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THE BERRY-ESSEEN BOUND FOR CHARACTER RATIOS

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ABSTRACT. Let λ be a partition of n chosen from the Plancherel measure of the symmetric group S_n, let $\chi^\lambda(12)$ be the irreducible character of the symmetric group parameterized by λ evaluated on the transposition (12), and let $\dim(\lambda)$ be the dimension of the irreducible representation parameterized by λ. Fulman recently obtained the convergence rate of $O(n^{-s})$ for any $0 < s < \frac{1}{2}$ in the central limit theorem for character ratios $\frac{(n-1)}{\sqrt{2}} \frac{\chi^\lambda(12)}{\dim(\lambda)}$ by developing a connection between martingale and character ratios, and he conjectures that the correct speed is $O(n^{-1/2})$. In this paper we confirm the conjecture via a refinement of Stein's method for exchangeable pairs.

1. Introduction and main result

Let $n \geq 1$, let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ be a partition of n, i.e., $\lambda_1 + \lambda_2 + \cdots + \lambda_p = n$, and write simply $\lambda \vdash n$. Denote by $\dim(\lambda)$ the number of standard Young tableaux associated with the shape λ. By the Robinson-Schensted-Knuth correspondence [18], we have

$$\sum_{\lambda \vdash n} \dim(\lambda)^2 = n!.$$

Thus we produce the so-called Plancherel measure

$$P(\{\lambda\}) = \frac{\dim(\lambda)^2}{n!}.$$

Recently there has been intensive interest in the statistical properties of partitions chosen from the Plancherel measure. We refer the reader to the surveys by Aldous and Diaconis [1], Defit [4] and the seminal papers of Borodin, Okounkov and Olshanski [2], Johansson [14], and Okounkov and Pandharipande [16] for details.

It turns out that the Plancherel measure can also be regarded as a probability measure on the irreducible representation of the symmetric group S_n. Observe that the irreducible representation of the symmetric group S_n is parameterized by partitions λ of n and $\dim(\lambda)$ is just the corresponding dimension of the irreducible representation.

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Let $\chi^\lambda(12)$ be the irreducible character parameterized by λ evaluated on the transposition (12). The quantity $\frac{\chi^\lambda(12)}{\dim(\lambda)}$ is called a character ratio and is crucial for analyzing the convergence rate of the random walk on the symmetric group generated by transpositions in Diaconis and Shahshahani [5]. In fact, Diaconis and Shahshahani prove that the eigenvalues for this random walk are the character ratios $\frac{\chi^\lambda(12)}{\dim(\lambda)}$, each occurring with multiplicity $\dim(\lambda)^2$. Character ratios also play an essential role in work on the moduli spaces of curves; see Eskin and Okounkov [6], Okounkov and Pandharipande [16].

Kerov [15] first studies the asymptotic behavior for character ratios and outlines the proof of the following central limit theorem:

$$\frac{(n-1)}{\sqrt{2}}\frac{\chi^\lambda(12)}{\dim(\lambda)} \xrightarrow{d} N(0,1).$$

A full proof of the result appears in Ivanov and Olshanski [13]; see also Hora [12] for another proof. A more probabilistic approach to Kerov's central limit theorem has recently been given by Fulman [7], in which a Stein's method for exchangeable pairs is used to obtain for all $n \ge 2, z \in R$,

$$|P(\frac{(n-1)}{\sqrt{2}}\frac{\chi^\lambda(12)}{\dim(\lambda)} \le z) - \Phi(z)| \le 40.1n^{-1/4}$$

where $\Phi(z)$ is the standard normal distribution function.

More recently Fulman [8] developed a connection between martingales and character ratios of the symmetric group, and thereby improved the above speed of convergence to $O(n^{-s})$ for any $s < \frac{1}{2}$. He also conjectured that the correct speed is $O(n^{-1/2})$.

The main aim of this note is to confirm the following conjecture.

Theorem 1.1. *We have*

$$\sup_z |P(\frac{(n-1)}{\sqrt{2}}\frac{\chi^\lambda(12)}{\dim(\lambda)} \le z) - \Phi(z)| \le An^{-1/2} \tag{1.1}$$

where A is an absolute constant.

The proof of Theorem 1.1 will be given in Section 2. The main technique is a refinement of Stein's method for exchangeable pairs (see Theorem 2.1 below). Recall that two random variables W, W^* are called exchangeable if (W, W^*) and (W^*, W) have the same joint distribution function. In order to apply Stein's approach for exchangeable pairs, one needs to construct a W^* such that (W, W^*) is exchangeable and the difference $W - W^*$ is small. Fulman [7] uses the theory of harmonic functions on Bratelli diagrams and shows how it can be applied to generate a natural exchangeable pair (W, W^*). The basic idea is to use a reversible Markov chain on the set of partitions of size n whose stationary distribution is the Plancherel measure. Let λ^* be obtained from λ by one step in the chain, and then set $(W, W^*) = (W(\lambda), W^*(\lambda))$. This construction also has the merit of being applicable to more general groups [9] and to measures arising from symmetric functions [10].

In the setting of Theorem 1.1, we let $W = \frac{(n-1)}{\sqrt{2}}\frac{\chi^\lambda(12)}{\dim(\lambda)}$. Let parents$(\lambda, \mu)$ denote the set of partitions above both λ, μ in the Young lattice (this set has size 0 or 1 unless $\lambda = \mu$), i.e.,

$$\text{parents}(\lambda, \mu) = \#\{\tau : \lambda \nearrow \tau, \mu \nearrow \tau\}.$$

Define

$$W^*(\lambda) = W(\lambda^*)$$

where, given λ, the partition λ^* is μ with probability

$$J(\lambda, \mu) = \frac{\dim(\mu)|\text{parents}(\lambda, \mu)|}{(n+1)\dim(\lambda)}.$$

Then it follows from Proposition 2.1 of Fulman [7] that (W, W^*) is an exchangeable pair.

2. Proof

The proof is based on the following refinement of Stein's result [20] for exchangeable pairs.

Theorem 2.1. *Let (W, W^*) be an exchangeable pair of real-valued random variables such that*

$$E^W(W^*) = (1-\tau)W \tag{2.2}$$

with $0 < \tau < 1$, where $E^W(W^*)$ denotes the conditional expected value of W^* given W. Assume $E(W^2) \le 1$. Then for any $a > 0$,

$$\sup_z |P(W \le z) - \Phi(z)|$$
$$\le \sqrt{E\Big(1 - \frac{1}{2\tau}E^W(\Delta^2)\Big)^2} + \frac{0.41a^3}{\tau} + 1.5a + \frac{1}{2\tau}E\Delta^2 I_{\{|\Delta| \ge a\}}, \tag{2.3}$$

where $\Delta = W - W^*$.

If Δ is bounded, say $|\Delta| \le a_0$ for a constant a_0, then (2.3) reduces to

$$\sup_z |P(W \le z) - \Phi(z)| \le \sqrt{E\Big(1 - \frac{1}{2\tau}E^W(\Delta^2)\Big)^2} + \frac{0.41a_0^3}{\tau} + 1.5a_0.$$

Similar results for the bounded case were obtained by Rinott and Rotar [17] and Rinott and Goldstein [11].

Theorem 1.1 is an easy consequence of Theorem 2.1.

Proof of Theorem 1.1. By [7], we can choose

$$\tau = \frac{2}{n+1}, \qquad \sqrt{E\Big(1 - \frac{1}{2\tau}E^W(\Delta^2)\Big)^2} \le \frac{\sqrt{3}}{2n^{1/2}}.$$

Let $a = 4e\sqrt{2}n^{-1/2}$. Then, by the proof of Proposition 4.6 in [7],

$$\begin{aligned} E\Delta^2 I_{\{|\Delta|>a\}} &\le 8P(|\Delta| > a) \\ &\le 8P(\max(\lambda_1, \lambda_1') > 2e\sqrt{n}) \\ &\le 16e^{-2e\sqrt{n}}, \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{2\tau}E\Delta^2 I_{\{|\Delta|>a\}} &\le 4(n+1)e^{-2e\sqrt{n}} \\ &\le n^{-1/2}4(n+1)^{3/2}e^{-2e\sqrt{n}} \\ &\le 0.05n^{-1/2}. \end{aligned}$$

Therefore, by Theorem 2.1,

$$\begin{aligned}
&\sup_z |P(W \le z) - \Phi(z)| \\
&\le \frac{\sqrt{3}}{2n^{1/2}} + 0.205(n+1)(4e\sqrt{2})^3 n^{-3/2} + 4e\sqrt{2}n^{-1/2} + 0.05n^{-1/2} \\
&\le An^{-1/2},
\end{aligned}$$

where A is an absolute constant. □

We remark that if one uses

$$P(\lambda_1 \ge k) \le \binom{n}{k}/k!$$

for $1 \le k \le n$ (see Lemma 1.4.1 in [19]) and chooses $a = \delta n^{-1/2}$ with $\delta > 0$ properly, then the constant A can be reduced to 150.

Now we turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. For any measurable function f with $E\{(|W|+1)|f(W)|\} < \infty$, exchangeability and (2.2) imply

$$\begin{aligned}
0 &= E\{(W - W^*)(f(W) + f(W^*))\} \\
&= 2E\{f(W)(W - W^*)\} + E\{(W - W^*)(f(W^*) - f(W))\} \\
&= 2\tau E\{Wf(W)\} - E\{(W - W^*)(f(W) - f(W^*))\},
\end{aligned}$$

and hence

$$E\{Wf(W)\} = \frac{1}{2\tau} E\{(W - W^*)(f(W) - f(W^*))\}. \tag{2.4}$$

Now let $f = f_z$ be the solution of the following Stein equation:

$$f_z'(x) - xf_z(x) = I_{\{x \le z\}} - \Phi(z). \tag{2.5}$$

It is known (see [20, p.22]) that f is given by

$$f_z(x) = \begin{cases} \sqrt{2\pi} e^{x^2/2} \Phi(x)[1 - \Phi(z)] & \text{if } x \le z, \\ \\ \sqrt{2\pi} e^{x^2/2} \Phi(z)[1 - \Phi(x)] & \text{if } x \ge z, \end{cases}$$

satisfying

$$|xf_z(x)| \le 1, \ 0 < f_z(x) \le \sqrt{2\pi}/4, \tag{2.6}$$

$$|f_z'(x)| \le 1, \ |f_z'(x) - f_z'(y)| \le 1, \tag{2.7}$$

$$|(x+u)f_z(x+u) - xf_z(x)| \le (|x| + \sqrt{2\pi}/4)|u| \tag{2.8}$$

for all real x, y, and u. For the proofs of the above inequalities, we refer to [20, p.23] for (2.6) and the first inequality of (2.7), and to Chen and Shao [3] for the second inequality of (2.7). (2.8) is a consequence of (2.6), (2.7) and the mean value theorem.

By (2.5), we have

$$\begin{aligned} P(W \le z) - \Phi(z) &= Ef_z'(W) - EWf_z(W) \\ &= Ef_z'(W) - \frac{1}{2\tau}E\{(W-W^*)(f_z(W)-f_z(W^*))\} \\ &= E\Big\{f_z'(W)\big(1-\frac{1}{2\tau}\Delta^2\big)\Big\} \\ &\quad -\frac{1}{2\tau}E\{\Delta(f_z(W)-f_z(W-\Delta)-\Delta f_z'(W))\} \\ &:= J_1 + J_2. \end{aligned} \tag{2.9}$$

It follows from (2.6) that

$$\begin{aligned} |J_1| &= |E\Big\{f_z'(W)\big(1-\frac{1}{2\tau}E^W(\Delta^2)\big)\Big\}| \\ &\le E|1-\frac{1}{2\tau}E^W(\Delta^2)| \\ &\le \sqrt{E\Big(1-\frac{1}{2\tau}E^W(\Delta^2)\Big)^2}. \end{aligned} \tag{2.10}$$

To bound J_2, write

$$\begin{aligned} &E\{\Delta(f_z(W)-f_z(W-\Delta)-\Delta f_z'(W))\} \\ &= E\Big\{\Delta\int_{-\Delta}^0 (f_z'(W+t)-f_z'(W))dt\Big\} \\ &= E\Big\{\Delta I_{\{|\Delta|>a\}}\int_{-\Delta}^0 (f_z'(W+t)-f_z'(W))dt\Big\} \\ &\quad +E\Big\{\Delta I_{\{|\Delta|\le a\}}\int_{-\Delta}^0 (f_z'(W+t)-f_z'(W))dt\Big\} \\ &:= J_{2,1}+J_{2,2}. \end{aligned} \tag{2.11}$$

By (2.7),

$$|J_{2,1}| \le E\Delta^2 I_{\{|\Delta|>a\}}. \tag{2.12}$$

Using (2.5) again, we have

$$\begin{aligned} J_{2,2} &= E\Big\{\Delta I_{\{|\Delta|\le a\}}\int_{-\Delta}^0 ((W+t)f_z(W+t)-Wf_z(W))dt\Big\} \\ &\quad +E\Big\{\Delta I_{\{|\Delta|\le a\}}\int_{-\Delta}^0 (I_{\{W+t\le z\}}-I_{\{W\le z\}})dt\Big\} \\ &:= J_{2,2,1}+J_{2,2,2}. \end{aligned} \tag{2.13}$$

By (2.8),

$$\begin{aligned} |J_{2,2,1}| &\le E\Big\{\Delta I_{\{|\Delta|\le a\}}\int_{-\Delta}^0 (|W|+\sqrt{2\pi}/4)|t|dt\Big\} \\ &\le E\Big\{0.5|\Delta|^3 I_{\{|\Delta|\le a\}}(|W|+\sqrt{2\pi}/4)\Big\} \\ &\le 0.5a^3(\sqrt{2\pi}/4+E|W|) \\ &\le 0.5a^3(\sqrt{2\pi}/4+1) \le 0.82a^3. \end{aligned} \tag{2.14}$$

As for $J_{2,2,2}$, observe that

$$\begin{aligned} J_{2,2,2} &\le E\Big\{\Delta I_{\{0\le\Delta\le a\}}\int_{-\Delta}^{0} I_{\{z\le W\le z-t\}}dt\Big\} \\ &\le E\big(\Delta^2 I_{\{0\le\Delta\le a\}}I_{\{z\le W\le z+a\}}\big) \\ &\le 3a\tau, \qquad (2.15) \end{aligned}$$

where in the last inequality we used the concentration inequality in Lemma 2.1 below.

Similarly, we have

$$J_{2,2,2} \ge -3a\tau.$$

This proves Theorem 2.1. □

Lemma 2.1. *Under the assumption of Theorem* 2.1*, we have*

$$E\big(\Delta^2 I_{\{0\le\Delta\le a\}}I_{\{z\le W\le z+a\}}\big) \le 3a\tau \qquad (2.16)$$

for $a > 0$.

Proof. Let

$$f(x) = \begin{cases} -1.5a & \text{for } x \le z-a, \\ x-z-a/2 & \text{for } z-a \le x \le z+2a, \\ 1.5a & \text{for } x \ge z+2a. \end{cases}$$

By (2.4),

$$\begin{aligned} 3a\tau &\ge 2\tau E(Wf(W)) \\ &= E\{(W-W^*)(f(W)-f(W^*))\} \\ &= E\Big\{\Delta\int_{-\Delta}^{0} f'(W+t)dt\Big\} \\ &\ge E\Big\{\Delta\int_{-\Delta}^{0} I_{\{|t|\le a\}}I_{\{z\le W\le z+a\}}f'(W+t)dt\Big\} \\ &= E\Big(|\Delta|\min(a,|\Delta|)I_{\{z\le W\le z+a\}}\Big) \\ &\ge E\Big(\Delta^2 I_{\{0\le\Delta\le a\}}I_{\{z\le W\le z+a\}}\Big) \end{aligned}$$

as desired. □

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