

EXTENSIONS OF ORTHOSYMMETRIC LATTICE BIMORPHISMS

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(Communicated by Joseph A. Ball)

ABSTRACT. Let E be an Archimedean vector lattice, let E° be its Dedekind completion and let B be a Dedekind complete vector lattice. If $\Psi_0 : E \times E \rightarrow B$ is an orthosymmetric lattice bimorphism, then there exists a lattice bimorphism $\Psi : E^\circ \times E^\circ \rightarrow B$ that not just extends Ψ_0 but also has to be orthosymmetric. As an application, we prove the following: Let A be an Archimedean d -algebra. Then the multiplication in A can be extended to a multiplication in A° , the Dedekind completion of A , in such a fashion that A° is again a d -algebra with respect to this extended multiplication. This gives a positive answer to the problem posed by C. B. Huijsmans in 1990.

1. INTRODUCTION

The standard extension theorem under a domination hypothesis is, of course, the classical Hahn-Banach theorem. Its method of proof was used almost immediately by L.V. Kantorovich [9] to prove that a positive linear map from a dominating subspace E of a vector lattice A (i.e., for each $x \in A$ there exists $y \in E$ such that $|x| \leq y$) into a Dedekind complete vector lattice B can be extended to a positive linear of A into B . Moreover, Luxemburg and Schep [16] showed that vector lattice homomorphism of a dominating vector sublattice extends not just positively but so as to preserve the lattice operations. They did show, however, that the vector lattice homomorphic extensions were extreme points of the set of positive extensions.

Lipecki, alone and with coauthors [10, 11, 12, 13, 14, 15], took up the question of describing the extreme points of the set of positive extensions. They obtained a fairly simple characterization of them, in [10], and used this to produce vector lattice homomorphic extensions. Surprisingly enough, to the best of our knowledge, no attention has been paid in the literature to the corresponding problem of the bilinear map, except in the paper of Grobler and Labuschagne [7]. Here they proved that if E and F are majorizing vector sublattices of the vector lattices A and B and if C is a Dedekind complete vector lattice, then every lattice bimorphism $\Psi_0 : E \times F \rightarrow C$ can be extended to a lattice bimorphism Ψ to $A \times B$ into C . The question arises as to whether Ψ still satisfies the property (AF) when Ψ_0 has in addition the property (AF). The answer is affirmative (Theorem 1).

Received by the editors February 10, 2004 and, in revised form, January 13, 2005.

2000 *Mathematics Subject Classification*. Primary 06F25, 47B65.

Key words and phrases. d -algebra, f -algebra, lattice homomorphism, lattice bimorphism.

The author thanks Professor S. J. Bernau for providing the bibliographic information of [2].

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In the theory of partially ordered structures, vector lattices have been studied extensively. Indeed many kinds of vector lattices appear at many places in classical functional analysis. Less attention has been paid to lattice ordered algebras, and the history of that attention is more recent. Nonetheless, for the study of a positive operator and the structure theory of vector lattices, Archimedean f -algebras have played an important role in the last decades. They, often due to the order continuity of the multiplication, are well understood. They are commutative, and their multiplication uniquely extends to an f -algebra multiplication on the Dedekind completion. Almost f -algebras and d -algebras are not as well behaved and are less commonly found in applications. However, Archimedean almost f -algebras also are commutative (see [3], [4] and [8]), and their multiplication can be extended, though not uniquely to the Dedekind completion (see [4]).

In this paper we study the corresponding problem for d -algebras, which is a question posed by Huijsmans in [8] (last paragraph of section 7): can the multiplication of d -algebra be extended to its Dedekind completion? Recently, in [6] Chil has given a positive answer to the above question. The disadvantage of his approach is that the proof is long, quite involved and is not intrinsic, i.e., does not take place in the d -algebra product itself, because it relies heavily on a local representation theorem made by K. Boulabiar and M. A. Toumi (see [5], Theorem 1, Theorem 4), which make a connection between f -algebra and d -algebra products.

In this paper we present an elementary, intrinsic and short proof. Interestingly, it deals with lattice bimorphisms rather than d -algebra multiplication. We take it for granted that the reader is familiar with the notions of vector lattices (or Riesz spaces) and operators between them. For terminology, notations and concepts that are not explained in this paper, we refer to the standard monographs [1] and [19].

2. DEFINITIONS AND NOTATIONS

We shall assume throughout this paper that all vector lattices (or Riesz spaces) under consideration are Archimedean. A vector subspace E of a vector lattice A is said to be majorizing (dominating) A if for each $x \in A$ there exists $a \in E$ such that $|x| \leq a$. A linear mapping T defined on the vector lattice A with values in the vector lattice B is called *positive* if $T(A^+) \subset B^+$ (notation: $T \in \mathcal{L}^+(A, B)$ or $T \in \mathcal{L}^+(A)$ if $A = B$). The linear mapping $T \in \mathcal{L}^+(A, B)$ is called *lattice* (or *Riesz*) *homomorphism* (notation: $T \in \text{Hom}(A, B)$ or $T \in \text{Hom}(A)$ if $A = B$) whenever $a \wedge b = 0$ implies $T(a) \wedge T(b) = 0$.

Next, we recall the definitions and some basic facts about f -algebras. For more information about this field, we refer the reader to [1]. The (real) algebra A which is simultaneously a vector lattice is called a *lattice ordered algebra* (ℓ -*algebra*). In ℓ -algebra A we denote the collection of all nilpotent elements of A by $N(A)$. The ℓ -algebra A is referred to as semiprime if $N(A) = \{0\}$. The ℓ -algebra A is called an *f -algebra* if A verifies the property that $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge b = ca \wedge b = 0$. Any f -algebra is automatically commutative and has positive squares. Every unital f -algebra (i.e., an f -algebra with a unit element) is semiprime.

Also we need the following definitions. The ℓ -algebra A is called a *d -algebra* whenever it follows from $a \wedge b = 0$ and $c \geq 0$ that $ac \wedge bc = ca \wedge cb = 0$ (equivalently, whenever $|ab| = |a||b|$). In other words, the multiplications by positive elements in the d -algebra A are lattice homomorphisms. Contrary to the f -algebras, d -algebras need not be commutative nor have positive squares. For elementary theory of

d -algebras we refer to [3] and [17]. The ℓ -algebra A is called an *almost f -algebra* whenever it follows from $a \wedge b = 0$ that $ab = 0$. The present paragraph of this section deals with some facts about universally complete vector lattice spaces. The vector lattice A is called *universally complete* if A is a Dedekind complete vector lattice and every positive orthogonal system in A has a supremum in A . Every vector lattice A has a universal completion A^u ; this means that there exists a unique (up to a Riesz isomorphism) universally complete vector lattice A^u such that A can be identified with an order dense Riesz subspace of A^u . The vector lattice A^u is equipped by a canonical multiplication, under which A^u is an f -algebra with unit element. See [1, Section 8, Exercise 13] for an interesting approach to the existence of the universal completion using orthomorphisms. We end this section with some definitions of bilinear maps on vector lattices. Let A, B and C be vector lattices. A bilinear map Ψ from $A \times B$ into C is said to be *positive* whenever $a \in A^+$ and $b \in B^+$ imply $\Psi(a, b) \in C^+$ (equivalently $|\Psi(a, b)| \leq \Psi(|a|, |b|)$ for all $a \in A$ and $b \in B$). The bilinear map Ψ is called a *lattice (Riesz) bimorphism* whenever the partial operators

$$\Psi(a, \cdot) : \begin{array}{ccc} B & \rightarrow & C, \\ c & \mapsto & \Psi(a, c) \end{array} \quad \text{and} \quad \Psi(\cdot, b) : \begin{array}{ccc} A & \rightarrow & C, \\ c & \mapsto & \Psi(b, c) \end{array}$$

are lattice homomorphisms for every $a \in A^+$ and $b \in B^+$ (equivalently $|\Psi(a, b)| = \Psi(|a|, |b|)$ for all $a \in A$ and $b \in B$). A bilinear map Ψ from $A \times A$ into C is said to be *orthosymmetric (or to have the property (AF))* if $a \wedge b = 0$ implies $\Psi(a, b) = 0$.

3. THE MAIN RESULTS

Grobler and Labuschagne [7] proved that if E and F are vector lattices and if C is a Dedekind complete vector lattice, then every lattice bimorphism $\Psi_0 : E \times F \rightarrow C$ can be extended to a lattice bimorphism Ψ to $E^\circ \times F^\circ$ into C , where E° (resp. F°) is the Dedekind completion of E (resp. F). The question arises as to whether Ψ still satisfies the property (AF) when Ψ_0 has, in addition, the property (AF). The answer is affirmative. To reach this aim we need the following result.

Proposition 1. *Let E be a vector lattice, let B be a Dedekind complete vector lattice and let $\Psi_0 : E \times E \rightarrow B$ be a symmetric lattice bimorphism. If $\Psi : E^\circ \times E^\circ \rightarrow B$ is a lattice bimorphism extension of Ψ_0 to $E^\circ \times E^\circ$ into B , then for all $x, y \in E^\circ$ such that $x \wedge y = 0$, and for all $x_i \in E$ (resp $y_i \in E$) such that $0 \leq x_i \leq x$ (resp $0 \leq y_i \leq y$), we have*

$$\Psi(x, y_i) = \Psi(y_i, x) = \Psi(x_i, y) = \Psi(y, x_i) = \Psi(x_i, y_i) = 0.$$

Proof. Since $\Psi_0 : E \times E \rightarrow B$ is a symmetric lattice bimorphism, then $\Psi(x_i, y_i) = 0$. Let $S_0 = \Psi(\cdot, y_i)$ be the partial operator from $E + \mathbb{R}x$ to B defined by

$$S_0(a + \alpha x) = \Psi(a + \alpha x, y_i) \quad (\alpha \in \mathbb{R}, a \in E).$$

Then

$$\begin{aligned} S_0(a + \alpha x) &= \Psi(a, y_i) + \alpha \Psi(x, y_i) \\ &= \Psi_0(a, y_i) + \alpha \Psi(x, y_i) \\ &= \Psi_0(y_i, a) + \alpha \Psi(x, y_i) \\ &= \Psi(y_i, a) + \alpha \Psi(x, y_i) \quad (\alpha \in \mathbb{R}, a \in E). \end{aligned}$$

Since E° is a Dedekind completion vector lattice, it follows that $E^\circ = \{y_i\}^{dd} \oplus \{y_i\}^d$, where $\{y_i\}^{dd}$ (resp. $\{y_i\}^d$) is the band projection generated by y_i (resp. the band projection orthogonal of $\{y_i\}^{dd}$). Then $a = a_1 + a_2$, where $a_1 \in \{y_i\}^{dd}$ and $a_2 \in \{y_i\}^d$. Hence

$$S_0(a + \alpha x) = \Psi(y_i, a_1) + \Psi(y_i, a_2) + \alpha\Psi(x, y_i) \quad (\alpha \in \mathbb{R}, a \in E).$$

Since $x \wedge y = 0$, $0 \leq y_i \leq y$, $a_1 \in \{y_i\}^{dd}$, $a_2 \in \{y_i\}^d$, then $a_1 \wedge x = x \wedge y_i = a_2 \wedge y_i = a_1 \wedge a_2 = 0$. Therefore,

$$|\Psi(y_i, a_1)| \wedge |\Psi(y_i, a_2)| = |\Psi(y_i, a_1)| \wedge |\Psi(x, y_i)| = |\Psi(y_i, a_2)| \wedge |\Psi(x, y_i)| = 0$$

and so

$$|S_0(a + \alpha x)| = |\Psi(y_i, a_1)| + |\Psi(y_i, a_2)| + |\alpha| |\Psi(x, y_i)| \quad (\alpha \in \mathbb{R}, a \in E).$$

Then

$$(1) \quad |S_0(a + \alpha x)| \geq |\alpha| |\Psi(x, y_i)| \quad (\alpha \in \mathbb{R}, a \in E).$$

Let T_1 be the linear map from $E + \mathbb{R}x$ to B defined by

$$T_1(a + \alpha x) = \alpha\Psi(x, y_i) \quad (\alpha \in \mathbb{R}, a \in E).$$

Then T_1 satisfies, by inequality (1),

$$T_1(a + \alpha x) \leq |S_0(a + \alpha x)| = |\Psi(a + \alpha x, y_i)| = \Psi(|a + \alpha x|, y_i).$$

Since B is a Dedekind complete vector lattice, the classical Hahn-Banach proof gives a linear extension T of T_1 such that

$$T(u) \leq \Psi(|u|, y_i) \quad (u \in E^\circ).$$

Thus $\Psi(\cdot, y_i) + T$ and $\Psi(\cdot, y_i) - T$ are both positive extensions of $\Psi_0(\cdot, y_i)$, and $\Psi(\cdot, y_i)$ is an extreme point of the set of positive extensions of $\Psi_0(\cdot, y_i)$ to E° (see [2], Theorem 4.4). So $\Psi(x, y_i) = 0$. Using the same argument, we have, for all $x, y \in E^\circ$ such that $x \wedge y = 0$, and for all $x_i \in E$ (resp. $y_i \in E$) such that $0 \leq x_i \leq x$ (resp. $0 \leq y_i \leq y$)

$$\Psi(y_i, x) = \Psi(x_i, y) = \Psi(y, x_i) = 0,$$

and we are done. \square

A standard argument leads immediately to the following.

Theorem 1. *Let E be a vector lattice and let B be a Dedekind complete vector lattice. If $\Psi_0 : E \times E \rightarrow B$ is a lattice bimorphism that satisfies the property (AF), then every lattice extension $\Psi : E^\circ \times E^\circ \rightarrow B$ of Ψ_0 preserves the property (AF).*

Proof. Let $x, y \in E^\circ$ such that $x \wedge y = 0$, and let $K = A \cap I_x$, where I_x is the order ideal generated by x in E° . By the previous result and since K is order dense in I_x , the partial operator $\Psi(\cdot, y)_{/K} : K \rightarrow B$ is null in K (that is, $\Psi(a, y) = 0$, for all $a \in K$). Hence $\Psi(\cdot, y)_{/K} : K \rightarrow B$ is a positive order continuous map, so it has a unique positive extension to I_x . Since the null map on I_x and $\Psi(\cdot, y)_{/I_x}$ are two positive extensions of $\Psi(\cdot, y)_{/K}$, it follows that $\Psi(x, y) = 0$, and the proof is complete. \square

The previous theorem will be applied to d -algebras but is in this more context of independent interest.

For the rest of the paper we shall fix the following notation and assumptions. Let $(A, *)$ be a d -algebra. Let $\Psi_0 : A \times A \rightarrow A; (a, b) \mapsto a * b$ be the lattice bimorphism associated with the d -algebra product of A and let $\Psi : A^\circ \times A^\circ \rightarrow A^\circ$ be a lattice bimorphism extension of Ψ_0 . Hence we construct a new multiplication also denoted by $*$. So it remains to show that the new extended product must be associative.

In order to hit this mark, we need the following theorem.

Theorem 2. $(a * b^\wedge) * c = a * (b^\wedge * c)$, for all $0 \leq a, c \in A, 0 \leq b^\wedge \in A^\circ$.

Proof. Let $0 \leq a, c \in A$. Let the operator $T_{a,c} : A \rightarrow A; (b) \mapsto a * b * c$. Then $T_{a,c}$ is a lattice homomorphism, so it can be extended as a lattice homomorphism in two different ways, which are

$$T_1 : A^\circ \rightarrow A^\circ \quad ; \quad (b^\wedge) \mapsto (a * b^\wedge) * c$$

and

$$T_2 : A^\circ \rightarrow A^\circ \quad ; \quad (b^\wedge) \mapsto a * (b^\wedge * c).$$

We claim that $T_1 + T_2$ is a lattice homomorphism. Indeed, if $b_1 \wedge b_2 = 0$ in A° , we have in the unital f -algebra universal completion A^u of A , where the f -algebra multiplication is denoted by juxtaposition

$$\begin{aligned} & ((T_1 + T_2)(b_1)) ((T_1 + T_2)(b_2)) \\ &= [((a * b_1) * c) + (a * (b_1 * c))] [((a * b_2) * c) + (a * (b_2 * c))] \\ &= ((a * b_1) * c) (a * (b_2 * c)) + (a * (b_1 * c)) ((a * b_2) * c) \\ &= \Psi_{a,c}(b_1, b_2) + \Psi_{a,c}(b_2, b_1), \end{aligned}$$

where

$$\begin{aligned} \Psi_{a,c}^0 : A \times A &\rightarrow A^u \\ (u, v) &\mapsto (a * u * c) (a * v * c) \end{aligned}$$

is of course a lattice homomorphism and

$$\begin{aligned} \Psi_{a,c} : A^\circ \times A^\circ &\rightarrow A^u \\ (u, v) &\mapsto ((a * u) * c) (a * (v * c)) \end{aligned}$$

is a lattice homomorphism extension of $\Psi_{a,c}^0$ to $A^\circ \times A^\circ$. Observe that $\Psi_{a,c}^0$ satisfies the property (AF). Hence, by the previous theorem, $\Psi_{a,c}$ also satisfies the property (AF). Then

$$((T_1 + T_2)(b_1)) ((T_1 + T_2)(b_2)) = \Psi_{a,c}(b_1, b_2) + \Psi_{a,c}(b_2, b_1) = 0.$$

Since any unital f -algebra is semiprime, $((T_1 + T_2)(b_1)) \wedge ((T_1 + T_2)(b_2)) = 0$ and so $T_1 + T_2$ is a lattice homomorphism. Let us take $S = \frac{T_1}{2} + \frac{T_2}{2}$. Therefore, S is a lattice extension to A° of $T_{a,c}$, which implies that S is an extreme extension. Hence $T_1 = S = T_2$. Finally, we have $(a * b^\wedge) * c = a * (b^\wedge * c)$ for all $0 \leq a, c \in A, 0 \leq b^\wedge \in A^\circ$, and we are done \square

The following corollary is now an immediate consequence.

Corollary 1. *Let $(A, *)$ be a d -algebra and let A° be its Dedekind completion. Then A° can be equipped with a d -algebra multiplication that extends the multiplication of A .*

A simple combination of Theorem 1 and Corollary 1 gives the following result.

Corollary 2. *Let A be a commutative d -algebra and let A° be its Dedekind completion. Then A° can be equipped with a commutative d -algebra multiplication that extends the multiplication of A .*

Remark 1. We remark that the product made of any lattice bimorphism extension of the lattice bimorphism associated with the d -algebra is associative. So the extension obtained in Corollary 1 is clearly far from being unique. This is illustrated by the following examples.

Example 1. Let A be the vector lattice of all real bounded sequences which have to the maximum a finite number of different values. It follows that the Dedekind completion of A denoted by A° is the vector lattice of all real bounded sequences. For $x \in A$, put $\underline{x} := \inf \{x_n, n \in \mathbb{N}\}$. Under the multiplication

$$f * g = \left(\underline{f} \right) g \quad (f, g \in A),$$

A is a d -algebra. For any $v \in \beta\mathbb{N} \setminus \mathbb{N}$ the multiplication $*$ of A extends to a d -algebra multiplication $*$ on $A^\circ (\equiv \ell^\infty(\mathbb{N}) \equiv C(\beta\mathbb{N}))$ by

$$f * g = \left\{ \inf_{t \in \beta\mathbb{N} \setminus \{v\}} \{(\beta f)(t)\} \right\} g \quad (f, g \in A^\circ).$$

Example 2. Let A be the vector lattice of all real stationary sequences. Its Dedekind completion A° is the vector lattice of all real bounded sequences. We consider $A \times A$ with the coordinatewise vector space operations and partial ordering. We equip $A \times A$ with the following multiplication:

$$\begin{pmatrix} f \\ g \end{pmatrix} * \begin{pmatrix} f' \\ g' \end{pmatrix} = \begin{pmatrix} \left(\lim_{n \rightarrow \infty} f(n) \right) f' \\ \left(\lim_{n \rightarrow \infty} f(n) \right) g' \end{pmatrix} \quad (f, g, f', g' \in A).$$

For any $v \in \beta\mathbb{N} \setminus \mathbb{N}$ the multiplication $*$ of $A \times A$ extends to a d -algebra multiplication $*$ on $A^\circ \times A^\circ$ by

$$\begin{pmatrix} f \\ g \end{pmatrix} * \begin{pmatrix} f' \\ g' \end{pmatrix} = \begin{pmatrix} ((\beta f)(v)) f' \\ ((\beta f)(v)) g' \end{pmatrix} \quad (f, g, f', g' \in A^\circ).$$

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