

## A NOTE ON PERIODIC POINTS OF ORDER PRESERVING SUBHOMOGENEOUS MAPS

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ABSTRACT. Let  $\mathbb{R}_+^n$  be the standard closed positive cone in  $\mathbb{R}^n$  and let  $\Gamma(\mathbb{R}_+^n)$  be the set of integers  $p \geq 1$  for which there exists a continuous, order preserving, subhomogeneous map  $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ , which has a periodic point with period  $p$ . It has been shown by Akian, Gaubert, Lemmens, and Nussbaum that  $\Gamma(\mathbb{R}_+^n)$  is contained in the set  $B(n)$  consisting of those  $p \geq 1$  for which there exist integers  $q_1$  and  $q_2$  such that  $p = q_1 q_2$ ,  $1 \leq q_1 \leq \binom{n}{m}$ , and  $1 \leq q_2 \leq \binom{m}{\lfloor m/2 \rfloor}$  for some  $1 \leq m \leq n$ . This note shows that  $\Gamma(\mathbb{R}_+^n) = B(n)$  for all  $n \geq 1$ .

### 1. INTRODUCTION

Let  $K$  be a polyhedral cone in a finite dimensional real vector space equipped with the norm topology and let  $f: K \rightarrow K$  be a continuous map that preserves the ordering induced by  $K$ . Suppose, in addition, that  $f: K \rightarrow K$  is subhomogeneous, that is,  $\lambda f(x) \leq f(\lambda x)$  for all  $0 \leq \lambda \leq 1$  and  $x \in K$ . It was proved by Akian et al. [2, Theorem 2.1] that every bounded orbit of  $f: K \rightarrow K$  converges to a periodic orbit and, moreover, that the period of each periodic point of  $f$  is bounded by

$$(1) \quad \beta_N = \max_{1 \leq m \leq N} \binom{N}{m} \binom{m}{\lfloor m/2 \rfloor} = \frac{N!}{\lfloor \frac{N}{3} \rfloor! \lfloor \frac{N+1}{3} \rfloor! \lfloor \frac{N+2}{3} \rfloor!} \sim \frac{3^{N+1} \sqrt{3}}{2\pi N},$$

where  $N$  is the number of facets of the polyhedral cone  $K$ . In fact, it follows from [2, Theorem 5.1] that the set of possible periods of periodic points of continuous, order preserving, subhomogeneous maps  $f: K \rightarrow K$ , denoted  $\Gamma(K)$ , is contained in a finite set  $B(N)$  consisting of  $p \geq 1$  for which there exist integers  $q_1$  and  $q_2$  such that  $p = q_1 q_2$ ,  $1 \leq q_1 \leq \binom{N}{m}$ , and  $1 \leq q_2 \leq \binom{m}{\lfloor m/2 \rfloor}$  for some  $1 \leq m \leq N$  (compare [2, Section 7]). On the other hand, if the polyhedral cone  $K$  is the standard positive cone given by  $\mathbb{R}_+^n = \{x \in \mathbb{R}^n: x_i \geq 0 \text{ for } 1 \leq i \leq n\}$ , then it follows from [2, Theorem 2.2] that  $\Gamma(\mathbb{R}_+^n)$  contains the set  $A(n)$  consisting of those  $p \geq 1$  for which there exist integers  $q_1$  and  $q_2$  such that  $p = \text{lcm}(q_1, q_2)$ ,  $1 \leq q_1 \leq \binom{n}{m}$ , and  $1 \leq q_2 \leq \binom{m}{\lfloor m/2 \rfloor}$  for some  $1 \leq m \leq n$ . As  $\mathbb{R}_+^n$  has  $n$  facets, we thus have the following inclusions:

$$A(n) \subset \Gamma(\mathbb{R}_+^n) \subset B(n) \quad \text{for all } n \geq 1.$$

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It has been shown in [2, Section 7] that the largest element of  $A(n)$  has the same asymptotics as  $\beta_n$  in (1), which is the largest element of  $B(n)$ . Knowing these facts it is natural to ask if one could completely determine the finite set  $\Gamma(\mathbb{R}_+^n)$  and, in particular, to see if  $\Gamma(\mathbb{R}_+^n) = A(n)$  or if  $\Gamma(\mathbb{R}_+^n) = B(n)$ . The purpose of this note is to show that  $\Gamma(\mathbb{R}_+^n)$  and  $B(n)$  are equal. The idea of the proof can be sketched as follows. First we take  $q_1$ , where  $1 \leq q_1 \leq \binom{n}{m}$ , distinct parts of the cone, such that every element in these  $q_1$  parts has exactly  $m$  nonzero coordinates. Subsequently we select in each of the  $q_1$  parts,  $q_2$  distinct points, where  $1 \leq q_2 \leq \binom{m}{\lfloor m/2 \rfloor}$ , and construct a continuous, order preserving, subhomogeneous map  $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$  which has the set of  $q_1 q_2$  points as a periodic orbit. It is known that  $f$  has to map parts into parts and that the periodic orbit can have at most  $\binom{m}{\lfloor m/2 \rfloor}$  points in a part of the cone that consists of elements with exactly  $m$  nonzero coordinates (see [10, Theorem 5.2] and [2, Corollary 4.4]).

Before going into the details of the proof it is useful to recall several definitions. Let  $X$  be a finite dimensional real topological vector space. A subset  $K$  of  $X$  is called a *closed cone* if it is a closed convex subset of  $X$  such that  $\lambda K \subset K$  for all  $\lambda \geq 0$  and  $K \cap (-K) = \{0\}$ . A closed cone  $K$  is called a *polyhedral cone* if there exist finitely many linear functionals  $\varphi_1, \dots, \varphi_m$  such that  $K = \{x \in X: \varphi_i(x) \geq 0 \text{ for } 1 \leq i \leq m\}$ . In other words,  $K$  is the intersection of finitely many closed half-spaces. A *face* of a polyhedral cone is a set of the form  $F = K \cap \{x \in X: \varphi(x) = 0\}$ , where  $\varphi: X \rightarrow \mathbb{R}$  is a linear functional such that  $K \subset \{x \in X: \varphi(x) \geq 0\}$ . The *dimension* of a face is the dimension of its linear span and is denoted by  $\dim(F)$ . A face  $F$  is called a *facet* if  $\dim(F) = \dim(K) - 1$ .

A cone  $K$  in a vector space  $X$  induces a partial ordering  $\leq$  on  $X$  by  $x \leq y$  if  $y - x \in K$ . A map  $f: K \rightarrow K$  is called *order preserving* if for each  $x, y \in K$  with  $x \leq y$  we have that  $f(x) \leq f(y)$ . The map  $f: K \rightarrow K$  is said to be *subhomogeneous* if  $\lambda f(x) \leq f(\lambda x)$  for all  $0 \leq \lambda \leq 1$  and  $x \in K$ . A point  $x \in K$  is called a *periodic point* of  $f: K \rightarrow K$  if  $f^p(x) = x$  for some integer  $p \geq 1$ . The minimal such  $p \geq 1$  is said to be the *period* of  $x$  under  $f$ .

Maps that are order preserving and subhomogeneous arise in various areas of mathematics, including the theory of discrete event systems [3, 5], optimal control and game theory [1], idempotent analysis [8], nonlinear Perron-Frobenius theory [4, 12], and in the analysis of monotone dynamical systems [6, 7, 9, 11, 12, 13]. A particularly interesting class of continuous, order preserving, homogeneous maps on the standard positive cone are the so-called min-max maps. To define this class of maps it is convenient to first introduce the following notation: for  $a, b \in \mathbb{R}$  we write  $a \wedge b$  to denote  $\min\{a, b\}$  and  $a \vee b$  to denote  $\max\{a, b\}$ . A *min-max map*  $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$  is a map such that each coordinate function  $f_i(x)$  is of the form

$$f_i(x) = \bigvee_{j=1}^m \left( \bigwedge_{k \in I_j} a_{jk}^i x_k \right),$$

where each  $I_j$  is a nonempty subset of  $\{1, \dots, n\}$  and every  $a_{jk}^i > 0$ . Here the integer  $m$  and the sets  $I_j$  can depend on  $i$ . A simple example is the map  $f: \mathbb{R}_+^3 \rightarrow \mathbb{R}_+^3$  given by

$$f \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (2x_2 \wedge x_1) \vee (2x_1 \wedge x_3) \\ 2x_3 \wedge x_1 \\ 2x_1 \wedge x_2 \end{pmatrix}.$$

It is easy to verify that  $x = (1, 2, 0)$  is a periodic point of  $f$  with period 4. We shall use the min-max maps to prove the equality  $\Gamma(\mathbb{R}_+^n) = B(n)$  by exhibiting for each  $p \in B(n)$  a min-max map on  $\mathbb{R}_+^n$  with a periodic point of period  $p$ .

2. PROOF OF THE EQUALITY

Having recalled these definitions we now prove the result of this note.

**Theorem 2.1.** *For each  $n \geq 1$  the sets  $\Gamma(\mathbb{R}_+^n)$  and  $B(n)$  are equal.*

*Proof.* It has been shown in [2, Theorem 5.1] that  $\Gamma(\mathbb{R}_+^n) \subset B(n)$  for all  $n \geq 1$ . To prove the opposite inclusion let  $p \in B(n)$  and suppose that  $p = q_1q_2$ , where  $q_1$  and  $q_2$  are integers such that  $1 \leq q_1 \leq \binom{n}{m}$  and  $1 \leq q_2 \leq \binom{m}{\lfloor m/2 \rfloor}$  for some  $1 \leq m \leq n$ . We need to construct a continuous, order preserving, subhomogeneous map  $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$  and a periodic point of  $f$  of period  $p$ .

We begin by constructing a periodic orbit of size  $q_2$  in  $\{1, 2\}^m$ . Let  $w^1, \dots, w^{q_2}$  be vectors in  $\{1, 2\}^m$  such that each  $w^k$  has exactly  $\lfloor m/2 \rfloor$  coordinates equal to 2 and put  $w^{q_2+1} = w^1$ . Note that there are  $\binom{m}{\lfloor m/2 \rfloor}$  such elements in  $\{1, 2\}^m$  and hence we can do this for all possible values of  $q_2$ . It is easy to verify that  $\{w^1, \dots, w^{q_2}\}$  is a periodic orbit of the min-max map  $g: \mathbb{R}_+^m \rightarrow \mathbb{R}_+^m$  given by

$$g(z)_i = \bigvee_{k: w_i^{k+1}=2} \left( \bigwedge_{j: w_j^k=2} z_j \right), \quad 1 \leq i \leq m \text{ and } z \in \mathbb{R}_+^m.$$

Indeed, one can check that  $g(w^k) = w^{k+1}$  for all  $1 \leq k \leq q_2$ .

The idea is to copy the points  $w^1, \dots, w^{q_2}$  into  $q_1$  distinct parts of the cone  $\mathbb{R}_+^n$  and subsequently to define a min-max map  $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$  that cycles through the  $q_1q_2$  points. To do this we first select  $q_1$  distinct subsets  $I_a$  of  $\{1, \dots, n\}$  of size  $m$ . Each set  $I_a$  corresponds to a part  $P_a$  of the cone  $\mathbb{R}_+^n$  by defining

$$P_a = \{x \in \mathbb{R}_+^n : x_i \neq 0 \text{ if and only if } i \in I_a\}.$$

Now we copy the set  $\{w^1, \dots, w^{q_2}\}$  into each of the  $q_1$  parts  $P_a$  by inserting zero coordinates in the correct places. To do this formally it is convenient to introduce the following function. For  $1 \leq a \leq q_1$  and  $1 \leq k \leq m$  let  $\nu(a, k)$  be the index of the  $k$ th nonzero coordinate of the elements of  $P_a$ . So for each  $1 \leq a \leq q_1$  we define the copying function  $\eta_a: \mathbb{R}_+^m \rightarrow \mathbb{R}_+^n$  by

$$\eta_a(z)_i = \begin{cases} z_k & \text{if } \nu(a, k) = i, \\ 0 & \text{otherwise.} \end{cases}$$

By applying the functions  $\eta_a$  to the set  $\{w^1, \dots, w^{q_2}\}$  we obtain  $p = q_1q_2$  distinct points in  $\mathbb{R}_+^n$ . For  $1 \leq a \leq q_1$  and  $1 \leq b \leq q_2$  we let  $y^{a,b} \in \{0, 1, 2\}^n$  denote the vector given by  $y^{a,b} = \eta_a(w^b)$ . Thus,  $y^{a,b}$  is the copy of  $w^b$  in the part  $P_a$ .

To complete the proof we define a min-max map  $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$  such that

$$(2) \quad f(y^{a,b}) = \begin{cases} y^{a+1,b} & \text{if } 1 \leq a < q_1, \\ y^{1,b+1} & \text{if } a = q_1, \end{cases}$$

for each  $1 \leq a \leq q_1$  and  $1 \leq b \leq q_2$ . (Here the index  $b$  is counted modulo  $q_2$ .) It is clear that if  $f$  satisfies (2), then  $y^{a,b}$  is a periodic point of  $f$  with period  $p = q_1q_2$ .

Before we define the min-max map  $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$  we introduce the following notation. For each  $x \in \mathbb{R}_+^n$  we let  $x|_{I_a}$  denote the vector in  $\mathbb{R}_+^n$  with coordinates  $(x|_{I_a})_j = x_{\nu(a,j)}$  for  $1 \leq j \leq m$ . Now let  $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$  be given by

$$f(x)_i = \left[ \bigvee_{1 \leq k < q_1} \left( \bigwedge_{j \in I_k} 2x_j \right) \wedge \eta_{k+1}(x|_{I_k})_i \right] \vee \left[ \left( \bigwedge_{j \in I_{q_1}} 2x_j \right) \wedge \eta_1(g(x|_{I_{q_1}}))_i \right]$$

for  $1 \leq i \leq n$  and  $x \in \mathbb{R}_+^n$ . As  $f$  is a min-max map, it is continuous, order preserving, and homogeneous. To see that (2) holds for  $f$ , we remark that  $\bigwedge_{j \in I_k} 2y_j^{a,b} = 0$  whenever  $k \neq a$  and  $\bigwedge_{j \in I_k} 2y_j^{a,b} = 2$  otherwise. We also have that  $y|_{I_a}^{a,b} = w^b$  and therefore

$$f(y^{a,b})_i = \left( \bigwedge_{j \in I_a} 2y_j^{a,b} \right) \wedge \eta_{a+1}(w^b)_i = 2 \wedge y_i^{a+1,b} = y_i^{a+1,b},$$

if  $1 \leq a < q_1$ . On the other hand, if  $a = q_1$ , then

$$f(y^{a,b})_i = \left( \bigwedge_{j \in I_{q_1}} 2y_j^{q_1,b} \right) \wedge \eta_1(g(w^b))_i = 2 \wedge \eta_1(w^{b+1})_i = y_i^{1,b+1}$$

and this completes the proof.  $\square$

We conclude by noting that there are other ways to cycle through the points  $y^{a,b}$  by using different min-max functions. In fact the only restriction on the periodic orbit is that if two of its points are in the same part of the cone, then their images should also be contained in a single part of the cone.

#### REFERENCES

- [1] M. Akian and S. Gaubert, Spectral theorem for convex monotone homogeneous maps, and ergodic control. *Nonlinear Anal.* **52**(2), (2003), 637–679. MR1938367 (2003i:93085)
- [2] M. Akian, S. Gaubert, B. Lemmens, and R. D. Nussbaum, Iteration of order preserving subhomogeneous maps on a cone, *Math. Proc. Cambridge Philos. Soc.*, to appear.
- [3] F. Baccelli, G. Cohen, G. J. Olsder, and J. P. Quadrat, *Synchronization and Linearity: An Algebra for Discrete Event Systems*, Wiley Ser. Probab. Statist. Probab. Statist., Wiley: Chichester, 1992. MR1204266 (94b:93001)
- [4] S. Gaubert and J. Gunawardena, The Perron-Frobenius theory for homogeneous, monotone functions. *Trans. Amer. Math. Soc.*, **356**(12), (2004), 4931–4950. MR2084406
- [5] J. Gunawardena, From max-plus algebra to nonexpansive mappings: a nonlinear theory for discrete event systems. *Theoret. Comput. Sci.* **293**(1), (2003), 141–167. MR1957616 (2004b:93083)
- [6] M. W. Hirsch, Positive equilibria and convergence in subhomogeneous monotone dynamics, in *Comparison Methods and Stability Theory* (X. Liu and D. Siegel ed.), pp. 169–188, Lecture Notes in Pure and Appl. Math., 162, Dekker, New York, 1994. MR1291618 (95f:34090)
- [7] J. F. Jiang, Sublinear discrete-time order-preserving dynamical systems. *Math. Proc. Cambridge Philos. Soc.* **119**(3), (1996), 561–574. MR1357065 (96h:34090)
- [8] V. N. Kolokoltsov and V. P. Maslov, *Idempotent Analysis and Applications*. Kluwer Acad. Press, 1997. MR1447629
- [9] U. Krause and P. Ranft, A limit set trichotomy for monotone nonlinear dynamical systems. *Nonlinear Anal.* **19**(4), (1992), 375–392. MR1178411 (93i:58081)
- [10] B. Lemmens and M. Scheutzwow, On the dynamics of sup-norm nonexpansive maps, *Ergodic Theory Dynam. Systems* **25**(3), (2005), 861–871. MR2142949
- [11] R. D. Nussbaum, Hilbert’s projective metric and iterated nonlinear maps. *Mem. Amer. Math. Soc.* **75** (391), (1988), 1–137. MR0961211 (89m:47046)

- [12] R. D. Nussbaum, Iterated nonlinear maps and Hilbert's projective metric II. *Mem. Amer. Math. Soc.* **79**(401), (1989), 1–118. MR0963567 (90c:47109)
- [13] P. Takáč, Asymptotic behavior of discrete-time semigroups of sublinear, strongly increasing mappings with applications to biology. *Nonlinear Anal.* **14**(1), (1990), 35–42. MR1028245 (90j:47088)

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