdot g^{-1} \cdot f_{j1}(x) \), where \( k \) is chosen such that \( \delta \leq -r_1^{-k} a_1 b < \delta/r_1 \). We get a map \( h_*(x) = f_{i1}^{-1} \cdot g_2 \cdot f_{j1}(x) \in \mathcal{N} \) such that
\[ \frac{\delta}{2} \leq h_*(0) - h_*(0) \leq (1/r_1 + 1/2)\delta. \]

Hence for any \( \delta > 0 \) and any \( x \in D \), there are two points \( y_1, y_2 \in D \) such that
\[ \frac{\delta}{2} \leq y_1 - x \leq (1/r_1 + 1/2)\delta, \quad \frac{\delta}{2} \leq x - y_2 \leq (1/r_1 + 1/2)\delta. \]

Therefore \( \overline{D} = \mathbb{R} \).

\[ \square \]

**References**


(93d:28014)


(92j:28008)


(82h:49020)


Institute for Mathematics, Arndt University, 17487 Greifswald, Germany
E-mail address: bandt@uni-greifswald.de

Institute for Mathematics, Arndt University, 17487 Greifswald, Germany
E-mail address: nvh0@yahoo.com

Department of Mathematics, Tsinghua University, P.O. Box 100084, Beijing, People’s Republic of China
E-mail address: HRao@math.tsinghua.edu.cn