THE BERRY-ESSEEN BOUND FOR CHARACTER RATIOS

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Abstract. Let $\lambda$ be a partition of $n$ chosen from the Plancherel measure of the symmetric group $S_n$, let $\chi^\lambda(12)$ be the irreducible character of the symmetric group parameterized by $\lambda$ evaluated on the transposition $(12)$, and let $\dim(\lambda)$ be the dimension of the irreducible representation parameterized by $\lambda$. Fulman recently obtained the convergence rate of $O(n^{-s})$ for any $0 < s < \frac{1}{2}$ in the central limit theorem for character ratios $(n-1)^{1/2} \frac{\chi^\lambda(12)}{\dim(\lambda)}$ by developing a connection between martingale and character ratios, and he conjectures that the correct speed is $O(n^{-1/2})$. In this paper we confirm the conjecture via a refinement of Stein’s method for exchangeable pairs.

1. INTRODUCTION AND MAIN RESULT

Let $n \geq 1$, let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ be a partition of $n$, i.e., $\lambda_1 + \lambda_2 + \cdots + \lambda_p = n$, and write simply $\lambda \vdash n$. Denote by dim($\lambda$) the number of standard Young tableaux associated with the shape $\lambda$. By the Robinson-Schensted-Knuth correspondence [18], we have

$$\sum_{\lambda \vdash n} \dim(\lambda)^2 = n!.$$ 

Thus we produce the so-called Plancherel measure

$$P(\{\lambda\}) = \frac{\dim(\lambda)^2}{n!}.$$ 

Recently there has been intensive interest in the statistical properties of partitions chosen from the Plancherel measure. We refer the reader to the surveys by Aldous and Diaconis [1], Defit [4] and the seminal papers of Borodin, Okounkov and Olshanski [2], Johansson [14], and Okounkov and Pandharipande [16] for details.

It turns out that the Plancherel measure can also be regarded as a probability measure on the irreducible representation of the symmetric group $S_n$. Observe that the irreducible representation of the symmetric group $S_n$ is parameterized by partitions $\lambda$ of $n$ and $\dim(\lambda)$ is just the corresponding dimension of the irreducible representation.
Let $\chi_{\lambda}(12)$ be the irreducible character parameterized by $\lambda$ evaluated on the transposition $(12)$. The quantity $\frac{\chi_{\lambda}(12)}{\dim(\lambda)}$ is called a character ratio and is crucial for analyzing the convergence rate of the random walk on the symmetric group generated by transpositions in Diaconis and Shahshahani [5]. In fact, Diaconis and Shahshahani prove that the eigenvalues for this random walk are the character ratios $\frac{\chi_{\lambda}(12)}{\dim(\lambda)}$, each occurring with multiplicity $\dim(\lambda)^2$. Character ratios also play an essential role in work on the moduli spaces of curves; see Eskin and Okounkov [6], Okounkov and Pandharipande [16].

Kerov [15] first studies the asymptotic behavior for character ratios and outlines the proof of the following central limit theorem:

$$\frac{(n-1)}{\sqrt{2}} \frac{\chi_{\lambda}(12)}{\dim(\lambda)} \xrightarrow{d} N(0,1).$$

A full proof of the result appears in Ivanov and Olshanski [13]; see also Hora [12] for another proof. A more probabilistic approach to Kerov’s central limit theorem has recently been given by Fulman [7], in which a Stein’s method for exchangeable pairs is used to obtain for all $n \geq 2, z \in \mathbb{R}$,

$$|P\left(\frac{(n-1)}{\sqrt{2}} \frac{\chi_{\lambda}(12)}{\dim(\lambda)} \leq z\right) - \Phi(z)| \leq 40.1 n^{-1/4}$$

where $\Phi(z)$ is the standard normal distribution function.

More recently Fulman [8] developed a connection between martingales and character ratios of the symmetric group, and thereby improved the above speed of convergence to $O(n^{-s})$ for any $s < \frac{1}{2}$. He also conjectured that the correct speed is $O(n^{-1/2}).$

The main aim of this note is to confirm the following conjecture.

**Theorem 1.1.** We have

$$\sup_{z} |P\left(\frac{(n-1)}{\sqrt{2}} \frac{\chi_{\lambda}(12)}{\dim(\lambda)} \leq z\right) - \Phi(z)| \leq A n^{-1/2}$$

where $A$ is an absolute constant.

The proof of Theorem 1.1 will be given in Section 2. The main technique is a refinement of Stein’s method for exchangeable pairs (see Theorem 2.1 below). Recall that two random variables $W, W^*$ are called exchangeable if $(W, W^*)$ and $(W^*, W)$ have the same joint distribution function. In order to apply Stein’s approach for exchangeable pairs, one needs to construct a $W^*$ such that $(W, W^*)$ is exchangeable and the difference $W - W^*$ is small. Fulman [7] uses the theory of harmonic functions on Bratelli diagrams and shows how it can be applied to generate a natural exchangeable pair $(W, W^*)$. The basic idea is to use a reversible Markov chain on the set of partitions of size $n$ whose stationary distribution is the Plancherel measure. Let $\lambda^*$ be obtained from $\lambda$ by one step in the chain, and then set $(W, W^*) = (W(\lambda), W^*(\lambda))$. This construction also has the merit of being applicable to more general groups [9] and to measures arising from symmetric functions [10].

In the setting of Theorem 1.1 we let $W = \frac{(n-1)}{\sqrt{2}} \frac{\chi_{\lambda}(12)}{\dim(\lambda)}$. Let parents($\lambda, \mu$) denote the set of partitions above both $\lambda, \mu$ in the Young lattice (this set has size 0 or 1.}
unless $\lambda = \mu$), i.e.,
\[
\text{parents}(\lambda, \mu) = \# \{ \tau : \lambda \not\sim \tau, \mu \not\sim \tau \}.
\]
Define
\[
W^*(\lambda) = W(\lambda^*)
\]
where, given $\lambda$, the partition $\lambda^*$ is $\mu$ with probability
\[
J(\lambda, \mu) = \frac{\dim(\mu) |\text{parents}(\lambda, \mu)|}{(n + 1) \dim(\lambda)}.
\]
Then it follows from Proposition 2.1 of Fulman [7] that $(W, W^*)$ is an exchangeable pair.

2. Proof

The proof is based on the following refinement of Stein’s result [20] for exchangeable pairs.

**Theorem 2.1.** Let $(W, W^*)$ be an exchangeable pair of real-valued random variables such that
\[
E^W(W^*) = (1 - \tau)W
\]
with $0 < \tau < 1$, where $E^W(W^*)$ denotes the conditional expected value of $W^*$ given $W$. Assume $E(W^2) \leq 1$. Then for any $a > 0$,
\[
\sup_z |P(W \leq z) - \Phi(z)| \leq \sqrt{\frac{2}{\pi}} a + 0.41 \tau^3 + 1.5a + \frac{1}{2\tau} E\Delta^2 I_{\{\Delta > a\}}
\]
where $\Delta = W - W^*$.

If $\Delta$ is bounded, say $|\Delta| \leq a_0$ for a constant $a_0$, then (2.3) reduces to
\[
\sup_z |P(W \leq z) - \Phi(z)| \leq \sqrt{\frac{2}{\pi}} a + 0.41a_0^3 + 1.5a_0.
\]
Similar results for the bounded case were obtained by Rinott and Rotar [17] and Rinott and Goldstein [11].

Theorem 1.1 is an easy consequence of Theorem 2.1.

**Proof of Theorem 1.1.** By [7], we can choose
\[
\tau = \frac{2}{n + 1}, \quad \sqrt{\frac{2}{\pi}} a + 0.41 \tau^3 + 1.5a + \frac{1}{2\tau} E\Delta^2 I_{\{\Delta > a\}} \leq \sqrt{\frac{3}{2n^{1/2}}}
\]
Let $a = 4e \sqrt{2n^{-1/2}}$. Then, by the proof of Proposition 4.6 in [7],
\[
E\Delta^2 I_{\{\Delta > a\}} \leq 8P(\Delta > a) \leq 8P(\max(\lambda_1, \lambda_1') > 2e \sqrt{n}) \leq 16e^{-2e \sqrt{n}},
\]
and hence
\[
\frac{1}{2\tau} E\Delta^2 I_{\{\Delta > a\}} \leq 4(n + 1)e^{-2e \sqrt{n}} \leq n^{-1/2}4(n + 1)^{3/2}e^{-2e \sqrt{n}} \leq 0.05n^{-1/2}.
\]
Therefore, by Theorem 2.1,
\[ \sup_z |P(W \leq z) - \Phi(z)| \]
\[ \leq \frac{\sqrt{3}}{2n^{1/2}} + 0.205(n + 1)(4\varepsilon \sqrt{2})^3n^{-3/2} + 4\varepsilon \sqrt{2}n^{-1/2} + 0.05n^{-1/2} \]
\[ \leq An^{-1/2}, \]
where \( A \) is an absolute constant. \( \square \)

We remark that if one uses
\[ P(\lambda_1 \geq k) \leq \binom{n}{k} / k! \]
for \( 1 \leq k \leq n \) (see Lemma 1.4.1 in [19]) and chooses \( a = \delta n^{-1/2} \) with \( \delta > 0 \) properly, then the constant \( A \) can be reduced to 150.

Now we turn to the proof of Theorem 2.1.

**Proof of Theorem 2.1.** For any measurable function \( f \) with \( E\{|W| + 1\}|f(W)| < \infty \), exchangeability and (2.2) imply
\[ 0 = E\{(W - W^*)(f(W) + f(W^*))\} \]
\[ = 2E\{f(W)(W - W^*)\} + E\{(W - W^*)(f(W^*) - f(W))\} \]
\[ = 2\tau E\{Wf(W)\} - E\{(W - W^*)(f(W) - f(W^*))\}, \]
and hence
\[ E\{Wf(W)\} = \frac{1}{2\tau} E\{(W - W^*)(f(W) - f(W^*))\}. \]

Now let \( f = f_z \) be the solution of the following Stein equation:
\[ f'_z(x) - xf_z(x) = I_{(x \leq z)} - \Phi(z). \]

It is known (see [20, p.22]) that \( f \) is given by
\[ f_z(x) = \begin{cases} \sqrt{2\pi e^{-x^2/2}}\Phi(x)[1 - \Phi(z)] & \text{if } x \leq z, \\ \sqrt{2\pi e^{-x^2/2}}\Phi(z)[1 - \Phi(x)] & \text{if } x \geq z, \end{cases} \]
satisfying
\[ |xf_z(x)| \leq 1, \quad 0 < f_z(x) \leq \sqrt{2\pi}/4, \]
\[ |f'_z(x)| \leq 1, \quad |f'_z(x) - f'_z(y)| \leq 1, \]
\[ |(x + u)f_z(x + u) - xf_z(x)| \leq ((|x| + \sqrt{2\pi}/4)|u| \]
for all real \( x, y, \) and \( u \). For the proofs of the above inequalities, we refer to [20, p.23] for (2.6) and the first inequality of (2.7), and to Chen and Shao [3] for the second inequality of (2.7). (2.8) is a consequence of (2.6), (2.7) and the mean value theorem.
By (2.5), we have
\[
P(W \leq z) - \Phi(z) = E f'_z(W) - E W f_z(W)
\]
\[
= E f'_z(W) - \frac{1}{2\tau} E \{(W - W^*)(f_z(W) - f_z(W^*))\}
\]
\[
= E \left\{ f'_z(W) \left( 1 - \frac{1}{2\tau} \Delta^2 \right) \right\}
\]
\[
- \frac{1}{2\tau} E \{ \Delta(f_z(W) - f_z(W - \Delta) - \Delta f'_z(W)) \}
\]
(2.9)
:= J_1 + J_2.

It follows from (2.6) that
\[
|J_1| = |E \left\{ f'_z(W) \left( 1 - \frac{1}{2\tau} E W(\Delta^2) \right) \right\}|
\]
\[
\leq E \left\{ 1 - \frac{1}{2\tau} E W(\Delta^2) \right\}
\]
(2.10)
\[
\leq \sqrt{E \left\{ 1 - \frac{1}{2\tau} E W(\Delta^2) \right\}^2}.
\]

To bound \( J_2 \), write
\[
E \{ \Delta(f_z(W) - f_z(W - \Delta) - \Delta f'_z(W)) \}
\]
\[
= E \left\{ \Delta \int_{-\Delta}^0 (f'_z(W + t) - f'_z(W))dt \right\}
\]
\[
= E \left\{ \Delta I_{|\Delta|>a} \int_{-\Delta}^0 (f'_z(W + t) - f'_z(W))dt \right\}
\]
\[
+ E \left\{ \Delta I_{|\Delta|\leq a} \int_{-\Delta}^0 (f'_z(W + t) - f'_z(W))dt \right\}
\]
(2.11)
:= J_{2,1} + J_{2,2}.

By (2.7),
\[
|J_{2,1}| \leq E \Delta^2 I_{|\Delta|>a}.
\]

Using (2.5) again, we have
\[
J_{2,2} = E \left\{ \Delta I_{|\Delta|\leq a} \int_{-\Delta}^0 ((W + t)f_z(W + t) - W f_z(W))dt \right\}
\]
\[
+ E \left\{ \Delta I_{|\Delta|\leq a} \int_{-\Delta}^0 (I_{W+t\leq z} - I_{W\leq z})dt \right\}
\]
(2.13)
:= J_{2,2,1} + J_{2,2,2}.

By (2.8),
\[
|J_{2,2,1}| \leq E \left\{ \Delta I_{|\Delta|\leq a} \int_{-\Delta}^0 (|W| + \sqrt{2\pi}/4)|t|dt \right\}
\]
\[
\leq E \left\{ 0.5|\Delta|^3 I_{|\Delta|\leq a} (|W| + \sqrt{2\pi}/4) \right\}
\]
\[
\leq 0.5a^3(\sqrt{2\pi}/4 + E|W|)
\]
(2.14)
\[
\leq 0.5a^3(\sqrt{2\pi}/4 + 1) \leq 0.82a^3.
\]
As for $J_{2,2,2}$, observe that

\[
J_{2,2,2} \leq E\left\{ \Delta I_{\{0 \leq \Delta \leq a\}} \int_{-\Delta}^{0} I_{\{z \leq W \leq z-t\}} dt \right\} \\
\leq E(\Delta^2 I_{\{0 \leq \Delta \leq a\}} I_{\{z \leq W \leq z+a\}}) \\
\leq 3a\tau.
\]

(2.15)

where in the last inequality we used the concentration inequality in Lemma 2.1 below.

Similarly, we have

\[
J_{2,2,2} \geq -3a\tau.
\]

This proves Theorem 2.1 \qed

**Lemma 2.1.** Under the assumption of Theorem 2.1, we have

\[
E(\Delta^2 I_{\{0 \leq \Delta \leq a\}} I_{\{z \leq W \leq z+a\}}) \leq 3a\tau
\]

for $a > 0$.

**Proof.** Let

\[
f(x) = \begin{cases} 
-1.5a & \text{for } x \leq z - a, \\
 x - z - a/2 & \text{for } z - a \leq x \leq z + 2a, \\
1.5a & \text{for } x \geq z + 2a.
\end{cases}
\]

By (2.4),

\[
3a\tau \geq 2\tau E(W f(W)) \\
= E\{(W - W^*)(f(W) - f(W^*))\} \\
= E\left\{ \Delta \int_{-\Delta}^{0} f'(W + t) dt \right\} \\
\geq E\left\{ \Delta \int_{-\Delta}^{0} I_{\{|t| \leq a\}} I_{\{z \leq W \leq z+a\}} f'(W + t) dt \right\} \\
= E\left|\Delta\right| \min(\left|\Delta\right|, I_{\{z \leq W \leq z+a\}}) \\
\geq E\left\{ \Delta^2 I_{\{0 \leq \Delta \leq a\}} I_{\{z \leq W \leq z+a\}} \right\}
\]

as desired. \qed

**References**


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