PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 6, Pages 1631–1634
S 0002-9939(05)08283-3
Article electronically published on December 15, 2005

ON THE LÉVY CONSTANTS FOR QUADRATIC IRRATIONALS

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(Communicated by David E. Rohrlich)

ABSTRACT. We prove that the set of Lévy constants for quadratic irrationals is dense in $[\log \frac{\sqrt{5}+1}{2}, \infty)$.

1. Introduction

Let x be an irrational number and let $[a_0(x); a_1(x), a_2(x), \cdots]$ be its regular continued fraction expansion. For any $n \geq 1$, we denote by $p_n(x)/q_n(x) := [a_0(x); a_1(x), a_2(x), \cdots, a_n(x)]$ the nth convergent of x.

With the conventions $p_{-1}(x) = 1$, $q_{-1}(x) = 0$, $p_0(x) = a_0(x)$, $q_0(x) = 1$, we have

$$p_{n+1}(x) = a_{n+1}(x) \cdot p_n(x) + p_{n-1}(x), \quad n \geq 0, \tag{1.1}$$

$$q_{n+1}(x) = a_{n+1}(x) \cdot q_n(x) + q_{n-1}(x), \quad n \geq 0. \tag{1.2}$$

The famous Theorem of P. Lévy [6] states that

$$\lim_{n\to\infty} \frac{\log q_n(x)}{n} = \frac{\pi^2}{12 \log 2}$$

for almost all $x \in \mathbb{R}$ in the sense of Lebesgue.

Definition 1.1. For any irrational number $x \in \mathbb{R}$, if $\lim_{n\to\infty} \frac{\log q_n(x)}{n}$ exists, we say x has a Lévy constant and denote the limit by $\beta(x)$.

Notice that if x has a Lévy constant, then $\beta(x) \geq \log \frac{\sqrt{5}+1}{2}$.

H. Jager and P. Liardet [4] (see also C. Faivre [2]) proved that every quadratic irrational has a Lévy constant. Let

$$\mathbf{B} = \{\beta(x) : \ x \text{ is a quadratic irrational}\}.$$

E. P. Golubeva [3] showed that $\frac{\pi^2}{12 \log 2}$ is a limit point of $\mathbf{B}$; she also presented some evidence that $\log \frac{\sqrt{5}+1}{2}$ may be an isolated point of $\mathbf{B}$. However, we prove that:

Theorem 1.2. $\mathbf{B}$ *is dense in* $[\log \frac{\sqrt{5}+1}{2}, \infty)$.

Received by the editors January 28, 2005.
2000 *Mathematics Subject Classification.* Primary 11K50, 11J70.
Key words and phrases. Lévy constant, quadratic irrational.
The author was supported in part by the Kua-Shi-Ji Foundation of Educational Committee and NSFC (10571138). This work was done during the author's visit to the LAMFA, CNRS UMR 6140, Amiens; he would like to thank the institution for their warm hospitality.

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Remark 1.3. If $x = [a_0; a_1, a_2, \cdots, \overline{a_{r+1}, a_{r+2}, \cdots, a_{r+s}}]$, from [4] and [1], we have

$$\lim_{n\to\infty} \frac{1}{n} \log q_n(x) = \frac{1}{s} \log \rho(W) \tag{1.3}$$

where

$$W = \begin{pmatrix} a_{r+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{r+2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{r+s} & 1 \\ 1 & 0 \end{pmatrix}$$

and $\rho(\cdot)$ denotes the spectral radius of the matrix. Using (1.3), it may be shown that Theorem 1.2 holds, but this nice theorem has not been acknowledged before and the method we used does give a short, simple and self-contained proof of Theorem 1.2.

2. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2.

For any $n \geq 1$ and $(a_1, a_2, \cdots, a_n) \in \mathbb{N}^n$, let $q_n(a_1, a_2, \cdots, a_n)$ be the denominator of the finite continued fraction $[0; a_1, a_2, \cdots, a_n]$. The following lemma is proved in [7]; we present it here for completeness.

Lemma 2.1. *For any $n \geq 1$ and $1 \leq k \leq n$, we have*

$$\frac{a_k + 1}{2} \leq \frac{q_n(a_1, a_2, \cdots, a_n)}{q_{n-1}(a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n)} \leq a_k + 1 \ (q_0 := 1).$$

Proof. By (1.2),

$$\frac{q_k(a_1, \cdots, a_k)}{q_{k-1}(a_1, \cdots, a_{k-1})} = \frac{a_k q_{k-1}(a_1, \cdots, a_{k-1}) + q_{k-2}(a_1, \cdots, a_{k-2})}{q_{k-1}(a_1, \cdots, a_{k-1})} \leq a_k + 1,$$

$$\frac{q_k(a_1, \cdots, a_k)}{q_{k-1}(a_1, \cdots, a_{k-1})} \geq a_k \geq \frac{a_k + 1}{2},$$

$$\begin{aligned}
\frac{q_{k+1}(a_1, \cdots, a_{k+1})}{q_k(a_1, \cdots, a_{k-1}, a_{k+1})} &= \frac{a_{k+1} q_k(a_1, \cdots, a_k) + q_{k-1}(a_1, \cdots, a_{k-1})}{a_{k+1} q_{k-1}(a_1, \cdots, a_{k-1}) + q_{k-2}(a_1, \cdots, a_{k-2})} \\
&= \frac{(a_{k+1} a_k + 1) q_{k-1}(a_1, \cdots, a_{k-1}) + a_{k+1} q_{k-2}(a_1, \cdots, a_{k-2})}{a_{k+1} q_{k-1}(a_1, \cdots, a_{k-1}) + q_{k-2}(a_1, \cdots, a_{k-2})} \\
&\leq \frac{a_{k+1} a_k q_{k-1}(a_1, \cdots, a_{k-1})}{a_{k+1} q_{k-1}(a_1, \cdots, a_{k-1})} \\
&\quad + \frac{q_{k-1}(a_1, \cdots, a_{k-1}) + a_{k+1} q_{k-2}(a_1, \cdots, a_{k-2})}{a_{k+1} q_{k-1}(a_1, \cdots, a_{k-1}) + q_{k-2}(a_1, \cdots, a_{k-2})} \\
&\leq a_k + 1.
\end{aligned}$$

$$\begin{aligned}
&\frac{q_{k+1}(a_1, \cdots, a_{k+1})}{q_k(a_1, \cdots, a_{k-1}, a_{k+1})} \\
&\quad = \frac{(a_{k+1} a_k + 1) q_{k-1}(a_1, \cdots, a_{k-1}) + a_{k+1} q_{k-2}(a_1, \cdots, a_{k-2})}{a_{k+1} q_{k-1}(a_1, \cdots, a_{k-1}) + q_{k-2}(a_1, \cdots, a_{k-2})} \\
&\quad \geq \frac{(a_{k+1} a_k + 1) q_{k-1}(a_1, \cdots, a_{k-1})}{(a_{k+1} + 1) q_{k-1}(a_1, \cdots, a_{k-1})} \\
&\quad = \frac{a_{k+1} a_k + 1}{a_{k+1} + 1} \geq \frac{a_k + 1}{2}.
\end{aligned}$$

Using (1.2) and induction, we get the desired result. □

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The following is the proof of Theorem 1.2.

Proof. For any $\log \frac{\sqrt{5}+1}{2} < \lambda < \infty$ and any $0 < \epsilon < \lambda - \log \frac{\sqrt{5}+1}{2}$, choose $N \in \mathbb{N}$ large enough such that

$$e^{N\left(\lambda+\epsilon-\log \frac{\sqrt{5}+1}{2}\right)+\log \frac{\sqrt{5}+1}{2}} - 1 \geq 2 \cdot e^{N\left(\lambda-\epsilon-\log \frac{\sqrt{5}+1}{2}\right)+\log \frac{\sqrt{5}+1}{2}}.$$

Choose $b \in \mathbb{N}$ such that

$$2 \cdot e^{N\left(\lambda-\epsilon-\log \frac{\sqrt{5}+1}{2}\right)+\log \frac{\sqrt{5}+1}{2}} - 1 \leq b \leq e^{N\left(\lambda+\epsilon-\log \frac{\sqrt{5}+1}{2}\right)+\log \frac{\sqrt{5}+1}{2}} - 1. \tag{2.1}$$

Let $x \in (0,1)$ such that $a_n(x) = b$ if $n = kN$ for some $k \in \mathbb{N}$, and $a_n(x) = 1$ otherwise. x has a purely periodic continued fraction expansion, thus by Lagrange's theorem (see [5], page 56), x is a quadratic irrational.

For any $n \geq N$, there exists $k \in \mathbb{N}$ such that $kN \leq n < (k+1)N$. By Lemma 2.1,

$$q_n(x) \leq q_{n-k}(1,1,\cdots,1)(b+1)^k \leq c_1 \left(\frac{\sqrt{5}+1}{2}\right)^{n-k} (b+1)^k, \tag{2.2}$$

$$q_n(x) \geq q_{n-k}(1,1,\cdots,1)\left(\frac{b+1}{2}\right)^k \geq c_2 \left(\frac{\sqrt{5}+1}{2}\right)^{n-k} \left(\frac{b+1}{2}\right)^k, \tag{2.3}$$

where c_1, c_2 in (2.2) and (2.3) are positive constants which do not depend on n. Thus by (2.1), we have

$$\begin{aligned}
\beta(x) &\leq \limsup_{n\to\infty} \frac{\log\left(c_1\left(\frac{\sqrt{5}+1}{2}\right)^{n-k}(b+1)^k\right)}{n} \\
&= \limsup_{n\to\infty} \frac{k\left(\log(b+1) - \log\frac{\sqrt{5}+1}{2}\right)}{n} + \log\frac{\sqrt{5}+1}{2} \\
&\leq \frac{1}{N}\left(\log(b+1) - \log\frac{\sqrt{5}+1}{2}\right) + \log\frac{\sqrt{5}+1}{2} \leq \lambda + \epsilon,
\end{aligned}$$

$$\begin{aligned}
\beta(x) &\geq \liminf_{n\to\infty} \frac{\log\left(c_2\left(\frac{\sqrt{5}+1}{2}\right)^{n-k}\left(\frac{b+1}{2}\right)^k\right)}{n} \\
&= \liminf_{n\to\infty} \frac{k\left(\log(b+1) - \log\frac{\sqrt{5}+1}{2}\right)}{n} + \log\frac{\sqrt{5}+1}{2} \\
&\geq \liminf_{k\to\infty} \frac{k\cdot\left(\log\frac{b+1}{2} - \log\frac{\sqrt{5}+1}{2}\right)}{(k+1)N} + \log\frac{\sqrt{5}+1}{2} \\
&= \frac{1}{N}\left(\log\frac{b+1}{2} - \log\frac{\sqrt{5}+1}{2}\right) + \log\frac{\sqrt{5}+1}{2} \geq \lambda - \epsilon.
\end{aligned}$$

Therefore

$$\beta(x) \in [\lambda-\epsilon, \lambda+\epsilon].$$

□

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Acknowledgment

The author is grateful to C. Baxa and C. Faivre for sending him the papers [1] and [2].

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