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A PERTURBED ELEMENTARY OPERATOR AND RANGE-KERNEL ORTHOGONALITY

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Abstract. Let $B(\mathcal{H})$ denote the algebra of operators on a Hilbert $\mathcal{H}$. If A_j and $B_j \in B(\mathcal{H})$ are commuting normal operators, and C_j and $D_j \in B(\mathcal{H})$ are commuting quasi-nilpotents such that $A_jC_j - C_jA_j = B_jD_j - D_jB_j = 0$, then define $M_j, N_j \in B(\mathcal{H})$ and $\mathcal{E}, E \in B(B(\mathcal{H}))$ by $M_j = A_j + C_j$, $N_j = B_j + D_j$, $\mathcal{E}(X) = A_1XA_2 + B_1XB_2$ and $E(X) = M_1XM_2 + N_1XN_2$. It is proved that $E^{-1}(0) \subseteq H_0(\mathcal{E}) = \mathcal{E}^{-1}(0)$ and $X \in E^{-1}(0) \Longrightarrow ||X|| \leq k\text{dist}(X, \mathcal{E}(B(\mathcal{H})))$, where $k \geq 1$ is some scalar and $H_0(\mathcal{E})$ is the quasi-nilpotent part of the operator $\mathcal{E}$.

1. Introduction and notation

For a Banach space operator T, $T \in B(\mathcal{V})$, the kernel $T^{-1}(0)$ and the range $T(\mathcal{V})$ are said to have a *k-gap* for some real number $k \geq 1$, denoted $T^{-1}(0)\perp_k T(\mathcal{V})$, if

$$y \in T^{-1}(0) \Longrightarrow ||y|| \leq k\text{dist}(y, T(\mathcal{V})))$$

[9, Definition]. A subspace $\mathcal{M}$ of the Banach space $\mathcal{V}$ is said to be *orthogonal* to a subspace $\mathcal{N}$ of $\mathcal{V}$, in the sense of G. Birkhoff and R. C. James (see [11, p. 93]), if $||m|| \leq ||m + n||$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$. This asymmetric definition of orthogonality coincides with the usual definition of orthogonality in the case in which $\mathcal{V} = \mathcal{H}$ is a Hilbert space. A 1-gap between $T^{-1}(0)$ and $T(\mathcal{V})$ corresponds to the *range-kernel orthogonality* for the operator T in the sense of [1, 2]. The following implications are straightforward to see:

$$T^{-1}(0)\perp_k T(\mathcal{V}) \Longrightarrow T^{-1}(0)\cap\overline{T(\mathcal{V})} = \{0\} \Longrightarrow T^{-1}(0)\cap T(\mathcal{V}) = \{0\} \Longrightarrow \text{asc}(T) \leq 1,$$

where $\overline{T(\mathcal{V})}$ denotes the closure of $T(\mathcal{V})$ and $\text{asc}(T)$ denotes the *ascent of* T. A k-gap between $T^{-1}(0)$ and $T(\mathcal{V})$ does not imply that $T(\mathcal{V})$ is closed, or even when $T(\mathcal{V})$ is closed that $\mathcal{V} = T^{-1}(0) \oplus T(\mathcal{V})$ (see for example [1], [2] and [21]).

Let $\mathcal{V} = B(\mathcal{H})$ be the algebra of operators on a Hilbert space $\mathcal{H}$, and let $\delta_{AB} \in B(B(\mathcal{H}))$ denote the *generalized derivation* $\delta_{AB}(X) = AX - XB$. If $A, B \in B(\mathcal{H})$ are normal (or, more generally, if $A, B \in B(\mathcal{H})$ satisfy the *Putnam-Fuglede commutativity property* that $\delta_{AB}^{-1}(0) \subseteq \delta_{A^*B^*}^{-1}(0)$), then $\delta_{AB}^{-1}(0) \perp \delta_{AB}(B(\mathcal{H}))$ (see [1], [7], [17] and some of the references cited there). This orthogonality extends to

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the *elementary operator* $\triangle_{AB} \in B(B(\mathcal{H}))$, $\triangle_{AB}(X) = AXB - X$. Cyclic subnormal operators $A, B \in B(\mathcal{H})$ do not satisfy the property $\delta_{AB}^{-1}(0) \subseteq \delta_{A^*B^*}^{-1}(0)$. However, $\delta_{AA}^{-1}(0) \perp \delta_{AA}(B(\mathcal{H}))$ for a cyclic subnormal operator A [17], and this orthogonality fails in the absence of the hypothesis that the subnormal A is cyclic [12]. Observe that cyclic subnormal operators are *essentially normal*, which implies that $\pi(A)$ is a normal operator (whenever A is a cyclic subnormal), where π denotes the Calkin map. Since an operator $X \in B(H)$ in the commutant of a cyclic subnormal operator is subnormal [24], and since $||X|| = ||\pi(X)||$ for a subnormal operator X, it is seen that $\delta_{AA}^{-1}(0) \perp \delta_{AA}(B(\mathcal{H}))$ is consequent to the Putnam-Fuglede commutativity property $\delta_{\pi(A)\pi(A)}^{-1}(0) \subseteq \delta_{\pi(A^*)\pi(A^*)}^{-1}(0)$. We generalize this argument to prove a result (see Theorem 3.4 *infra*) which not only subsumes a number of the extant results on the range-kernel orthogonality of the operators δ_{AB} and $\triangle_{AB}$ but also leads to some new ones. Our principal objective in this note is to study the range-kernel orthogonality of the operators δ_{AB} and $\triangle_{AB}$ under perturbation by suitable quasi-nilpotents. For this we consider the elementary operator $\mathcal{E}_{\mathbf{AB}}(X) = A_1XA_2 + B_1XB_2$, where $\mathbf{A} = (A_1, A_2)$ and $\mathbf{B} = (B_1, B_2)$ are commuting tuples of normal operators. We prove that if C_j, D_j are quasi-nilpotent operators in $B(\mathcal{H})$ such that (the commutators)

$$[A_j, C_j] = [B_j, D_j] = [C_j, D_j] = 0$$

for $j = 1, 2$, $M_j = A_j + C_j$ and $N_j = B_j + D_j$, and the elementary operator $\mathbf{E_{MN}}$ is defined by $\mathbf{E_{MN}}(X) = M_1XM_2 + N_1XN_2$, then $\mathbf{E}_{\mathbf{MN}}^{-1}(0) \subseteq \mathcal{E}_{\mathbf{AB}}^{-1}(0) = H_0(\mathcal{E}_{\mathbf{AB}})$ and $\mathbf{E}_{\mathbf{MN}}^{-1}(0) \perp_k \mathcal{E}_{\mathbf{AB}}(B(\mathcal{H}))$, where $H_0(\mathcal{E}_{\mathbf{AB}})$ denotes *the quasi-nilpotent part* of the operator $\mathcal{E}_{\mathbf{AB}}$.

In the following d_{AB} shall denote either of the (already introduced) operators δ_{AB} and $\triangle_{AB}$. For an operator $T \in B(\mathcal{V})$, we shall denote the spectrum, the point spectrum, the approximate point spectrum, the surjectivity spectrum, the isolated points of the spectrum and the spectral radius of T by $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, $\sigma_{su}(T)$, $\mathrm{iso}\sigma(T)$ and $r(T)$, respectively. Recall that T is *normaloid* if $r(T) = ||T||$. The commutator $AB - BA$ of $A, B \in B(\mathcal{V})$ will be denoted by $[A, B]$. The operator $T \in B(\mathcal{V})$ is said to be Fredholm if $T(\mathcal{V})$ is closed and both the *deficiency indices* $\dim(T^{-1}(0))$ and co-$\dim(T(\mathcal{V}))$ are finite, and then the *index* of T is defined to be $\mathrm{ind}(T) = \dim(T^{-1}(0)) - \text{co-}\dim(T(\mathcal{V}))$. The essential spectrum $\sigma_e(T)$ of T is the complement in $\mathbb{C}$ (= the set of complex numbers) of the Fredholm set of T. The ascent of T, $\mathrm{asc}(T)$, is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$.

The *quasinilpotent part* $H_0(T - \lambda)$ and the *analytic core* $K(T - \lambda)$ of $T - \lambda$ are defined by

$$H_0(T - \lambda) = \{x \in \mathcal{V} : \lim_{n \longrightarrow \infty} ||(T - \lambda)^n x||^{\frac{1}{n}} = 0\}$$

and

$$\begin{gathered} K(T - \lambda) = \{x \in \mathcal{V} : \text{there exists a sequence } \{x_n\} \subset \mathcal{V} \text{ and } \delta > 0 \text{ for which} \\ x = x_0, (T - \lambda)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, ...\}. \end{gathered}$$

We note that $H_0(T - \lambda)$ and $K(T - \lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T - \lambda)$ such that $(T - \lambda)^{-p}(0) \subseteq H_0(T - \lambda)$ for all $p = 0, 1, 2, ...$ and $(T - \lambda)K(T - \lambda) = K(T - \lambda)$ [19].

We shall be making but a passing reference to the concepts of *a decomposable operator*, *Bishop's property* (β), *the decomposition property* (δ), and *the single-valued extension property*. The interested reader is referred to the monograph [18] for information on these concepts. The operator $T \in B(\mathcal{V})$ is a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty \to B(\mathcal{V})$ for which $\Phi(1) = I$ and $\Phi(Z) = T$, where $C^\infty(\mathtt{C})$ is the Fréchet algebra of all infinitely differentiable functions on $\mathtt{C}$ (endowed with its usual topology of uniform convergence on compact sets for the functions and their partial derivatives) and Z is the identity function on $\mathtt{C}$ (see [5] and [18]).

2. Elementary operator $d_{A_1A_2}$

A Banach space operator H, $H \in B(\mathcal{V})$, is said to be *hermitian* if the numerical range of H is a subset of the set $\mathtt{R}$ of real numbers (equivalently, if $||exp(i\alpha H)|| = 1$ for all $\alpha \in \mathtt{R}$; see [4, p. 55]). An operator $A \in B(\mathcal{V})$ is *normal* if there exist hermitian elements $H, K \in B(\mathcal{V})$ such that $A = H + iK$ and $[H, K] = 0$. Let $A_j = H_j + iK_j$, $j = 1, 2$, be normal elements of $B(\mathcal{V})$. Then $\delta_{A_1A_2} = \delta_{H_1H_2} + i\delta_{K_1K_2}$, where $||exp(i\alpha\delta_{H_1H_2})|| = ||exp(i\alpha\delta_{K_1K_2})|| = 1$ for all $\alpha \in \mathtt{R}$ and $[\delta_{H_1H_2}, \delta_{K_1K_2}] = 0$. Hence $\delta_{A_1A_2} \in B(B(\mathcal{V}))$ is normal, and it follows from [13, Theorem A] that $\delta^{-1}_{A_1A_2}(0) \perp \delta_{A_1A_2}(B(\mathcal{V}))$. The normality of A_1 and A_2 is not sufficient to guarantee the normality of $\triangle_{A_1A_2}$ (see [9, Example 2.1]), and an argument similar to that for the case $\delta_{A_1A_2}$ cannot be applied to conclude that $\triangle^{-1}_{A_1A_2}(0) \perp \triangle_{A_1A_2}(B(\mathcal{V}))$. However, if $A_1, A_2 \in B(\mathcal{H})$ are normal, then $\triangle^{-1}_{A_1A_2}(0) = \triangle^{-1}_{A_1^*A_2^*}(0)$. (This is the well-known Putnam-Fuglede theorem for the elementary operator $\triangle_{A_1A_2}$; see, for example, [8].) Define the Hilbert space $\hat{\mathcal{H}}$ by $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$. Let the normal operator $\hat{A}$, the operator $\hat{X} \in B(\hat{\mathcal{H}})$ and the operator $\triangle_{\hat{A}\hat{A}^*} \in B(B(\hat{\mathcal{H}}))$ be defined, respectively, by $\hat{A} = A_1 \oplus A_2^*$, $\hat{X} = \left[\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right]$ and $\triangle_{\hat{A}\hat{A}^*}(\hat{X}) = \hat{A}\hat{X}\hat{A}^* + \hat{X}$. Then $\triangle^{-1}_{\hat{A}\hat{A}^*}(0) = \triangle^{-1}_{\hat{A}^*\hat{A}}(0)$, and it follows from [7, Theorem 1] that $\triangle^{-1}_{\hat{A}\hat{A}^*}(0) \perp (\triangle_{\hat{A}\hat{A}^*}(B(\mathcal{H})) \cup \triangle_{\hat{A}^*\hat{A}}(B(\mathcal{H})))$, and hence that $\triangle^{-1}_{A_1A_2}(0) \perp (\triangle_{A_1A_2}(B(\mathcal{H})) \cup \triangle_{A_1^*A_2^*}(B(\mathcal{H})))$. We have proved the following proposition (see [1, 2, 7, 9, 16, 17, 22]).

Proposition 2.1. *If $A_1, A_2 \in B(\mathcal{H})$ are normal, then*

$$d^{-1}_{A_1A_2}(0) \perp (d_{A_1A_2}(B(\mathcal{H})) \cup d_{A_1^*A_2^*}(B(\mathcal{H}))).$$

Let $\mathcal{A}$ denote a C^*-algebra with norm $||.||_{\mathcal{A}}$.

Corollary 2.2. *If a_1, a_2 are normal elements of $\mathcal{A}$, then*

$$d^{-1}_{a_1a_2}(0) \perp (d_{a_1a_2}(\mathcal{A}) \cup d_{a_1^*a_2^*}(\mathcal{A})).$$

Proof. There exists a Hilbert space $\mathcal{H}$ and a $*$-isometric isomorphism $\psi : \mathcal{A} \longrightarrow B(\mathcal{H})$ preserving order such that $A_1 = \psi(a_1)$ and $A_2 = \psi(a_2)$ are normal elements of $B(\mathcal{H})$. Let $x \in d^{-1}_{a_1a_2}(0)$; set $\psi(x) = X$. Then $X \in d^{-1}_{A_1A_2}(0)$, and

$$\begin{aligned} ||x||_{\mathcal{A}} = ||X|| &\leq \min(||X + d_{A_1A_2}(Y)||, ||X + d_{A_1^*A_2^*}(Y)||) \\ &= \min(||x + d_{a_1a_2}(y)||_{\mathcal{A}}, ||x + d_{a_1^*a_2^*}(y)||_{\mathcal{A}}) \end{aligned}$$

for every $y \in \mathcal{A}$. □

Recall from Kittaneh [17] that if $A \in B(\mathcal{H})$ is a cyclic subnormal operator, then $\delta^{-1}_{AA}(0) \perp \delta_{AA}(B(\mathcal{H}))$. This orthogonality fails in the absence of the hypothesis that A is cyclic [12]. For the elementary operator $\triangle_{AA}$, we have the following corresponding result.

Proposition 2.3. *If A is a cyclic subnormal operator, then to each $X \in \triangle_{AA}^{-1}(0)$ such that $X^2 \neq 0$ there corresponds a scalar $k = k(X)$, $0 < k \leq 1$, such $k||X|| \leq ||\triangle_{AA}(Y) + X||$ for every $Y \in B(\mathcal{H})$.*

Proof. Let $X \in \triangle_{AA}^{-1}(0)$. Then $AXA = X \Longrightarrow AXAXA = X^2A \Longrightarrow AX^2 = X^2A \Longrightarrow \delta_{AA}(X^{2m}) = 0$ and $AX^{2m-1}A = X^{2m-1}$ for every integer $m \geq 1$. Hence $||X^{2m}|| \leq ||\delta_{AA}(Z) + X^{2m}||$ for every integer $m \geq 1$ and $Z \in B(\mathcal{H})$. Choose $Z = YAX^{2m-1}$; then $||X^{2m}|| \leq ||\triangle_{AA}(Y) + X||||X^{2m-1}||$ for every $Y \in B(\mathcal{H})$ and all integers $m \geq 1$. Let $n = 2m$; since $\lim_{n \longrightarrow \infty} ||X^n||^{\frac{1}{n}} = \lim_{m \longrightarrow \infty} ||(X^2)^m||^{\frac{1}{2m}}$, and since X^{2m} is subnormal for every value of m, $r(X) = \sqrt{||X^2||}$. Set $||\triangle_{AA}(Y) + X|| = C$. Then $||X^{2m}|| \leq C||X^{2m-2}||||X||$ for every $m \geq 1$, and it follows from

$$\frac{||X^2||}{||X||}.\frac{||X^4||}{||X^2||||X||}. \cdots .\frac{||X^{2m}||}{||X^{2m-2}||||X||} \leq C^m$$

that

$$\frac{||X^{2m}||^{\frac{1}{m}}}{||X||} \leq C$$

for every integer $m \geq 1$. Hence $\frac{r(X)^2}{||X||} \leq C$. Define k by $\frac{r(X)^2}{||X||^2} = k$; then $0 < k \leq 1$ and $k||X|| \leq C$. □

The following (simple) theorem generalizes Proposition 2.1, Kittaneh's result on cyclic subnormal operators and Proposition 2.3. Let $\pi : B(\mathcal{H}) \longrightarrow B(\mathcal{H})/K(\mathcal{H})$ denote the Calkin map, and let $\mathcal{C}$ denote the Calkin algebra $B(\mathcal{H})/K(\mathcal{H})$. Recall that an operator $A \in B(\mathcal{H})$ is *essentially normal* if $\pi([A, A^*]) = 0$.

Theorem 2.4. *Let $A_1, A_2 \in B(\mathcal{H})$ be essentially normal operators. Set $d_{\pi(A_1)\pi(A_2)} = \mathbf{D}$ and $d_{\pi(A_1^*)\pi(A_2^*)} = \mathbf{D}_*$. Then $\mathbf{D}^{-1}(0) \perp (\mathbf{D}(\mathcal{C}) \cup \mathbf{D}_*(\mathcal{C}))$.*

Proof. Since the operators $\pi(A_1)$ and $\pi(A_2)$ are normal elements of the C^*-algebra $\mathcal{C}$, Corollary 2.2 applies. □

Theorem 2.4 applies in particular to operators $A_1, A_2 \in B(\mathcal{H})$ for which $d_{A_1A_2}^{-1}(0) \subseteq d_{A_1^*A_2^*}^{-1}(0)$. Such operators satisfy the Putnam-Fuglede commutativity property: $d_{A_1A_2}(X) = 0 \Longrightarrow d_{A_1^*A_2^*}(X) = 0$. Observe that if $d_{A_1A_2}(X) = 0 \Longrightarrow d_{A_1^*A_2^*}(X) = 0$, then $\mathcal{H}_1 = \overline{ranX}$ reduces A_1, $\mathcal{H}_2 = ker^{\perp}X$ reduces A_2, and $A_{11} = A_1|_{\mathcal{H}_1}$ and $A_{21} = A_2|_{\mathcal{H}_2}$ are normal operators (see [8]). Following an argument similar to that leading to the statement of Proposition 2.1, it is seen that $d_{A_1A_2}^{-1}(0) \perp (d_{A_1A_2}(B(\mathcal{H})) \cup d_{A_1^*A_2^*}(B(\mathcal{H})))$. Theorem 2.4 says more. Let

$$\mathcal{S} = \{T \in B(\mathcal{H}) : ||\pi(T)|| = ||T||\}.$$

Examples of operators $T \in \mathcal{S}$ are provided by hyponormal operators ($|T^*|^2 \leq |T|^2$), and paranormal operators ($||Tx||^2 \leq ||T^2x||||x||$ for every $x \in \mathcal{H}$) which have no eigenvalues of finite multiplicity. (Indeed, every normaloid $T \in B(\mathcal{H})$ such that T has no eigen-values of finite multiplicity is in $\mathcal{S}$, for the reason that for such an operator boundary $\sigma(T) \subseteq \sigma_a(T) =$ the left essential spectrum of T (see [6, 1.8.11 Proposition]), and hence $||T|| = r(T) = r(\pi(T)) = ||\pi(T)||$.)

Corollary 2.5. *If $A_1, A_2 \in B(\mathcal{H})$ are essentially normal operators, then:*

(i) *to each non-trivial $X \in d^{-1}_{A_1A_2}(0)$ there corresponds a scalar $k = k(X)$, $0 \leq k \leq 1$, such that*

$$k||X|| \leq min\{||d_{A_1A_2}(Y) + X||, ||d_{A_1^*A_2^*}(Y) + X||\}$$

for every $Y \in B(\mathcal{H})$;

(ii) $\quad (d^{-1}_{A_1A_2}(0) \cap \mathcal{S}) \perp (d_{A_1A_2}(B(\mathcal{H})) \cup d_{A_1^*A_2^*}(B(\mathcal{H})))$.

Proof. Observe that if $X \in d^{-1}_{A_1A_2}(0)$, then $\pi(X) \in d^{-1}_{\pi(A_1)\pi(A_2)}(0)$. The operators A_1 and A_2 being essentially normal, it follows from an application of Theorem 2.4 that

$$\begin{aligned} ||\pi(X)|| &\leq \min(||\pi(d_{A_1A_2}(Y) + X)||, ||\pi(d_{A_1^*A_2^*}(Y) + X)||) \\ &\leq \min(||d_{A_1A_2}(Y) + X||, ||d_{A_1^*A_2^*}(Y) + X||) \end{aligned}$$

for every $Y \in B(\mathcal{H})$ and $X \in d^{-1}_{A_1A_2}(0)$. The proof of (i) follows if we set $k = \frac{||\pi(X)||}{||X||}$. Evidently, $k = 1$ is independent of X in the case in which $X \in \mathcal{S}$; this proves (ii). □

Corollary 2.5 applies in particular to n-multicyclic hyponormal operators (which, by the Berger-Shaw inequality [14, Theorem 4.4], are essentially normal). Again, Corollary 2.5 holds for all $X \in \delta^{-1}_{A_1A_2}(0)$ in the case in which A_1, A_2 are cyclic subnormal operators. In such a case $B = A_1 \oplus A_2$ is a cyclic subnormal operator and every $Z \in \delta^{-1}_{BB}(0)$ is subnormal (and hence in $\mathcal{S}$) by a well-known result of Yoshino [24]. Recall that $A \in B(\mathcal{H})$ is p-hyponormal, $0 < p \leq 1$, if $|A^*|^{2p} \leq |A|^{2p}$. (A 1-hyponormal operator is hyponormal.) Since an n-multicyclic p-hyponormal operator satisfies

$$\text{trace}(|A|^{2p} - |A^*|^{2p})^{\frac{1}{p}} \leq \frac{n}{\pi}\text{area}\sigma(A),$$

[23, Theorem], $\pi(|A|^{2p} - |A^*|^{2p}) = 0$. Observe that A is p-hyponormal implies $\pi(A)$ is p-hyponormal, and then $\pi(|A|^{2p}) = \{|\pi(A)|^2\}^p$. (A p-hyponormal is q-hyponormal for every $0 < q \leq p$ (see [23] and [10]), and we may assume without loss of generality that $p = 2^{m-1}$ for some natural number m.) Hence $\{|\pi(A)|^2\}^p = \{|\pi(A^*)|^2\}^p$, which implies that $\pi(A)$ is normal.

Corollary 2.6. *If $A_1, A_2 \in B(H)$ are n-multicyclic p-hyponormal operators, then*

$$(d^{-1}_{A_1A_2}(0) \cap \mathcal{S}) \perp (d_{A_1A_2}(B(\mathcal{H})) \cup d_{A_1^*A_2^*}(B(\mathcal{H}))).$$

It is worth mentioning at this juncture that p-hyponormal operators belong to the set $\mathcal{S}$. To see this, recall that p-hyponormal operators are normaloid and to each p-hyponormal operator A there corresponds a hyponormal operator $\tilde{A}$ such that $\sigma_e(A) = \sigma_e(\tilde{A})$ and $||A|| = ||\tilde{A}||$ [10], which implies that $||\pi(\tilde{A})|| = ||\pi(A)|| = ||\tilde{A}|| = ||A||$.

3. PERTURBED ELEMENTARY OPERATOR $\mathcal{E}_{\mathbf{AB}}$

Let $\mathbf{A} = (A_1, A_2)$ and $\mathbf{B} = (B_1, B_2)$ be commuting tuples of normal operators in $B(\mathcal{H})$, and let C_j, D_j be quasi-nilpotent operators in $B(\mathcal{H})$ such that

$$[A_j, C_j] = [B_j, D_j] = [C_j, D_j] = 0$$

for $j = 1, 2$. Define $M_j, N_j \in B(\mathcal{H})$ by $M_j = A_j + C_j, N_j = B_j + D_j$ and the elementary operator $\mathbf{E_{MN}}$ by $\mathbf{E_{MN}}(X) = M_1XM_2 + N_1XN_2$. We prove that

$\mathbf{E}_{\mathbf{MN}}^{-1}(0) \subseteq \mathcal{E}_{\mathbf{AB}}^{-1}(0) = H_0(\mathcal{E}_{\mathbf{AB}})$ and $\mathbf{E}_{\mathbf{MN}}^{-1}(0)\perp_k\mathcal{E}_{\mathbf{AB}}(B(\mathcal{H}))$. But before that we give a simple proof of the following known result (see [9, 16, 22]).

Lemma 3.1. $\mathcal{E}_{\mathbf{AB}}^{-1}(0)\perp_k\mathcal{E}_{\mathbf{AB}}(B(\mathcal{H}))$.

Proof. A familiar argument (see the proof of [9, Theorem 2.7]; see also the argument leading to Proposition 2.1) shows that to prove the lemma it will suffice to prove that $\phi^{-1}(0) \perp_k \phi(B(\mathcal{H}))$, where $\phi \in B(B(\mathcal{H}))$ is the operator $\phi(X) = AXA^* + BXB^*$ for commuting normal operators A and B.

The *Berberian extension theorem* [14, p. 43] says that given an operator $T \in B(\mathcal{H})$, there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and an isometric $*$-isomorphism $T \longrightarrow T^0$ preserving order such that $\sigma(T) = \sigma(T^0)$ and $\sigma_a(T) = \sigma_a(T^0) = \sigma_p(T^0)$. For simplicity, let us denote the Berberian extension T^0 of T by the (lower case) letter t. Thus a, b are commuting normal operators with $\sigma(a) = \sigma_p(a)$, $\sigma(b) = \sigma_p(b)$, and $X \in \phi^{-1}(0)$ implies that $\phi(x) = axa^* + bxb^* = 0$. We divide the proof into the cases $0 \notin \sigma(b)$ and $0 \in \sigma(b)$. If $0 \notin \sigma(b)$, then

$$\phi(x) = b(ab^{-1}xb^{*^{-1}}a^* + x)b^* = b\triangle_{tt^*}(x)b^*,$$

where $t = ab^{-1}$ is normal. Evidently, $||x|| \leq ||\triangle_{tt^*}(z) + x||$ for every $x \in \triangle_{tt^*}^{-1}(0) = \phi^{-1}(0)$ and $z \in B(\mathcal{K})$. Choose $z = byb^*$; then $||x|| \leq ||\phi(y)+x||$ for every $x \in \phi^{-1}(0)$ and $y \in B(\mathcal{K})$.

Assume now that $0 \in \sigma(b) = \sigma_p(b)$. Then $b = 0 \oplus b_{22}$ (with respect to the decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, say), where b_{22} is invertible. Since $[a, b] = 0$, $a = a_{11} \oplus a_{22}$ (with respect to $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$). Letting x have the corresponding matrix representation $x = [x_{jr}]_{j,r=1}^2$, it then follows that

$$||\phi(z) + x|| = \left|\left|\begin{bmatrix} a_{11}z_{11}a_{11}^* + x_{11} & a_{11}z_{12}a_{22}^* + x_{12} \\ a_{22}z_{21}a_{11}^* + x_{21} & a_{22}z_{22}a_{22}^* + b_{22}z_{22}b_{22}^* + x_{22} \end{bmatrix}\right|\right|$$

for every $z = [z_{jr}]_{j,r=1}^2 \in B(\mathcal{K})$. Since b_{22} is invertible, the norm of the jr-th entry in the above matrix is $\geq ||x_{jr}||$, which (see [3]) implies that the norm on the right-hand side in the above equality is $\geq \frac{1}{4}\sum_{j,r=1}^2 ||x_{jr}|| \geq \frac{1}{4}||x||$. Letting $k = 4$, we have

$$||x|| \leq k||\phi(z) + x||$$

for every $x \in \phi^{-1}(0)$ and $z \in B(\mathcal{K})$. Let $Z = \left[\begin{smallmatrix} 0 & Y \\ 0 & 0 \end{smallmatrix}\right]$, $Y \in B(\mathcal{H})$, and choose $z = Z^0$; then $||X|| = ||x|| \leq k||\phi(z) + x|| = k||\phi(Y) + X||$. ☐

The constant k, although an improvement on the constant in [9, Theorem 2.7], is not the best possible. That distinction goes to $k = 1$, and is achieved in the case in which $A_j^{-1}(0) \cap B_j^{-1}(0) = \{0\}$ (see [22, Theorem 2.4] and [9, Corollary 2.8]).

The following corollary is immediate from Lemma 3.1.

Corollary 3.2. $asc(\mathcal{E}_{\mathbf{AB}}) \leq 1$.

Corollary 3.3. $\mathcal{E}_{\mathbf{AB}}^{-1}(0) = \mathcal{E}_{\mathbf{A}^*\mathbf{B}^*}^{-1}(0)$, *where* $\mathbf{A}^* = (A_1^*, A_2^*)$ *and* $\mathbf{B}^* = (B_1^*, B_2^*)$. *Consequently,* $\mathcal{E}_{\mathbf{AB}}^{-1}(0)\perp_k\mathcal{E}_{\mathbf{A}^*\mathbf{B}^*}(B(\mathcal{H}))$.

Proof. Since $\text{asc}(\mathcal{E}_{\mathbf{AB}}) \leq 1$ and $\mathbf{A}$ and $\mathbf{B}$ are commuting tuples of normal operators, $\mathcal{E}_{\mathbf{AB}}$ satisfies the Putnam-Fuglede theorem: $\mathcal{E}_{\mathbf{AB}}(X) = 0 \Longrightarrow \mathcal{E}_{\mathbf{A}^*\mathbf{B}^*}(X) = 0$ [20]. Now apply Lemma 3.1 to $\mathcal{E}_{\mathbf{A}^*\mathbf{B}^*}$. ☐

Theorem 3.4. (i) $\mathbf{E}_{\mathbf{MN}}^{-1}(0) \subseteq \mathcal{E}_{\mathbf{AB}}^{-1}(0) = H_0(\mathcal{E}_{\mathbf{AB}})$.

(ii) $\mathbf{E}_{\mathbf{MN}}^{-1}(0)\perp_k(\mathcal{E}_{\mathbf{AB}}(B(\mathcal{H})) \cup \mathcal{E}_{\mathbf{A}^*\mathbf{B}^*}(B(\mathcal{H})))$.

Proof. The proof of part (ii) follows from part (i), via an application of Lemma 3.1 and Corollary 3.3; we prove part (i).

Let L_E and $R_E \in B(B(\mathcal{H}))$ denote, respectively, the operator of *left multiplication by* E, $L_E(X) = EX$, and the operator of *right multiplication by* E, $R_E(X) = XE$. Define $\phi_{EF}(X) = L_E X R_F$. Let $X \in \mathbf{E}_{\mathbf{MN}}^{-1}(0)$. Then $\mathcal{E}_{\mathbf{AB}}(X) = -\Phi(X)$, where $\Phi = \phi_{A_1C_2} + \phi_{C_1A_2} + \phi_{C_1C_2} + \phi_{B_1D_2} + \phi_{D_1B_2} + \phi_{D_1D_2}$. Recall from [5, 4.3.8 Lemma] that the sum of two commuting quasi-nilpotent operators, also the product of two commuting operators one of which is quasi-nilpotent, is quasi-nilpotent. Since $[L_E, R_F] = 0$, and since the operators L_E and R_E are quasi-nilpotent whenever E is quasi-nilpotent, the operator Φ is quasi-nilpotent, which implies that $\lim_{n\longrightarrow\infty} ||\mathcal{E}_{\mathbf{AB}}^n(X)||^{\frac{1}{n}} = 0$, and hence that $X \in H_0(\mathcal{E}_{\mathbf{AB}})$.

The operators A_j and B_j, $j = 1, 2$, being normal, the operators L_{A_1}, L_{B_1}, R_{A_2} and R_{B_2} are generalized scalar operators. Furthermore, since $[A_j, B_j] = 0$, $[L_{A_j}, R_{B_j}] = [L_{A_1}R_{A_2}, L_{B_1}R_{B_2}] = 0$. Observe that L_{A_1} and R_{A_2}, similarly L_{B_1} and R_{B_2}, are commuting generalized scalar operators with two commuting spectral distributions. Hence $\mathcal{E}_{\mathbf{AB}} = L_{A_1}R_{A_2} + L_{B_1}R_{B_2}$ is a generalized scalar operator (see 4.3.3 Theorem, 4.4.2 Proposition and 4.4.3 Theorem of [5]), which implies that there exists an integer $p \geq 1$ such that $H_0(\mathcal{E}_{\mathbf{AB}}) = \mathcal{E}_{\mathbf{AB}}^{-p}(0)$ [5, 4.4.5 Theorem]. By Corollary 3.2, $\operatorname{asc}(\mathcal{E}_{\mathbf{AB}}) \leq 1$; hence $H_0(\mathcal{E}_{\mathbf{AB}}) = \mathcal{E}_{\mathbf{AB}}^{-1}(0)$, which implies that $X \in \mathcal{E}_{\mathbf{AB}}^{-1}(0)$. □

Let $\pi(\mathbf{A})$ and $\pi(\mathbf{B})$ denote the tuples $(\pi(A_1), \pi(A_2))$ and $(\pi(B_1), \pi(B_2))$, respectively, and let (as in Section 2) $\mathcal{S} = \{T \in B(\mathcal{H}) : ||T|| = ||\pi(T)||\}$. The following corollary is an immediate consequence of Theorem 3.4.

Corollary 3.5. *If* A_j, B_j, $j = 1, 2$, *are essentially normal operators in* $B(\mathcal{H})$, *then* $(\mathbf{E}_{\mathbf{MN}}^{-1}(0) \cap \mathcal{S}) \perp_k (\mathcal{E}_{\mathbf{AB}}(B(\mathcal{H})) \cup \mathcal{E}_{\mathbf{A^*B^*}}(B(\mathcal{H})))$.

The fact that $\mathcal{E}_{\mathbf{AB}}$ is a generalized scalar operator (in particular, a decomposable operator [18, p. 97]) has a number of consequences. We describe just one such consequence below.

As a decomposable operator, $\mathcal{E}_{\mathbf{AB}}$ satisfies both *Bishop's property* (β) and the *decomposition property* (δ). In particular, both $\mathcal{E}_{\mathbf{AB}}$ and the conjugate operator $\mathcal{E}_{\mathbf{AB}}^*$ have *the single-valued extension property* [18, Theorem 2.5.19] and $\sigma(\mathcal{E}_{\mathbf{AB}}) = \sigma_a(\mathcal{E}_{\mathbf{AB}}) = \sigma_{su}(\mathcal{E}_{\mathbf{AB}})$ [18, Proposition 1.3.2]. Furthermore, isolated points of $\sigma(\mathcal{E}_{\mathbf{AB}})$ are poles of the resolvent of $\mathcal{E}_{\mathbf{AB}}$. This is seen as follows. If $\lambda \in \operatorname{iso}\sigma(\mathcal{E}_{\mathbf{AB}})$, then (see [19] and [15])

$$B(\mathcal{H}) = H_0(\mathcal{E}_{\mathbf{AB}} - \lambda) \oplus K(\mathcal{E}_{\mathbf{AB}} - \lambda),$$

which since $\mathcal{E}_{\mathbf{AB}} - \lambda$ is (again) generalized scalar implies that

$$B(\mathcal{H}) = (\mathcal{E}_{\mathbf{AB}} - \lambda)^{-p}(0) \oplus K(\mathcal{E}_{\mathbf{AB}} - \lambda)$$

for some integer $p \geq 1$ [5, 4.4.5 Theorem]. Hence $(\mathcal{E}_{\mathbf{AB}} - \lambda)^p(B(\mathcal{H})) = K(\mathcal{E}_{\mathbf{AB}} - \lambda)$ and

$$B(\mathcal{H}) = (\mathcal{E}_{\mathbf{AB}} - \lambda)^{-p}(0) \oplus (\mathcal{E}_{\mathbf{AB}} - \lambda)^p(B(\mathcal{H})),$$

i.e. λ is a pole of the resolvent of $\mathcal{E}_{\mathbf{AB}}$ [15, Proposition 50.2]. Apparently, $(\mathcal{E}_{\mathbf{AB}} - \lambda)^{-p}(0) \perp (\mathcal{E}_{\mathbf{AB}} - \lambda)^p(B(\mathcal{H}))$ at every $\lambda \in \operatorname{iso}\sigma(\mathcal{E}_{\mathbf{AB}})$. In the particular case in which $\lambda = 0$, one may take $p = 1$. Can this be done for a general $0 \neq \lambda \in \operatorname{iso}\sigma(\mathcal{E}_{\mathbf{AB}})$? An affirmative answer here would imply the Putnam-Fuglede property $(\mathcal{E}_{\mathbf{AB}} - \lambda)^{-1}(0) = (\mathcal{E}_{\mathbf{A^*B^*}} - \overline{\lambda})^{-1}(0)$ for $\mathcal{E}_{\mathbf{AB}} - \lambda$ at every $\lambda \in \operatorname{iso}\sigma(\mathcal{E}_{\mathbf{AB}})$.

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