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ON THE OPEN SET CONDITION FOR SELF-SIMILAR FRACTALS

CHRISTOPH BANDT, NGUYEN VIET HUNG, AND HUI RAO

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Abstract. For self-similar sets, the existence of a feasible open set is a natural separation condition which expresses geometric as well as measure-theoretic properties. We give a constructive approach by defining a central open set and characterizing those points which do not belong to feasible open sets.

1. Introduction

1.1. **Self-similar sets.** Let $f_1, ..., f_m$ be contracting similarity maps on $\mathbb{R}^n$, that is,

$$|f_i(x) - f_i(y)| = r_i \cdot |x - y| \quad \text{for all } x, y \in \mathbb{R}^n,$$

where the $r_i \in (0, 1)$ are the contraction factors and $|\cdot|$ denotes the Euclidean norm. There is a unique compact set A which satisfies the set equation

$$A = f_1(A) \cup ... \cup f_m(A)$$

and is called the self-similar set generated by the maps f_i [3, 2]. The set A consists of similar copies $A_i = f_i(A)$ of itself, where each A_i consists of smaller copies $A_{ij} = f_i(f_j(A))$, and so on. For any integer n, we can consider the set S^n of words $\mathbf{i} = i_1...i_n$ from the alphabet $S = \{1, ..., m\}$. Writing $f_{\mathbf{i}} = f_{i_1} \dots f_{i_n}$ and $A_{\mathbf{i}} = f_{\mathbf{i}}(A)$, we can express A as

$$A = \bigcup \{A_{\mathbf{i}} \mid \mathbf{i} \in S^n\} \ .$$

When n tends to infinity, this induces a continuous map $\pi : S^\infty \to A$ from the set S^∞ of sequences $i_1 i_2 i_3 ...$ onto the self-similar set, the so-called address map.

The open set condition (OSC). The f_i are said to satisfy the OSC if there exists a nonempty open set $V \subset \mathbb{R}^n$ such that

$$\bigcup_{i=1}^{m} f_i(V) \subseteq V \quad \text{and} \quad f_i(V) \cap f_j(V) = \emptyset \ \text{ for } i \neq j. \tag{1}$$

We call V a *feasible open set* of the f_i, or of A. The OSC controls the overlap of the A_i. It was introduced by P.A.P. Moran [5] in 1946 in order to show that the

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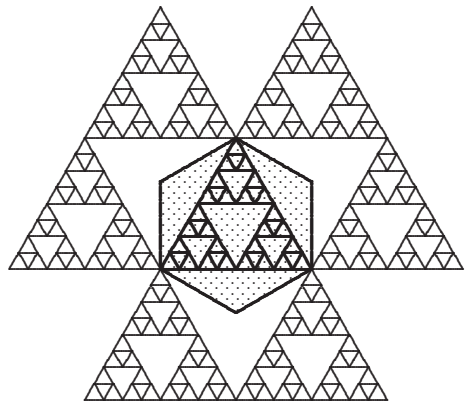

FIGURE 1. Some neighbor sets and central open set for the Sierpinski gasket.

canonical Hausdorff measure is positive on A. More recently, Schief [6] proved the converse: positive Hausdorff measure implies OSC. They also proved that OSC is equivalent to a combinatorial condition: there exists an integer N such that at most N incomparable pieces $A_{\mathbf{j}}$ of size $\geq \varepsilon$ can intersect the ε-neighborhood of a piece $A_{\mathbf{i}}$ of diameter ε.

The neighbor map condition. An algebraic equivalent for OSC was given by Bandt and Graf [1]. The inverse map $f_{\mathbf{i}}^{-1}$ is used to transform the small pieces $f_{\mathbf{i}}(A)$ and $f_{\mathbf{j}}(A)$ into A and $h(A) = f_{\mathbf{i}}^{-1} f_{\mathbf{j}}(A)$, respectively. Here $h(A)$ is "the potential neighbor set" of A if we imagine that the self-similar structure is extended outward from A. The second requirement of the OSC can be written as $f_{\mathbf{i}}(V) \cap f_{\mathbf{j}}(V) = \emptyset$ for $i_1 \neq j_1$, and this is equivalent to $V \cap f_{\mathbf{i}}^{-1} f_{\mathbf{j}}(V) = \emptyset$. If such an open set V exists, then the map $h = f_{\mathbf{i}}^{-1} f_{\mathbf{j}}$ cannot be near the identity map id. Let $S^* = \bigcup_{n \geq 1} S^n$. The maps in

$$\mathcal{N} = \{h = f_{\mathbf{i}}^{-1} f_{\mathbf{j}} |\ \mathbf{i}, \mathbf{j} \in S^*,\ i_1 \neq j_1\}$$

will be called *neighbor maps.* The algebraic formulation of OSC reads as follows.

There is a constant $\kappa > 0$ such that $\|h - id\| > \kappa$ for all neighbor maps h.

The norm of an affine mapping g on $\mathbb{R}^n$ is $\|g\| = \sup_{|x| \leq 1} |g(x)|$, as usual. Geometrically, the condition says that *compared to their size*, two pieces $A_{\mathbf{i}}$ and $A_{\mathbf{j}}$ cannot be arbitrarily close to each other.

2. The central open set

So far, no algorithm is known to construct feasible open sets. Here we give a constructive approach. Let us say that $x \in \mathbb{R}^n$ is a *forbidden point* for A if there is no feasible open set V containing x.

All points of a "neighbor set" $h(A) = f_{\mathbf{i}}^{-1} f_{\mathbf{j}}(A)$ are forbidden points for A. This will follow from Proposition 3 and can easily be proved directly. Thus an open set V cannot contain points of the set $H = \bigcup \{h(A) \mid h \in \mathcal{N}\}$. So let us define the *central open set* for $f_1, ..., f_m$ as

$$V_c = \{x \mid d(x, A) < d(x, H)\},$$

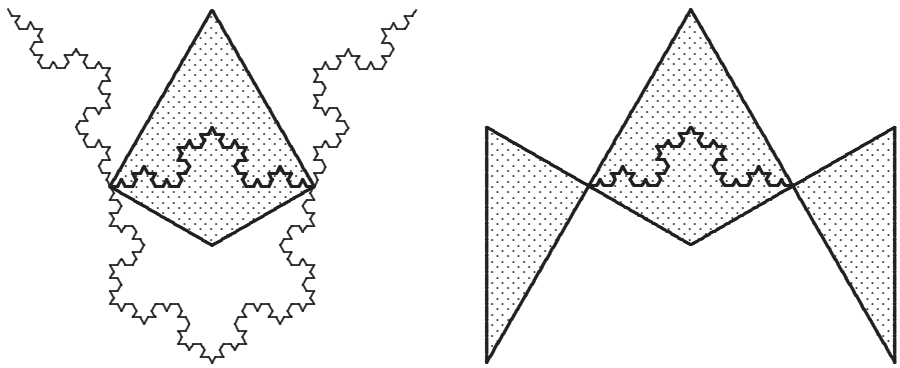

FIGURE 2. a) Neighbor sets and central open set for the Koch curve.
b) A feasible open set which contains the central one.

where $d(x, Y) = \inf\{|x-y| \mid y \in Y\}$ denotes the distance from a point to a set. Thus $d(x, H) = \inf\{d(x, h(A)| \; h \in \mathcal{N}\}$. The definition of V_c resembles the construction of a fundamental domain for a transformation group, where the neighbor maps play the part of the transformations.

Theorem 1. *If OSC holds, the central open set V_c is a feasible open set. If OSC does not hold, then V_c is empty.*

Proof. We show that V_c fulfils condition (1) whenever it is nonempty. It follows that if OSC does not hold, V_c must be empty. To verify $f_i(V_c) \subseteq V_c$, consider a point $x \in V_c$ and $y = f_i(x)$. Since f_i is a similarity map with factor r_i,

$$d(y, A) \leq d(y, f_i(A)) = r_i \cdot d(x, A) < r_i \cdot d(x, H) = d(y, f_i(H)) \leq d(y, H).$$

In the last step, we used $H \subseteq f_i(H)$ which follows from the fact that each $f_{\mathbf{i}}^{-1} f_{\mathbf{j}}(a)$ can be written as $f_i(f_i^{-1} f_{\mathbf{i}}^{-1} f_{\mathbf{j}}(a))$.

Next, let $i \neq j$ and $x \in V_c$. Then $d(x, A) < d(x, f_i^{-1} f_j(A))$ by definition of V_c. Applying f_i on both sides of the inequality and dividing by r_i, we obtain

$$d(f_i(x), f_i(A)) < d(f_i(x), f_j(A)) \; .$$

The points of $f_i(V_c)$ are nearer to A_i than to any other piece of A. A similar statement holds for the points of $f_j(V_c)$. This proves $f_i(V_c) \cap f_j(V_c) = \emptyset$.

If OSC holds, then by Schief [6] there is an open set V with $A \cap V \neq \emptyset$. Since V does not intersect $\overline{H}$ (cf. Proposition 3), each point in $V \cap A$ belongs to V_c, and so V_c is nonempty. □

Corollary 2. *OSC holds if and only if A is not contained in $\overline{H}$.*

M. Moran [4] even claimed that OSC holds if and only if H contains no dense subset of A, that is, $A \neq \overline{A \cap H}$. But since his proof contains a gap, his assertion remains open as long as we do not know more about H. So let us study the structure of forbidden points of A in greater detail.

3. The fixed points of neighbor maps

Hutchinson [3] characterized the self-similar set A as the closure of the set of fixed points of the $f_{\mathbf{i}}, \mathbf{i} \in S^*$. It turns out that the set of forbidden points has a similar structure. Let J denote the set of fixed points of neighbor maps,

$$J = \{x \in \mathbb{R}^n \mid h(x) = x \text{ for some } h \in \mathcal{N}\}.$$

Proposition 3. *All points of the closure $\overline{J}$ are forbidden points for A, and $H \subseteq \overline{J}$.*

Proof. Assume $x \in \overline{J}$ belongs to an open set V. Then V contains the fixed point y of a neighbor map $f_{\mathbf{i}}^{-1} f_{\mathbf{j}}$. Thus $f_{\mathbf{i}}(V) \cap f_{\mathbf{j}}(V)$ contains $f_{\mathbf{i}}(y)$, so V is not feasible.

To show that $H \subseteq \overline{J}$ we fix a point $b \in H$, say $b \in f_{\mathbf{i}}^{-1} f_{\mathbf{j}}(A)$. We assume that $|\mathbf{i}| < |\mathbf{j}|$ so that $h = f_{\mathbf{i}}^{-1} f_{\mathbf{j}}$ is contractive. Actually we may enlarge $\mathbf{j}$ as far as we want, for the only requirement is that $f_{\mathbf{i}}(b) \in f_{\mathbf{j}}(A)$.

Let c be the fixed point of the map $h = f_{\mathbf{i}}^{-1} f_{\mathbf{j}}$. For $a = h^{-1}(b)$ we have

$$|c - b| = |h(c) - h(a)| = r_h \cdot |c - a|.$$

We fix $\mathbf{i}$ and extend $\mathbf{j}$ to obtain a sequence of h_n such that $b \in h_n(A)$. In this procedure, one can show that c is bounded, r_h tends to 0, and a is bounded since it is in A. Therefore, the fixed points of h_n tend to b. □

Our problem here is whether there are forbidden points outside $\overline{J}$. For self-similar sets on the line, we will prove that there are no such points. When OSC holds, our statement is a bit more general. We start with a simple observation.

Lemma 4 (cf. [1], Prop. 1(i))**.** *Given a forbidden point x of A and an $\varepsilon > 0$, there is a neighbor map h with $|h(x) - x| < \varepsilon$.*

Proof. Let B denote the open ball with center x and radius $\varepsilon/2$, and consider $V = \bigcup_{\mathbf{i} \in S^*} f_{\mathbf{i}}(B)$. Since V is not a feasible open set, $f_i(V) \cap f_j(V) \neq \emptyset$ for some $i \neq j$. So there are $\mathbf{i}, \mathbf{j} \in S^*$ with $f_i f_{\mathbf{i}}(B) \cap f_j f_{\mathbf{j}}(B) \neq \emptyset$. The map $h = f_{i\mathbf{i}}^{-1} f_{j\mathbf{j}}$ fulfils the condition if h is contractive; otherwise h^{-1} fulfils the condition. □

Theorem 5. *Let the mappings $f_i(x) = u_i x + v_i$ on $\mathbb{C}^n$ with $u_i \in \mathbb{C}$ and $v_i \in \mathbb{C}^n$ satisfy OSC. Then any forbidden point belongs to $\overline{J}$.*

Proof. Any neighbor map can be written as

$$h(x) = rx + (1 - r)c \text{ or } h(x) = x + b \text{ with } r \in \mathbb{C} \text{ and } c, b \in \mathbb{C}^n.$$

Let x be a forbidden point. OSC means $\|h - id\| > \kappa$ for all neighbor maps and some $\kappa > 0$. Let $\varepsilon < \kappa/2$ be given, and take h from Lemma 4. h cannot be a translation since then $\|h - id\| = |h(x) - x|$. Thus

$$\varepsilon > |h(x) - x| = |1 - r| \cdot |c - x|.$$

By the definition of norm, there is a y with $|y| \leq 1$ such that

$$\kappa \leq |h(y) - y| = |1 - r| \cdot |c - y|.$$

Subtracting the two inequalities and dividing by $|1 - r|$, we obtain

$$(\kappa - \varepsilon)/|1 - r| < |c - y| - |c - x| \leq |x - y| \leq |x| + 1 \ .$$

Now we use the first inequality again:

$$|c-x| < \frac{\varepsilon}{|1-r|} < \frac{\varepsilon}{\kappa-\varepsilon}\cdot(|x|+1) < \frac{2\varepsilon}{\kappa}\cdot(|x|+1).$$

For $\varepsilon \to 0$ this shows that x is in $\overline{J}$. □

Theorem 6. *An IFS on $\mathbb{R}$ does not satisfy OSC if and only if $\overline{J} = \mathbb{R}$.*

Proof. We need only show that no OSC implies that $\overline{J} = \mathbb{R}$. Let $f_i(x) = r_i x + d_i$, $1 \le i \le m$. Without loss of generality, we may assume that $d_1 = 0$. It follows that $0 \in A$. Define $D = \{h(0) \mid h \in \mathcal{N}\}$.

First $D \subseteq \overline{J}$, for $h(0) \in h(A) \subset \overline{J}$ by Proposition 3. In the following, we will show that $\overline{D} = \mathbb{R}$ and hence $\overline{J} = \mathbb{R}$.

We can assume that $r_1 > 0$. Otherwise we consider the IFS formed by the f_{ij} with $i, j \in S$ for which J is not larger and $f_{11}(x) = r_1^2 x$.

Take any neighbor map $h = f_{\mathbf{i}}^{-1} f_{\mathbf{j}}(x)$, where $f_{\mathbf{i}}^{-1}(x) = a_1 x + b_1$ and $f_{\mathbf{j}}(x) = a_2 x + b_2$. Then $h(x) = a_1 a_2 x + a_1 b_2 + b_1$. Pick any $\delta > 0$; we will show that there exists a neighbor map $h^* \in \mathcal{N}$ such that

$$\delta/2 \le h^*(0) - h(0) \le (1/r_1 + 1/2)\delta.$$

Let $g(x) = ax+b$ be a neighbor map in $\mathcal{N}$. Denote $g^{-1}(x) = x/a - b/a = a'x + b'$. Since there is no OSC, we can choose g arbitrarily near to the identity map. Without loss of generality, we may assume that $a_1 b > 0$; otherwise we exchange g and g^{-1}. We choose g so near to the identity map that

$$|(a-1)a_1 b_2| < \delta/2,\ |a_1 b| < \delta \text{ and } |(a'-1)a_1 b_2| < \delta/2,\ |a_1 b'| < \delta.$$

Let $g_1(x) = (f_1^{-1})^k \cdot g \cdot f_1^k(x)$; then $g_1(x) \in \mathcal{N}$ and $g_1(x) = ax + r_1^{-k} b$. Choose the integer k such that

$$\delta \le r_1^{-k} a_1 b < \delta/r_1.$$

Let $h^*(x) = f_{\mathbf{i}}^{-1} \cdot g_1 \cdot f_{\mathbf{j}}(x)$. Then

$$\delta/2 < h^*(0) - h(0) = (a-1)a_1 b_2 + r_1^{-k} a_1 b < (1/r_1 + 1/2)\delta.$$

Likewise let $g_2(x) = (f_1^{-1})^k \cdot g^{-1} \cdot f_1^k(x)$, where k is chosen such that $\delta \le -r_1^{-k} a_1 b' < \delta/r_1$. We get a map $h_*(x) = f_{\mathbf{i}}^{-1} \cdot g_2 \cdot f_{\mathbf{j}}(x) \in \mathcal{N}$ such that

$$\delta/2 \le h(0) - h_*(0) \le (1/r_1 + 1/2)\delta.$$

Hence for any $\delta > 0$ and any $x \in D$, there are two points $y_1, y_2 \in D$ such that

$$\delta/2 \le y_1 - x \le (1/r_1 + 1/2)\delta, \quad \delta/2 \le x - y_2 \le (1/r_1 + 1/2)\delta.$$

Therefore $\overline{D} = \mathbb{R}$. □

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Institute for Mathematics, Arndt University, 17487 Greifswald, Germany
E-mail address: bandt@uni-greifswald.de

Institute for Mathematics, Arndt University, 17487 Greifswald, Germany
E-mail address: nvh0@yahoo.com

Department of Mathematics, Tsinghua University, P.O. Box 100084, Beijing, People's Republic of China
E-mail address: HRao@math.tsinghua.edu.cn