

## A CHARACTERISATION OF $0^\#$ IN TERMS OF FORCING

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ABSTRACT. We show that “saturation” of the universe with respect to forcing over  $L$  with partial orders on  $\omega_1$  is equivalent to the existence of  $0^\#$ .

If  $P$  is a constructible forcing notion, then we say that  $G \subseteq P$  is  $P$ -generic iff  $G$  is  $P$ -generic over  $L$ . The statement that all countable constructible forcings have generics is rather weak, and holds for example in  $L[R]$ , where  $R$  is a Cohen real over  $L$ . But it is not possible that all constructible forcings have generics: consider the forcing that collapses  $\omega_1$  to  $\omega$  with finite conditions.

**Definition.**  $V$  is  $L$ -saturated for  $\omega_1$ -forcings iff whenever  $P$  is a constructible forcing of  $L$ -cardinality  $\omega_1$  such that for any  $p \in P$  there is a  $P$ -generic containing  $p$  in some  $\omega_1$ -preserving extension of  $V$ , then there is a  $P$ -generic in  $V$ .

**Theorem 1.** *The following are equivalent:*

- (a)  $V$  is  $L$ -saturated for  $\omega_1$ -forcings.
- (b)  $0^\#$  exists.

*Proof.* (a)  $\rightarrow$  (b) The existence of  $0^\#$  is equivalent to the statement that every stationary constructible subset of  $\omega_1$  contains a CUB subset (see [2]). Now use the following:

*Fact* (Baumgartner; see [1]). If  $X$  is a stationary constructible subset of  $\omega_1$ , then there is a forcing  $P \in L$  of  $L$ -cardinality  $\omega_1$  which preserves cardinals over  $V$  and adds a CUB subset to  $X$ . ( $P$  adds a CUB subset of  $X$  using “finite conditions”.)

(b)  $\rightarrow$  (a) Assume that  $0^\#$  exists, and suppose that  $P$  is a constructible forcing of  $L$ -cardinality  $\omega_1$  such that every condition in  $P$  belongs to a generic in an  $\omega_1$ -preserving extension of  $V$ . We will show that there is a  $P$ -generic in  $V$ . Assume that the universe of  $P$  is exactly  $\omega_1$ . Let  $P$  be of the form  $t(\vec{i}, \omega_1, \vec{\alpha})$ , where  $\vec{i} < \omega_1 < \vec{\alpha}$  is a finite increasing sequence of indiscernibles and  $t$  is an  $L$ -term. We claim that if  $\vec{i} < k_0 < k_1$  are countable indiscernibles and  $G_{k_0}$  is  $P_{k_0}$ -generic over  $L$ , then there is  $G_{k_1}$  containing  $G_{k_0}$  which is  $P_{k_1}$ -generic over  $L$ , where  $P_k = t(\vec{i}, k, \vec{\alpha})$ . If not, then player  $I$  wins the open game  $\mathcal{G}(k_0, k_1, G_{k_0})$ , where  $I$  chooses constructible dense subsets of  $P_{k_1}$  and  $II$  responds with increasingly strong conditions meeting

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these dense sets which are compatible with all conditions in  $G_{k_0}$ . The latter is a property of the model  $L[G_{k_0}]$ . Let  $p \in P_{k_0}$  be a condition forcing that  $I$  wins  $\mathcal{G}(k_0, k_1, G_{k_0})$ . Then  $p$  forces that  $I$  wins  $\mathcal{G}(k_2, k_3, G_{k_2})$ , where  $k_2 < k_3$  are any indiscernibles  $\geq k_0$  and  $G_{k_2}$  denotes the  $P_{k_2}$ -generic. But now let  $G$  be a  $P$ -generic containing  $p$  in an  $\omega_1$ -preserving extension of  $V$ . As  $G$  preserves  $\omega_1$  over  $V$ , there are indiscernibles  $k_2 < k_3$  with  $k_0 \leq k_2$  such that  $G \cap k_2$  is  $P_{k_2}$ -generic and  $G \cap P_{k_3}$  is  $P_{k_3}$ -generic, so clearly player  $II$  has a winning strategy in the game  $\mathcal{G}(k_2, k_3, G \cap P_{k_2})$ , in contradiction to the choice of  $p$ .

Now it is easy to build a  $P$ -generic: List the countable indiscernibles greater than  $\vec{i}$  as  $j_0 < j_1 < j_2 < \dots$  and inductively choose  $P_{j_\alpha}$ -generic  $G_\alpha$  such that  $\alpha < \beta$  implies  $G_\alpha \subseteq G_\beta$ . At the first step,  $G_{j_0}$  is an arbitrary  $P_{j_0}$ -generic. By the previous paragraph there is no difficulty at the successor steps, where one extends  $G_{j_\alpha}$  to  $G_{j_{\alpha+1}}$ . At limit stages  $\lambda$ , the  $P_{j_\lambda}$ -genericity of the union  $G_{j_\lambda}$  of the  $G_{j_\alpha}$ ,  $\alpha < \lambda$ , follows by indiscernibility. The desired  $P$ -generic is the union of the  $G_{j_\alpha}$ ,  $\alpha < \omega_1$ .  $\square$

*Remark.* The proof of (a) implies (b) shows that the theorem still holds if “ $\omega_1$ -preserving extension” is taken to be “ $\omega_1$ -preserving set-generic extension” in the definition of  $L$ -saturation for  $\omega_1$ -forcings.

**Question.** Suppose that  $0^\#$  exists. Then does part (a) of the theorem hold (in the obvious sense) for constructible  $\omega_1^{+L}$ -forcings, i.e. constructible  $P$  of  $L$ -cardinality  $\omega_1^+$  of  $L$ ?

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