

ALMOST EVERYWHERE CONVERGENCE OF INVERSE FOURIER TRANSFORMS

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ABSTRACT. We show that if $\log(2 - \Delta)f \in L^2(\mathbb{R}^d)$, then the inverse Fourier transform of f converges almost everywhere. Here the partial integrals in the Fourier inversion formula come from dilates of a closed bounded neighbourhood of the origin which is star shaped with respect to 0. Our proof is based on a simple application of the Rademacher-Menshov Theorem. In the special case of spherical partial integrals, the theorem was proved by Carbery and Soria. We obtain some partial results when $\sqrt{\log(2 - \Delta)}f \in L^2(\mathbb{R}^d)$ and $\log \log(4 - \Delta)f \in L^2(\mathbb{R}^d)$. We also consider sequential convergence for general elements of $L^2(\mathbb{R}^d)$.

1. INTRODUCTION

We treat the almost everywhere convergence of partial integrals of inverse Fourier transforms on Euclidean space, for functions in L^2 with logarithmic Sobolev properties. The partial integrals are formed by integrating over dilates of a fixed closed bounded region V which is star shaped with respect to the origin and has the origin in its interior. Particular choices of V give rise to the familiar cases of spherical and polyhedral partial integrals. Our results are proved by a very simple application of the Rademacher-Menshov Theorem. In particular, we show that if the Fourier transform satisfies

$$(1) \quad \int_{\mathbb{R}^d} (\log(2 + |y|^2))^2 |\widehat{f}(y)|^2 dy < \infty,$$

then the partial integrals

$$S_R f(x) = \int_{RV} \widehat{f}(y) e^{2\pi i x \cdot y} dy$$

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converge almost everywhere as $R \rightarrow \infty$. If we reduce the power of the logarithmic factor, we have a partial result. We show that if

$$(2) \quad \int_{\mathbb{R}^d} \log(2 + |y|^2) \left| \widehat{f}(y) \right|^2 dy < \infty,$$

then $S_{R_n} f(x)$ converges almost everywhere as $R_n = n^{\log(n)} \rightarrow \infty$. When the logarithm is replaced by $\log \log$, we find that if

$$(3) \quad \int_{\mathbb{R}^d} (\log \log(4 + |y|^2))^2 \left| \widehat{f}(y) \right|^2 dy < \infty,$$

then $S_{r_m} f(x)$ converges almost everywhere as $m \rightarrow \infty$, for unbounded sequences (r_m) whose terms are in a second order lacunary set, as defined in [2].

The first case, with V a sphere, was done by Carbery and Soria [4, Theorem 3]. The introduction to their paper provides a broad description to the background of this area of Fourier analysis. See also [6] for some weighted norm estimates in the spherical case. The third case is a slight extension of the main result in [2], where they work with integrals over spheres.

Our contribution is the simplicity of the proof and the fact that it is independent of the geometry of V . The method seems to depend only on the Plancherel formula, and follows the same idea as used in [8].

2. THE RADEMACHER-MENSHOV THEOREM

Theorem 1. *Suppose that (X, μ) is a positive measure space. There is a positive constant c with the following property.*

For each orthogonal subset $\{P_n : n \in \mathbb{N}\}$ in $L^2(X, \mu)$ which satisfies

$$(4) \quad \sum_{n=1}^{\infty} (\log(n + 1))^2 \|P_n\|_2^2 < \infty,$$

the maximal function $\mathcal{M}(x) = \sup_{N \geq 1} \left| \sum_{n=1}^N P_n(x) \right|$ is in $L^2(X, \mu)$ and

$$(5) \quad \|\mathcal{M}\|_2 \leq c \left(\sum_{n=1}^{\infty} (\log(n + 1))^2 \|P_n\|_2^2 \right)^{1/2}.$$

In particular, when (4) holds, then the series $\sum_{n=1}^{\infty} P_n(x)$ converges almost everywhere on X .

See Theorem XIII.10.21 from [11], Proposition 2.3.1, and Theorem 2.3.2 from [1, Pages 79–80]. Here \log means logarithm with base 2. For an application in $L^2(\mathbb{R}^d)$, see part (b) of Lemma 5.1 in [5].

3. SETTING UP THE PARTIAL INTEGRALS

Suppose that V is a bounded closed subset of \mathbb{R}^d having 0 as an interior point and star shaped with respect to 0. Let $\beta = d(0, \partial V) > 0$. For each $R > 0$ dilate V to get $RV = \{Ry : y \in V\}$, so that the dilated set has measure

$$|RV| = R^d|V| \quad \text{and} \quad d(0, \partial(RV)) = R\beta.$$

Define partial integrals by

$$(6) \quad S_R f(x) = \int_{RV} \widehat{f}(y) e^{2\pi i x \cdot y} dy, \quad \forall f \in L^2(\mathbb{R}^d),$$

which give Fourier inversion in norm, $\lim_{R \rightarrow \infty} \|S_R f - f\|_2 = 0$. If $f \in L^2(\mathbb{R}^d)$ and $S_R f(x)$ converges almost everywhere as $R \rightarrow \infty$, then its limit equals $f(x)$ almost everywhere.

Now let $(R_n)_{n=1}^\infty$ be an unbounded increasing sequence of positive real numbers and fix an element $f \in L^2(\mathbb{R}^d)$. We think of the partial integrals $S_{R_n} f(x)$ as partial sums of the orthogonal expansion

$$(7) \quad S_{R_1} f(x) + \sum_{n=2}^\infty (S_{R_n} f(x) - S_{R_{n-1}} f(x)).$$

Define $P_n f \in L^2(\mathbb{R}^d)$ by setting

$$(8) \quad P_n f(x) = \begin{cases} S_{R_1} f(x) & \text{if } n = 1, \\ S_{R_n} f(x) - S_{R_{n-1}} f(x) & \text{if } n \geq 2. \end{cases}$$

Then the partial sums of (7) are

$$S_{R_n} f(x) = \sum_{k=1}^n P_k f(x), \quad \forall n \geq 1, x \in \mathbb{R}^d,$$

and $m \neq n$ implies that $P_m f \perp P_n f$. The Plancherel formula says that

$$(9) \quad \|P_n f\|_2^2 = \int_{R_n V \setminus R_{n-1} V} |\widehat{f}(y)|^2 dy.$$

4. CONVERGENT SUBSEQUENCES

Suppose that $f \in L^2(\mathbb{R}^d)$. Since $S_R f$ converges to f in norm, there exists a sequence $(R_n)_{n=0}^\infty$ with $\lim_{n \rightarrow \infty} S_{R_n} f(x) = f(x)$ almost everywhere. The Rademacher-Menshov Theorem gives a way of describing one such sequence.

Proposition 2. *Suppose $f \in L^2(\mathbb{R}^d)$. If an increasing unbounded sequence $0 = R_0 < R_1 < R_2 < \dots$ has the property*

$$(10) \quad \sum_{n=1}^\infty \|S_{R_n} f - S_{R_{n-1}} f\|_2^2 (\log(n+1))^2 < \infty,$$

then $\lim_{n \rightarrow \infty} S_{R_n} f(x) = f(x)$ almost everywhere. Furthermore, for each $f \in L^2(\mathbb{R}^d)$ there is an increasing unbounded sequence $(R_n)_{n=0}^\infty$ with property (10).

Proof. The first statement is a direct consequence of Theorem 1. It remains to prove the second statement. If \widehat{f} has bounded support, then it is integrable and the statement is immediate. Now suppose that \widehat{f} is not compactly supported. The function $R \mapsto F(R) = \|S_R f\|_2^2$ is continuous and its values are non-negative. If $R' < R''$, then $F(R') \leq F(R'')$, and $\lim_{R \rightarrow \infty} F(R) = \|f\|_2^2$. Let $(a_n)_{n=1}^\infty$ be a sequence of positive numbers with

$$\sum_{n=1}^\infty a_n = 1 \quad \text{and} \quad \sum_{n=1}^\infty (\log(n+1))^2 a_n < \infty.$$

There is an increasing unbounded sequence $(R_n)_{n=1}^\infty$ with the property

$$F(R_n)^2 = \|f\|_2^2 \sum_{m=1}^n a_m, \quad \forall n \geq 1.$$

In particular, $\|S_{R_{n+1}}f\|_2^2 - \|S_{R_n}f\|_2^2 = a_{n+1}\|f\|_2^2$, for all $n \geq 1$. Define the projections as in (8). Then we have that

$$\sum_{n=1}^{\infty} (\log(n+1))^2 \|P_n f\|_2^2 < \infty$$

and we can apply Theorem 1. □

The Cauchy-Schwarz inequality and the Plancherel formula imply that when a sequence of partial integrals converges, then the sequence can be perturbed slightly and still preserve convergence.

Lemma 3. *Suppose $(R_n)_{n=1}^{\infty}$ is an increasing unbounded sequence. For each $\rho > 0$ and $n \geq 1$ define the set*

$$E_{\rho}(n) = \{r > 0 : |r^d - R_n^d| \leq \rho\}.$$

For these sets and $f \in L^2(\mathbb{R}^d)$ there is the inequality,

$$\sup_{n \geq 1} \left(\sup_{r \in E_{\rho}(n)} |S_r f(x) - S_{R_n} f(x)| \right) \leq \|f\|_2 \sqrt{\rho|V|}, \quad \forall x \in \mathbb{R}^d.$$

Now fix $f \in L^2(\mathbb{R}^d)$ and suppose $(R_n)_{n=1}^{\infty}$ is an increasing unbounded sequence for which $S_{R_n} f(x)$ converges almost everywhere. Furthermore, let $E_{\rho} = \bigcup_{n=1}^{\infty} E_{\rho}(n)$. If $(r_m)_{m=1}^{\infty}$ is an increasing unbounded sequence whose terms belong to a set E_{ρ} , then $\lim_{m \rightarrow \infty} S_{r_m} f(x) = f(x)$, almost everywhere.

Proof. If $0 \leq R_n^d - r^d \leq \rho$, then $rV \subset R_n V$ and $|R_n V \setminus rV| \leq \rho|V|$, so that

$$(11) \quad |S_{R_n} f(x) - S_r f(x)| \leq \left(\int_{R_n V \setminus rV} |\widehat{f}(y)|^2 dy \right)^{1/2} \sqrt{\rho|V|}.$$

Since $f \in L^2(\mathbb{R}^d)$, the right-hand side tends to zero as $R_n \rightarrow \infty$. A similar argument applies to the case $0 \leq r^d - R_n^d \leq \rho$. □

We can apply the Rademacher-Menshov Theorem again to give a minor extension of Proposition 2.

Lemma 4. *Suppose that $f \in L^2(\mathbb{R}^d)$ and $(R_n)_{n=0}^{\infty}$ satisfy (10) of Proposition 2. If $(r_m)_{m=1}^{\infty}$ is an unbounded increasing sequence with the property that*

$$|\{m : R_n \leq r_m \leq R_{n+1}\}| \leq cn^{\gamma}, \quad \forall n \geq 1,$$

for some positive constants c and γ , then

$$\lim_{m \rightarrow \infty} S_{r_m} f(x) = f(x), \quad \text{almost everywhere.}$$

Proof. For each $n \geq 1$, suppose that there is a finite set of M_n real numbers arranged in the interval (R_n, R_{n+1}) , say

$$R_n = r_1(n) < \dots < r_{M_n}(n) = R_{n+1}$$

and define functions

$$Q_{k,n}(x) = S_{r_{k+1}(n)} f(x) - S_{r_k(n)} f(x), \quad 1 \leq k < M_n.$$

These functions form an orthogonal subset of $L^2(\mathbb{R}^d)$ and so the Rademacher-Menshov Theorem says that

$$\max_{1 \leq m < M_n} |S_{r_m(n)}f(x) - S_{R_n}f(x)| = \max_{1 \leq m < M_n} \left| \sum_{k=1}^m Q_{k,n}(x) \right|$$

has L^2 norm bounded by

$$c(\log M_n) \|S_{R_{n+1}}f - S_{R_n}f\|_2 = c(\log M_n) \|P_{n+1}f\|_2.$$

Suppose that

$$\log M_n \leq \gamma \log n = \log(n^\gamma), \quad \forall n \geq 2.$$

Because of (10) we see that

$$\sum_{n=1}^{\infty} \max_{1 \leq m \leq M_n} |S_{r_m(n)}f(x) - S_{R_n}f(x)|^2$$

is in $L^1(\mathbb{R}^d)$. We then have that as $n \rightarrow \infty$,

$$\max_{1 \leq m \leq M_n} |S_{r_m(n)}f(x) - S_{R_n}f(x)| \rightarrow 0, \text{ almost everywhere.}$$

□

5. THE MAIN RESULT

Proposition 5. *Suppose that $f \in L^2(\mathbb{R}^d)$ satisfies the condition (1). Then $\lim_{R \rightarrow \infty} S_R f(x) = f(x)$, almost everywhere on \mathbb{R}^d . Furthermore, there is a constant $c > 0$ so that for all $w \in \mathbb{R}^d$,*

$$(12) \quad \int_{|x-w| \leq 1} \left| \sup_{R>0} |S_R f(x)| \right|^2 dx \leq c \int_{\mathbb{R}^d} (\log(2 + |y|^2))^2 |\widehat{f}(y)|^2 dy.$$

Proof. Take the sequence $R_n = n^{1/d}$ in setting up (8) and let

$$\mathcal{M}f(x) = \sup_{n \geq 1} |S_{R_n}f(x)|, \quad \forall x \in \mathbb{R}^d.$$

When y is in the shell $R_n V \setminus R_{n-1} V$ it satisfies $|y| \geq (n-1)^{1/d} \beta$ and for large n there is a constant $c > 0$ for which

$$\log(n+1) \leq c \log(2 + |y|^2), \quad \forall y \in R_n V \setminus R_{n-1} V.$$

Combine this with (9) to see that

$$(\log(n+1))^2 \|P_n\|_2^2 \leq c \int_{R_n V \setminus R_{n-1} V} (\log(2 + |y|^2))^2 |\widehat{f}(y)|^2 dy.$$

Since f satisfies inequality (1), the sum of the terms on the right-hand side is finite. This verifies the hypothesis (4) in Theorem 1 and so $S_{R_n}f(x)$ converges almost everywhere as $n \rightarrow \infty$. Furthermore, we see that since (1) holds, then inequality (5) says that

$$(13) \quad \|\mathcal{M}f\|_2 \leq c \left(\int_{\mathbb{R}^d} (\log(2 + |y|^2))^2 |\widehat{f}(y)|^2 dy \right)^{1/2}.$$

We can dominate the maximal function over $R \geq 1$ by the maximal function over the sequence $(R_n)_{n=1}^\infty$ plus a remainder,

$$\sup_{R \geq 1} |S_R f(x)| \leq \mathcal{M}f(x) + \sup_{n > 0} \left(\sup_{R_n \leq r < R_{n+1}} |S_r f(x) - S_{R_n} f(x)| \right).$$

We chose the sequence $R_n = n^{1/d}$ so that the increments in the measure of the dilates of V are constant,

$$|R_n V \setminus R_{n-1} V| = n|V| - (n - 1)|V| = |V|.$$

If $R_n \leq r < R_{n+1}$, then $n \leq r^d < n + 1$ and $|r^d - n| = |r^d - R_n^d| \leq 1$, so that we can apply Lemma 3 with $\rho = 1$. Hence

$$(14) \quad \left\| \sup_{n > 0} \left(\sup_{R_n \leq r < R_{n+1}} |S_r f - S_{R_n} f| \right) \right\|_\infty \leq c \|f\|_2.$$

Combine inequalities (13) and (14) to prove (12). □

See [6, Chapter 2] for more sophisticated methods for estimating $S_{R_n} f(x) - S_r f(x)$.

6. THE CASE OF ONE POWER OF LOGARITHM

The first part of the method used above can be applied to other sequences.

Proposition 6. *Suppose that $f \in L^2(\mathbb{R}^d)$ satisfies the condition (2) and that $R_n = n^{\log n}$, for $n \geq 1$. Then $\lim_{n \rightarrow \infty} S_{R_n} f(x) = f(x)$, almost everywhere on \mathbb{R}^d .*

Proof. We have that $\log(R_n) = (\log n)^2$ and for large n there is a constant c for which

$$(15) \quad (\log(n + 1))^2 \|P_n f\|_2^2 \leq c \int_{R_n V \setminus R_{n-1} V} \log(2 + |y|^2) \left| \widehat{f}(y) \right|^2 dy.$$

Inequality (2) means that the sum of the terms on the right-hand side is finite and so Theorem 1 applies. □

Note that $n^{\log n} = 2^{(\log n)^2}$ grows slower than any unbounded geometric progression but faster than n^k , for each $k \in \mathbb{N}$. The measure of the shell $R_n V \setminus R_{n-1} V$ grows too rapidly to use the estimate from Lemma 3. However, Lemma 4 gives convergence for some other sequences.

Corollary 7. *Suppose that $f \in L^2(\mathbb{R}^d)$ satisfies (2) and $(r_m)_{m=1}^\infty$ is an unbounded increasing sequence with the property that*

$$\left| \left\{ m : n^{\log n} \leq r_m \leq (n + 1)^{\log(n+1)} \right\} \right| \leq cn^\gamma, \quad \forall n \geq 1,$$

for some positive constants c and γ . Then $\lim_{m \rightarrow \infty} S_{r_m} f(x) = f(x)$, almost everywhere.

7. ITERATED LOGARITHM

Fix $a > 1$ and define the geometric progression $R_n = a^n$, for all $n \geq 1$. For $y \in R_n V \setminus R_{n-1} V$ we have $|y| \geq a^{n-1} \beta$ and for large n there is a constant $\kappa > 0$ with

$$\log \log(4 + |y|^2) \geq \kappa \log(n + 1).$$

This means that for large n we have

$$\begin{aligned} \kappa^2 (\log(n + 1))^2 \int_{R_n V \setminus R_{n-1} V} |\widehat{f}(y)|^2 dy \\ \leq \int_{R_n V \setminus R_{n-1} V} (\log \log(4 + |y|^2))^2 |\widehat{f}(y)|^2 dy \end{aligned}$$

and we can again apply Theorem 1.

Corollary 8. *Suppose that $f \in L^2(\mathbb{R}^d)$ satisfies (3) and that $a > 1$ is fixed. Then $\lim_{n \rightarrow \infty} S_{a^n} f(x) = f(x)$, almost everywhere on \mathbb{R}^d .*

Remark 7.1. For lacunary spherical partial integrals there is a much stronger result in [3, Theorem B] and in [7].

Lemma 4 can be applied to the case of $R_n = a^n$.

Corollary 9. *Fix $a > 1$ and let $(r_m)_{m=1}^\infty$ be an unbounded increasing sequence with the property that*

$$|\{m : a^n \leq r_m \leq a^{n+1}\}| \leq cn^\gamma, \quad \forall n \geq 1,$$

for some positive constants c and γ . If $f \in L^2(\mathbb{R}^d)$ satisfies (3), then

$$\lim_{m \rightarrow \infty} S_{r_m} f(x) = f(x), \quad \text{almost everywhere.}$$

We can combine Corollary 8 with Lemma 3 and Corollary 9 to extend the result of [2] to the case of general V .

Corollary 10. *Fix $a > 1$ and suppose $f \in L^2(\mathbb{R}^d)$ satisfies (3). Let*

$$A = \{a^n(1 - a^{-k}) : n, k \in \mathbb{N}\}$$

and suppose that $(r_m)_{m=1}^\infty$ is an increasing unbounded sequence whose terms belong to A . Then $\lim_{m \rightarrow \infty} S_{r_m} f(x) = f(x)$, almost everywhere.

Proof. Let $R_n = a^n$ and consider the set E_1 , as defined in Lemma 3. We need to count how many elements are in $(A \setminus E_1) \cap [a^{n-1}, a^n]$, for each $n \geq 1$. That is, we count how many k satisfy

$$(16) \quad a^{nd} - a^{nd}(1 - a^{-k})^d > 1.$$

This is equivalent to the inequality

$$1 - (1 - a^{-k})^d > a^{-dn}$$

and the left-hand side is equal to $da^{-k}y^{d-1}$ for some $1 - a^{-k} \leq y \leq 1$. Taking logarithms, we see that if k satisfies the inequality (16), then we must have $k \leq cn$, for some constants c . This shows that $A \setminus E_1$ satisfies the criterion of Corollary 9. If a sequence has its values in A , then it is made up of subsequences in $A \cap E_1$ and $A \setminus E_1$. Apply Lemma 3 for $A \cap E_1$ and Corollary 9 for $A \setminus E_1$. \square

8. CAPACITY

We conclude with an extension of Theorem 1.3 of [9] to summation based on the set V . Following Definition 2 in [9], for each $0 < \alpha < d$ the $(\alpha, 2)$ -capacity of a subset $X \subset \mathbb{R}^d$ is

$$C_\alpha(X) = \inf \{ \|f\|_2^2 : f \in L^2_+(\mathbb{R}^d), \quad G_\alpha * f(x) \geq 1, \forall x \in X \}.$$

Here G_α is the Bessel kernel, with $\widehat{G}_\alpha(y) = (1 + |y|^2)^{-\alpha/2}$. Its properties are cataloged in [10, Section V.3]. Most importantly, $G_\alpha(x) \geq 0$ for all $x \neq 0$. Notice that if $f \in L^2_+(\mathbb{R}^d)$ and $X \subseteq \{x : G_\alpha * f(x) \geq \lambda\}$, then $C_\alpha(X) \leq \lambda^{-2} \|f\|_2^2$. Capacity is subadditive and sets of capacity zero have Lebesgue measure zero.

Let $R_n = n^{1/d}$ for each $n \geq 1$, as in the proof of Proposition 5, and define $\mathcal{M}f(x) = \sup_{n \geq 1} |S_{R_n} f(x)|$. Recall that this satisfies inequality (13).

Lemma 11. *Suppose that $\varphi \in L^2(\mathbb{R}^d)$ satisfies*

$$(17) \quad N(\varphi, \alpha) := \int_{\mathbb{R}^d} |\widehat{\varphi}(y)|^2 (1 + |y|^2)^\alpha (\log(2 + |y|))^2 dy < \infty,$$

for some $0 < \alpha < d$. There is a positive constant c so that

$$C_\alpha(\{x : \mathcal{M}\varphi(x) \geq \lambda\}) \leq c\lambda^{-2} N(\varphi, \alpha), \quad \forall \lambda > 0.$$

Proof. Since φ satisfies (17), there is a $\psi \in L^2(\mathbb{R}^d)$ with $\varphi = G_\alpha * \psi$ and

$$N(\varphi, \alpha) = N(\psi, 0) = \int_{\mathbb{R}^d} |\widehat{\psi}(y)|^2 (\log(2 + |y|))^2 dy < \infty.$$

Inequality (13) can be applied to both ψ and to $\varphi = G_\alpha * \psi$, so that the maximal functions satisfy $\mathcal{M}\psi \in L^2(\mathbb{R}^d)$ and $\mathcal{M}(G_\alpha * \psi) \in L^2(\mathbb{R}^d)$. Since G_α is positive, the observation on page 1419 of [9] can be adapted to our sequential maximal function so that

$$\mathcal{M}(G_\alpha * \psi)(x) \leq G_\alpha * (\mathcal{M}\psi)(x), \quad \forall x \in \mathbb{R}^d.$$

For each $\lambda > 0$ let

$$X_\lambda = \{x : \mathcal{M}(G_\alpha * \psi)(x) \geq \lambda\} \subseteq \{x : G_\alpha * (\mathcal{M}\psi)(x) \geq \lambda\}.$$

From the definition of capacity, $C_\alpha(X_\lambda) \leq \lambda^{-2} \|\mathcal{M}\psi\|_2^2 \leq c\lambda^{-2} N(\psi, 0)$. □

Proposition 12. *Suppose that $\varphi \in L^2(\mathbb{R}^d)$ satisfies (17) for some $0 < \alpha < d$. The set on which $S_R \varphi(x)$ does not converge to $\varphi(x)$, as $R \rightarrow \infty$, has $(\alpha, 2)$ -capacity zero.*

Proof. The argument based on Lemma 3 shows that it is enough to consider the convergence of $S_{R_n} \varphi(x)$ as $n \rightarrow \infty$. Let ψ be the function in the previous proof, so that $\varphi = G_\alpha * \psi$. For $\delta > 0$ let $H \in C_c^\infty(\mathbb{R}^d)$ satisfy

$$N(\psi - H, 0) = \int_{\mathbb{R}^d} |\widehat{\psi}(y) - \widehat{H}(y)|^2 (\log(2 + |y|))^2 dy < \delta.$$

We know that $\lim_{R \rightarrow \infty} S_R(G_\alpha * H)(x) = G_\alpha * H(x)$, for all x . For each $\eta > 0$,

$$\begin{aligned} & \left\{ x : \limsup_{n \rightarrow \infty} |S_{R_n} \varphi(x) - \varphi(x)| > \eta \right\} \\ & \subseteq \left\{ x : \sup_{n \geq 1} |S_{R_n}(\varphi - G_\alpha * H)(x)| > \frac{\eta}{2} \right\} \cup \left\{ x : |\varphi(x) - G_\alpha * H(x)| > \frac{\eta}{2} \right\}. \end{aligned}$$

Lemma 11 shows that

$$C_\alpha \left(\left\{ x : \sup_{n \geq 1} |S_{R_n} (G_\alpha * \psi - G_\alpha * H)(x)| > \frac{\eta}{2} \right\} \right) \leq 4c\eta^{-2}\delta.$$

Observe that $|G_\alpha * \psi - G_\alpha * H| \leq G_\alpha * |\psi - H|$. The definition of capacity shows that

$$C_\alpha \left(\left\{ x : |G_\alpha * \psi(x) - G_\alpha * H(x)| > \frac{\eta}{2} \right\} \right) \leq 4\eta^{-2} \|\psi - H\|_2^2 < 4\eta^{-2} c_d^2 \delta.$$

Letting $\delta \rightarrow 0$, we find that

$$C_\alpha \left(\left\{ x : \limsup_{n \rightarrow \infty} |S_{R_n} \varphi(x) - \varphi(x)| > \eta \right\} \right) = 0,$$

for every $\eta > 0$. The set of divergence is

$$\bigcup_{k \geq 1} \left\{ x : \limsup_{n \rightarrow \infty} |S_{R_n} \varphi(x) - \varphi(x)| > \frac{1}{k} \right\},$$

which is a countable union of sets of $(\alpha, 2)$ -capacity zero and so it also has $(\alpha, 2)$ -capacity zero. \square

One consequence of this proposition is that the partial inverse Fourier integrals of functions in Sobolev classes $L_\alpha^2(\mathbb{R}^d)$ converge pointwise, with the possible exception of sets with zero $(\alpha - \varepsilon, 2)$ -capacity, for every $\varepsilon > 0$.

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