PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 6, Pages 1647–1649
S 0002-9939(06)08367-5
Article electronically published on January 17, 2006

HOMOGENEOUS SOLUTIONS TO FULLY NONLINEAR ELLIPTIC EQUATIONS

NIKOLAI NADIRASHVILI AND YU YUAN

(Communicated by David S. Tartakoff)

ABSTRACT. We classify homogeneous degree $d \neq 2$ solutions to fully nonlinear elliptic equations.

In this note, we show that any homogeneous degree other than 2 solution to fully nonlinear elliptic equations must be "harmonic". Consider the fully nonlinear elliptic equation $F\left(D^2u\right) = 0$ with $\mu I \leq (F_{ij}) = \left(F_{M_{ij}}(M)\right) \leq \mu^{-1} I$. Nirenberg [N] derived the a priori $C^{2,\alpha}$ estimates for the above equation in dimension 2 in the 1950s. Krylov [K] and Evans [E] showed the same a priori estimates for the above equations in general dimensions under the assumption that F is convex. As a modest investigation of a priori estimates for general fully nonlinear elliptic equations without the convexity condition, we study the homogeneous solutions.

Theorem 0.1. *Let u be a continuous in $\mathbb{R}^n \setminus \{0\}$ homogeneous degree $d \neq 2$ solution to the elliptic equation $F\left(D^2u\right) = 0$ in $\mathbb{R}^n$ with $F \in C^1$. Then u is harmonic in a possible new coordinate system in $\mathbb{R}^n$, namely*

$$\sum_{i,j=1}^{n} F_{ij}(0)\, D_{ij}u(x) = 0.$$

Consequently, $u \equiv 0$ if $-(n-2) < d < 0$ or d is not an integer; otherwise, u is a homogeneous harmonic polynomial with integer degree d.

In contrast to the variational problem, Sverák and Yan [SY] constructed homogeneous degree less than 1 minimizers to some strongly convex functional. Also Safonov [S] earlier constructed homogeneous order $\alpha \in (0,1)$ solutions to linear non-divergence elliptic equations with variable coefficients.

As one simple application to special Lagrangian equations [HL] $F\left(D^2u\right) = \sum_{i=1}^{n} \arctan \lambda_i - c = 0$, where the λ_i's are the eigenvalues of the Hessian D^2u. It follows from our theorem that any homogeneous degree other than 2 solution must be a harmonic polynomial (and it also forces $c = 0$).

When $d \in [0, 1 + \alpha(n,\mu))$, our theorem also follows from Krylov-Safonov C^α estimates (cf. [CC, Corollary 5.7]). The missing case $d = 2$ is delicate. One only knows that any homogeneous degree 2 solution to the above fully nonlinear elliptic equation in dimension 3 is quadratic [HNY, p. 426].

Received by the editors November 5, 2004.
2000 *Mathematics Subject Classification.* Primary 35J60.
Both authors were partially supported by NSF grants, and the second author was also supported by a Sloan Research Fellowship.

Now we show our theorem.

Proof. We first consider the case that u is smooth in $\mathbb{R}^n \backslash \{0\}$. Set

$$\Sigma = \left\{ |x|^{d-2} D^2 u \left(\frac{x}{|x|} \right) | x \in \mathbb{R}^n \backslash \{0\} \right\}$$

and $\Gamma = \{M | F(M) = 0\}$. For the homogeneous order d function $u(x)$, $D^2 u(x) = |x|^{d-2} D^2 u \left(\frac{x}{|x|} \right)$. Let $|x| \to 0$ for $d > 2$ or $|x| \to \infty$ for $d < 2$; we see that $0 \in \Sigma$. Also u is a solution to $F(D^2 u) = 0$. Then the cone $\Sigma \subseteq \Gamma$. Now $F \in C^1$ and $(F_{ij}(0)) > 0$. We know that the unique tangent plane of Γ at 0 includes Σ. It follows that $\Sigma \perp (F_{ij}(0))$, or

$$\sum_{i,j=1}^{n} F_{ij}(0) D_{ij} u(x) = 0.$$

Without loss of generality, we assume $(F_{ij}(0)) = I$ throughout the proof. Then

$$0 = \triangle u(x) = |x|^{d-2} \left[d(d+n-2) u \left(\frac{x}{|x|} \right) + \triangle_{S^{n-1}} u \left(\frac{x}{|x|} \right) \right].$$

The remaining conclusion of the theorem follows.

Next we show the regularity of the viscosity solution u away from 0. Set $\lambda = d(d+n-2)$ and $\theta = x/|x|$. To start, we prove that $u(\theta)$ is a viscosity solution to

$$\triangle_{S^{n-1}} u + \lambda u = 0. \tag{0.1}$$

Let any smooth $\varphi(\theta)$ touch u from above at θ_0,

$$\begin{aligned} \varphi \geq u \quad & \text{in a neighborhood of } \theta_0, \\ \varphi(\theta_0) = u(\theta_0); & \end{aligned}$$

then

$$\begin{aligned} |x|^d \varphi \left(\frac{x}{|x|} \right) \geq |x|^d u \left(\frac{x}{|x|} \right) \quad & \text{in a neighborhood of } \theta_0, \\ |x|^d \varphi(\theta_0) = |x|^d u(\theta_0). & \end{aligned}$$

From our assumption that u is a viscosity (sub)solution, it follows that

$$F \left(D^2 \left(|x|^d \varphi \left(\frac{x}{|x|} \right) \right) \right) \geq 0$$

or

$$F \left(|x|^{d-2} D_x^2 \varphi(\theta) \right) \geq 0.$$

Let $|x| \to 0$ for $d > 2$ or $|x| \to \infty$ for $d < 2$ and we see that $F(0) \geq 0$. If we use the fact u is also a viscosity (super)solution, we can derive that $F(0) \leq 0$. So $F(0) = 0$, and

$$\frac{F \left(t D_x^2 \varphi(\theta) \right) - F(0)}{t} \geq 0.$$

Let $t \to 0$; we see that

$$\sum_{i,j=1}^{n} F_{ij}(0) D_{ij} \varphi \left(\frac{x}{|x|} \right) \geq 0$$

or

$$\triangle_{S^{n-1}} \varphi + \lambda \varphi \geq 0.$$

Thus u is a viscosity subsolution to (0.1). Similarly, u is a viscosity supersolution to the same equation.

Let N_ε be an ε neighborhood of any θ_0 on S^{n-1}, with ε small enough so that N_ε is in a narrow strip. Then there exists a positive smooth function h on N_ε such that

$$\triangle_{S^{n-1}} h + \lambda h \leq 0.$$

Let ψ be the smooth solution to (0.1) in N_ε with the boundary value u on ∂N_ε. Then $q = \frac{\psi - u}{h}$ is a viscosity solution to

$$\triangle_{S^{n-1}} q + 2\frac{\nabla h}{h} \cdot \nabla q + \frac{\triangle_{S^{n-1}} h + \lambda h}{h} q = 0,$$

where $\nabla h \cdot \nabla q$ simply denotes some linear combination of first-order derivatives of q in some local coordinates for N_ε, which we avoid for the sake of simple notation. Now that the coefficient $\frac{\triangle_{S^{n-1}} h + \lambda h}{h} \leq 0$, it follows from [W, Corollary 3.20] that $q = 0$ in N_ε. Therefore, u is smooth in N_ε and then on the whole S^{n-1}. □

References

[CC] Caffarelli, L. A. and Cabré, X., *Fully nonlinear elliptic equations.* American Mathematical Society Colloquium Publications, **43**, American Mathematical Society, Providence, RI, 1995. MR1351007 (96h:35046)

[E] Evans, L. C., *Classical solutions of fully nonlinear, convex, second-order elliptic equations,* Comm. Pure Appl. Math. **35** (1982), 333–363. MR0649348 (83g:35038)

[HNY] Han, Q., Nadirashvili, N., and Yuan, Yu, *Linearity of homogeneous order-one solutions to elliptic equations in dimension three,* Comm. Pure Appl. Math. **56** (2003), 425–432. MR1949137 (2003k:35043)

[HL] Harvey, R. and Lawson, H. B. Jr., *Calibrated geometry,* Acta Math. **148** (1982), 47–157. MR0666108 (85i:53058)

[K] Krylov, N. V., *Boundedly nonhomogeneous elliptic and parabolic equations,* Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 487–523 in Russian; English translation in Math. USSR Izv. **20** (1983), 459–492. MR0661144 (84a:35091)

[N] Nirenberg, L., *On nonlinear elliptic partial differential equations and Hölder continuity,* Comm. Pure Appl. Math. **6** (1953), 103–156. MR0064986 (16:367c)

[S] Safonov, M. V., *Unimprovability of estimates of Hölder constants for solutions of linear elliptic equations with measurable coefficients,* (Russian) Mat. Sb. (N.S.) **132**(174) (1987), no. 2, 275–288; translation in Math. USSR-Sb. **60** (1988), no. 1, 269–281. MR0882838 (88e:35049)

[SY] Sverák, V. and Yan, X., *Non-Lipschitz minimizers of smooth uniformly convex functionals,* Proc. Natl. Acad. Sci. USA **99** (2002), 15269–15276. MR1946762 (2003h:49066)

[W] Wang, L., *On the regularity theory of fully nonlinear parabolic equations I,* Comm. Pure Appl. Math. **45** (1992), 27–76. MR1135923 (92m:35126)

LATP, Centre de Mathématiques et Informatique, 39, rue F. Joliot-Curie, 13453 Marseille Cedex, France

E-mail address: `nicolas@cmi.univ-mrs.fr`

Department of Mathematics, Box 354350, University of Washington, Seattle, Washington 98195

E-mail address: `yuan@math.washington.edu`