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SPACES THAT ADMIT HYPERCYCLIC OPERATORS WITH HYPERCYCLIC ADJOINTS

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ABSTRACT. A continuous linear operator $T : X \to X$ is hypercyclic if there is an $x \in X$ such that the orbit $\{T^n x\}_{n\geq 0}$ is dense. A result of H. Salas shows that any infinite-dimensional separable Hilbert space admits a hypercyclic operator whose adjoint is also hypercyclic. It is a natural question to ask for what other spaces X does $\mathcal{L}(X)$ contain such an operator. We prove that for any infinite-dimensional Banach space X with a shrinking symmetric basis, such as c_0 and any ℓ_p $(1 < p < \infty)$, there is an operator $T : X \to X$, where both T and $T' : X' \to X'$ are hypercyclic.

1. Introduction and preliminaries

Let T be a continuous linear operator on a separated (real or complex) locally convex space X. We let $\mathrm{Orb}(T, x) \equiv \{T^n x : n \geq 0\}$ denote the orbit of $x \in X$ under T. Recall that T is hypercyclic if $\mathrm{Orb}(T, x)$ is dense in X for some (hypercyclic) $x \in X$. Thus, the existence of a hypercyclic operator on X requires that X is separable and, by a well-known result of Rolewicz [8], that X is infinite-dimensional. (More generally, no inductive limit of finite-dimensional spaces admits a hypercyclic operator [3].) In fact, every infinite-dimensional separable Fréchet space X carries a hypercyclic operator T [1, 3]. In particular, when X is a Hilbert space, Salas showed [9] that it is possible find such a T whose adjoint also is hypercyclic (see also [10]). This motivates us to study the following question: For what other types of spaces X does there exist an operator $T : X \to X$ where both T and $T' : X' \to X'$ are hypercyclic?

Definition 1. An operator $T : X \to X$ is said to be *dual hypercyclic* when both T and $T' : X' \to X'$ are hypercyclic. (We assume here that the dual X' is provided with the strong topology so, in particular, X' carries the norm topology when X is a Banach space (see Remark (i) for comments).)

By the discussion above, a necessary condition for the existence of a dual hypercyclic operator on X is that both X and X' are separable, so, for example, ℓ_1 does not support such an operator. An extra necessary condition is the following simple proposition (the proof runs parallel to that of a result of C. Kitai in her thesis).

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Proposition 1. *Let X be a locally convex space and assume that $T : X \to X$ is dual hypercyclic. Then, for any scalar λ, $T - \lambda$ and $T' - \lambda$ are one-to-one maps, $T - \lambda$ has a dense range and $T' - \lambda$ a $\sigma(X', X)$-dense range.*

Proof. We assume that X is complex, the real case follows similarly. We prove that $T - \lambda$ has a dense range, or equivalently, that $T' - \lambda$ is one-to-one. Let x be a hypercyclic vector for T. Then $\mathbb{C}(y) \equiv \{\langle T^n x, y\rangle\}_{n\geq 0}$ forms a dense set in $\mathbb{C}$ for any $y \neq 0$ in X'. Assume now that λ is an eigenvalue for T' and y a corresponding eigenvector. Then $\mathbb{C}(y) = \{\langle T^n x, y\rangle = \lambda^n \langle x, y\rangle\}$ which cannot be dense and hence a contradiction. Next, since the transpose of T' is T for the duality (X, X'), the arguments above complete the proof. □

Example 1. a) No unilateral (weighted) backward shift S on $\ell_2 = \ell_2(\mathbb{N})$ has a hypercyclic adjoint since $\ker S \neq 0$ (another argument is that S^* is a forward shift, which is not hypercyclic). $T \equiv S + \mathrm{Id}$ is hypercyclic for any such shift S with positive weights, but $T^* - 1 = S^*$ which has no dense range, so T^* is not hypercyclic. (See [10] for details on all this.)

b) Any non-trivial convolution operator T (such as any translation τ_a, $a \neq 0$, and any $P(D)$, $P \in \mathbb{C}[z_1, ..., z_n] \setminus \mathbb{C}$), on the Fréchet space $\mathscr{H}(\mathbb{C}^n)$ of n-variable entire functions (compact-open-topology) is hypercyclic [4, Theorem 5.1]. However, no such operator has a hypercyclic adjoint. Indeed, for any convolution operator T and $z_0 \in \mathbb{C}^n$, $\lambda \equiv Te^{\langle \cdot, z_0\rangle}(0)$, where $\langle z, \xi\rangle \equiv \sum z_i \xi_i$, is an eigenvalue for T.

c) $\omega = \prod_{n=0}^{\infty} \mathbb{C}$, and the ring of formal power-series $\hat{\mathscr{O}}[z_1, ..., z_n]$ in n-variables, provided with their natural product topologies (thus reflexive Fréchet spaces), do not support operators with hypercyclic adjoints. Indeed, their strong duals are inductive limits of finite-dimensional spaces.

We shall prove that any infinite-dimensional Banach space with a shrinking symmetric basis (see below) admits a dual hypercyclic operator.

We recall that the theory of cyclicity originates from the invariant subspace theory, and we close this Introduction with a reference to the survey article [6].

2. The result

We restate the result of Salas in [9] in the following way:

Proposition 2. *Let (u_n) denote the unit basis of $\ell_2(\mathbb{Z})$. There is a bounded sequence of scalars $(b_n)_{n\in\mathbb{Z}}$ (see* [9, p. 766]*), such that the (forward shift) operator $Wu_n \equiv b_{n+1}u_{n+1}$ and its adjoint $W^*u_n = b_n u_{n-1}$ (backward shift) both are hypercyclic.*

Of course, since any infinite-dimensional separable Hilbert space H is isomorphic to $\ell_2(\mathbb{Z})$, the result shows that any such H admits a hypercyclic operator whose adjoint is also hypercyclic. In view of our purposes, it is convenient to pass over to $\ell_2 = \ell_2(\mathbb{N})$ is instead of $\ell_2(\mathbb{Z})$ in the following way.

Let (e_n) denote the canonical unit basis of ℓ_2. Let $u : \mathbb{Z} \to \mathbb{N}$ be an arbitrary bijection (see e.g. Example 2), and let U denote the corresponding isomorphism $\ell_2(\mathbb{Z}) \to \ell_2$ defined by $U(u_n) \equiv e_{u(n)}$. Then $S = S_u \equiv UWU^{-1}$ and $S^* = U^{*-1}W^*U^*$ are hypercyclic on ℓ_2 and $Se_{u(n)} = b_{n+1}e_{u(n+1)}$. It is convenient to write out S explicitly. Let m_n denote the unique integer such that $u(m_n) = n$, i.e. $m_n \equiv u^{-1}(n)$, and let $\pi = \pi_u$ denote the permutation of $\mathbb{N}$ defined by $\pi(n) \equiv$

$u(m_n+1)$. Then S can be written

$$(1)\qquad Sx = S_u x = \sum_{n\geq 0} b_{m_n+1}(x, e_n)e_{\pi(n)} = \sum_{k\geq 0} b_{m_{\pi^{-1}(k)}+1}(x, e_{\pi^{-1}(k)})e_k,$$

and the adjoint S^* is obtained by similar expressions by noting that the adjoint of $x \mapsto (x, f)e$ is $y \mapsto (y, e)f$. (Here $(\cdot,\cdot)$ denotes the inner product in ℓ_2.)

We recall some fundamentals on Schauder bases of Banach spaces; for a comprehensive exposition of this theory we refer to [7]. If (e_n) denotes a (Schauder-) basis of a Banach space X, we shall tacitly assume that (f_n) denote the corresponding biorthogonal functionals in X', which are defined by $\langle e_n, f_m\rangle = \delta^n_m$. Recall that (f_n) is a basic sequence in X', and (e_n) is a *shrinking* basis iff (f_n) is a basis of X'. Thus, ℓ_1 lacks shrinking bases since the dual $\ell_1' \simeq \ell_\infty$ is not separable. Let us also recall that a basis (e_n) is called *symmetric* if $(e_{\pi(n)})$ and (e_n) are equivalent basic sequences for any permutation π of $\mathbb{N}$. (It then follows that $(e_{\pi(n)})$ also forms a basis.) A symmetric basis is necessarily unconditional and, by [7, Theorem 1.c.9], shrinking iff X has no subspace isomorphic to ℓ_1. The canonical unit bases of c_0 and of ℓ_p $(1 < p < \infty)$ are both shrinking and symmetric. (Other types of spaces that admit such bases are Orlicz and Lorentz spaces [7]; recall that ℓ_p is an Orlicz space.)

Lemma 1. *Let X be a Banach space with a symmetric basis (e_n). Then, for any bounded sequence (ω_n) of scalars and permutation π of $\mathbb{N}$, $Tx \equiv \sum_{n=0}^{\infty} \omega_n\langle x, f_n\rangle e_{\pi(n)}$ defines a continuous linear operator on X. The adjoint of T is given by $T'y = \sum \omega_n\langle e_{\pi(n)}, y\rangle f_n$ with weak* convergence.*

Proof. For any unconditional basis (e_n), $x = \sum\langle x, f_n\rangle e_n \mapsto \sum \omega_n\langle x, f_n\rangle e_n$ defines a bounded operator (see [7, Prop. 1.c.7]). Now if (e_n) is a symmetric basis, the principle of uniform boundedness gives that $x \mapsto \sum\langle x, f_n\rangle e_{\pi(n)}$ forms a bounded operator [7, p. 113]. In particular, $(e_{\pi(n)})$ is an unconditional basis, so, by piecing together the first and the latter part of the proof, we conclude that T is continuous. Hence T' exists and is continuous on X'. (f_n) forms a weak basis, which means that $y = \sum\langle e_n, y\rangle f_n$ weakly in X' for any $y \in X'$ [7, p. 8]. From this it is easily checked that $T'y = \sum \omega_n\langle e_{\pi(n)}, y\rangle f_n$ weakly. □

The series for T, in the lemma, converges unconditionally, and if (e_n) is shrinking, (f_n) is also a symmetric basis, and hence the series for T' converges unconditionally in norm.

Theorem 1. *Any infinite-dimensional Banach space X with a shrinking symmetric basis admits a dual hypercyclic operator $T : X \to X$.*

Proof. Let (e_n) be a normalized shrinking and symmetric basis in X. Consider the (non-linear) map $Q : \ell_2 \to X$ defined by $Q((\alpha_n)_{n\in\mathbb{N}}) \equiv \sum_0^\infty \alpha_n^2 e_n$. Since (e_n) is a bounded sequence and $(\alpha_n^2) \in \ell_1$, the series converges in X, and we shall prove that Q is continuous with dense range. The latter is obvious, so we concentrate on the continuity. Let B denote the set in X formed by $B \equiv \{\sum_0^\infty \alpha_n e_n : \sum|\alpha_n| \leq 1\}$. It is clear that B is bounded and absolutely convex, and we let X_B denote the normed space obtained from the span of B provided with the Minkowski functional $\|\cdot\|_B$. Clearly, $\operatorname{Im} Q \subseteq X_B$, and since B is bounded, the inclusion map $X_B \to X$ is continuous. Hence, it suffices to prove that $Q : \ell_2 \to X_B$ is continuous. By

Cauchy-Schwarz' inequality we obtain

$$\|Q(\alpha)-Q(\beta)\|_B = \inf\{\lambda>0 : \lambda^{-1}\sum_{n\geq 0}(\alpha_n^2-\beta_n^2)e_n \in B\}$$
$$\leq \sum_{n\geq 0}|(\alpha_n^2-\beta_n^2)| = \sum_{n\geq 0}|(\alpha_n-\beta_n)||(\alpha_n+\beta_n)| \leq \|\alpha-\beta\|_2\|\alpha+\beta\|_2,$$

hence the continuity.

Now, let $T = T_u$ be the operator on X defined by

$$Tx = T_u x \equiv \sum_{n=0}^{\infty} b_{m_n+1}^2 \langle x, f_n\rangle e_{\pi(n)}, \tag{2}$$

where b_n are those from Proposition 2, and m_n and the permutation $\pi = \pi_u$ come from the expression (1) for $S = S_u$. Lemma 1 shows that T is continuous. In view of (1) it is easily checked that $TQ = QS$, and hence T is hypercyclic. Indeed, the identity $TQ = QS$ gives that $Q\,\mathrm{Orb}(S,\alpha) = \mathrm{Orb}(T, Q(\alpha))$, and thus, if α is hypercyclic for S, then $Q\,\mathrm{Orb}(S,\alpha)$ forms a dense set (since Q is continuous and has a dense range), and hence, $Q(\alpha)$ is hypercyclic for T. In the same way we deduce that the adjoint $T' = T'_u : X' \to X'$, given by $T'y = \sum_0^\infty b_{m_n+1}^2\langle e_{\pi(n)}, y\rangle f_n$ (Lemma 1), is hypercyclic. Indeed, since (e_n) is normalized, (f_n) is bounded in X', and hence, by $Q' : \alpha \mapsto \sum \alpha_n^2 f_n$, we obtain a continuous map between ℓ_2 and X' such that $T'Q' = Q'S^*$. Since (e_n) is shrinking, (f_n) forms a basis of X' and consequently, Q' has a dense range. Thus $Q'(\alpha)$ forms a hypercyclic vector for T' for any such vector $\alpha \in \ell_2$ for S^*. □

Example 2. With u defined by $u(n) \equiv 2n-1$ for $n \geq 1$ and $u(n) \equiv -2n$ for $n \leq 0$ (i.e. $m_{2n-1} = n$, $m_{2n} = -n$; $\pi(n) = n \pm 2$ when n is odd, respectively even, $\neq 0$ and $\pi(0) = 1$), $T = T_u$ takes the form

$$Tx \equiv b_1^2\langle x, f_0\rangle e_1 + \sum_{n\geq 1} b_{n+1}^2\langle x, f_{2n-1}\rangle e_{2n+1} + b_{-n+1}^2\langle x, f_{2n}\rangle e_{2n-2},$$

and T'_u is obtained by using $[\langle\cdot, f\rangle e]' = \langle e, \cdot\rangle f$.

3. Remarks and questions

Remark. (i) We have pointed out that since $\mathcal{L}(\ell_\infty)$ lacks hypercyclic operators, ℓ_1 does not support any dual hypercyclic operator. However, if we allow the dual $\ell_1' \simeq \ell_\infty$ to be endowed with some other topology, i.e. different from the (strong) norm topology, we may find such an operator. Indeed, the proof of Theorem 1 shows that, for any (normalized) symmetric basis (e_n) (i.e. not necessarily shrinking), (2) defines a hypercyclic operator. So let $T : \ell_1 \to \ell_1$ denote the hypercyclic operator thus obtained when (e_n) is the (symmetric) unit basis of ℓ_1. Then the transpose T' exists for the duality (ℓ_1, ℓ_∞) and is thus continuous for $\sigma(\ell_\infty, \ell_1)$. The biorthogonal functionals in ℓ_∞ corresponding to the unit basis of ℓ_1 form a weak basis of ℓ_∞ (see the proof of Lemma 1). From this we deduce that $T' : \ell_\infty \to \ell_\infty$ is hypercyclic for the weak topology $\sigma(\ell_\infty, \ell_1)$. The general conclusion is: Any infinite-dimensional Banach space with a symmetric basis supports a hypercyclic operator T with T' $\sigma(X', X)$-hypercyclic.

(ii) We have proved that, given a (normalized) shrinking symmetric basis, then, for *any* choice of bijection u, we obtain by (2) an operator $T = T_u$ of Lemma 1 that is dual hypercyclic. The study of operators with *hypercyclic subspaces*, i.e.

infinite-dimensional closed subspaces whose non-zero vectors are hypercyclic, has become of great interest (see [6, p. 356] for further remarks). In [5] it is proved that a Banach space operator T, that is hereditarily hypercyclic (and thus hypercyclic), has a hypercyclic subspace iff the essential spectrum $\sigma_e(T)$ meets the closed unit disc. Hence, if $T : X \to X$ and T' are hereditarily hypercyclic, then T has a hypercyclic subspace iff T' does, since $\sigma_e(T) = \sigma_e(T')$. From this we conclude:

Proposition 3. *Both $T_u : X \to X$ (2) and $T'_u : X' \to X'$ have a hypercyclic subspace. Thus, every Banach space X with a shrinking symmetric basis admits an operator $T : X \to X$, where both T and T' have a hypercyclic subspace.*

Proof. It is easily checked that an operator of Lemma 1 is bijective iff (ω_n) is bounded away from zero—which is not satisfied for $(\omega_n = b^2_{m_n+1})$ in the definition (2) for T_u. Thus, by Proposition 1, $\operatorname{Im} T_u$ is not closed, so $0 \in \sigma_e(T_u) = \sigma_e(T'_u)$. Now, it is known that W and W^* both are hereditarily hypercyclic [2, p. 96], from which we deduce that so are T_u and T'_u. □

(iii) It is a natural question to ask if there is a hypercyclic self-adjoint Hilbert space operator $T : H \to H$, since it would then be dual hypercyclic. The answer is negative when H is complex, since, for any $x \in H$, $\{(T^n x, x)\}_{n \geq 0} \subseteq \mathbb{R}$ which is not dense in $\mathbb{C}$, so x cannot be hypercyclic for T. Next, the existence of a self-adjoint hypercyclic operator T on a real Hilbert space H would show that the well-known Hypercyclicity Criterion [2, 4, 6] is not necessary for hypercyclicity, which we recall is an open problem. Indeed, T extends to a self-adjoint operator $\tilde{T} : (x, y) \mapsto (Tx, Ty)$, on the complexification $\tilde{H}$ of H, and $\tilde{T}$ cannot satisfy the criterion by the discussion above, so T does not satisfy the criterion by [2, Cor. 2.8].

(iv) Natural questions are:

Question 1. Does there exist a dual hypercyclic operator that is not of the form given by Theorem 1?

Question 2. Does every separable Banach space with separable dual support a dual hypercyclic operator?

Of course, a positive answer to the second question gives also a positive answer to the first one. Further, most of the arguments in Section 2 extend to Fréchet spaces with basis, such as $\mathscr{H}(\mathbb{C}^n)$. However, the crucial point is to obtain a bounded basis whose biorthogonal functionals also form a bounded basis which, we think, is not possible. We close with the following:

Question 3. Does there exist any non-normable, say, Fréchet space X that supports a dual hypercyclic operator?

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