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ON INVARIANT DISTANCES ON ASYMPTOTIC TEICHMÜLLER SPACES

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ABSTRACT. In this paper, we will establish that any invariant distance on asymptotic Teichmüller space is a complete distance.

1. INTRODUCTION

By an *invariant distance* on a complex Banach manifold X, we mean a pseudodistance which satisfies the distance decreasing property for every holomorphic functions between X and the unit disc $\mathbb{D}$ in $\mathbb{C}$ (cf. §3.1). Invariant distances are powerful tools for studying the analytic structures of complex Banach manifolds (cf. [3] and [9]) and are also important objects of research in Teichmüller theory (see [6] and [10]). The Carathéodory distance and the Kobayashi distance are typical examples of invariant distances.

In [5], C. Earle, F. Gardiner and N. Lakic showed that every asymptotic Teichmüller space admits a structure of a complex Banach manifold. The aim of this paper is to show that any invariant distance is complete on asymptotic Teichmüller space. Namely, we will show

Theorem 1. *For every Riemann surface R, any invariant distance on $AT(R)$ is a complete distance.*

Recently, C. Earle, V. Markovic and D. Saric obtained that $AT(R)$ is embedded in a Banach space as a bounded domain (cf. §2.4). Therefore, the topology inherited from the Carathéodory distance $c_{AT(R)}$ coincides with that from the Banach-manifold structure (cf. Theorem IV.2.2 of [9]). In addition, we will obtain

Theorem 2. *There exist universal constants D_1 and D_2 with the following property: Let d be an invariant distance on $AT(R)$. Then*

$$c_{AT(R)}(p_1, p_2) \leq d(p_1, p_2) \leq D_1 c_{AT(R)}(p_1, p_2)$$

for all $p_1, p_2 \in AT(R)$ with $c_{AT(R)}(p_1, p_2) \leq D_2$.

Thus, the topology inherited from any invariant distance is *locally* biLipschitz equivalent to that from $c_{AT(R)}$.

In the case when given invariant distance is inner, the distance is *globally* biLipschitz equivalent to the Kobayashi distance: Recall that an *inner* distance is a pseudodistance defined from the lengths of rectifiable paths connecting two points

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(cf. §3.2; see also p. 128 of [9]). Theorem 2 asserts that a rectifiable path for the Kobayashi distance is also a rectifiable path for any invariant distance and vice versa, and these lengths are comparable (cf. Proposition 3.1). Since the Kobayashi distance is inner (cf. Corollary V.4.2 of [9]), we conclude

Corollary 1. *Any inner invariant distance on $AT(R)$ is biLipschitz equivalent to the Kobayashi distance on $AT(R)$ with biLipschitz constants universal.*

As an application of our theorems, we obtain information of the complex analytic structures of asymptotic Teichmüller spaces: A domain X in a Banach space B is said to be a *domain of holomorphy* (resp. an H^∞*-domain of holomorphy*) if there is no domain $X' \subset B$ containing X such that any holomorphic function (resp. bounded holomorphic function) on X is extended to X'. It is known that any bounded domain in a Banach space whose Carathéodory distance is complete is an H^∞*-domain of holomorphy*, and hence it is a domain of holomorphy (cf. Proposition 5.5 of [3]). Combining our Theorem 1 and Earle-Markovic-Saric's embedding theorem, we deduce

Corollary 2. *For every Riemann surface R, $AT(R)$ is biholomorphically equivalent to an H^∞-domain of holomorphy.*

Remark 1. The method in this paper also works for the Teichmüller space $T(R)$ and the Teichmüller space $T_0(R)$ of asymptotically conformal mappings for every Riemann surface R (cf. [4] and [5]).

Convention. Since $AT(R)$ consists of one point when R is analytically finite, *we concentrate only on hyperbolic Riemann surfaces throughout this paper.*

2. Notation

2.1. **Quasiconformal isotopies.** Let R be a Riemann surface. Let Γ be the Fuchsian group acting on $\mathbb{D}$ with $\mathbb{D}/\Gamma = R$ and denote by $\Lambda(\Gamma)$ the limit set of Γ. Then $\overline{R} = (\overline{\mathbb{D}} - \Lambda(\Gamma))/\Gamma$ is an orbifold with interior R and boundary $(\partial\mathbb{D} - \Lambda(\Gamma))/\Gamma$. We say that $(\partial\mathbb{D} - \Lambda(\Gamma))/\Gamma$ is the *ideal boundary* of R, and denote it by $\partial^{id}R$.

Let R and S be Riemann surfaces. Two quasiconformal mappings f and g from R to S are said to be *quasiconformally isotopic rel* $\partial^{id}R$ if there exist a constant $K \geq 1$ and a continuous family $\{g_t\}_{t\in[0,1]}$ of K-quasiconformal mappings such that $g_0 = f$, $g_1 = g$ and $g_t(p) = f(p) = g(p)$ for $(p, t) \in \partial^{id}R \times [0, 1]$.

2.2. **Differentials vanishing at infinity.** Let R be a Riemann surface and let Γ be a Kleinian group acting on a domain $D \subset \widehat{\mathbb{C}}$ with $D/\Gamma = R$. Let $L^\infty(D, \Gamma)$ denote the Banach space of L^∞-functions μ on D satisfying the compatibility condition $\mu \circ \gamma \cdot \left(\overline{\gamma'}/\gamma'\right) = \mu$ for all $\gamma \in \Gamma$. This compatibility condition allows us to identify the space $L^\infty(D, \Gamma)$ with the space $L^\infty(R)$ of bounded measurable $(-1, 1)$-forms on R. In particular, the absolute value $|\mu|$ is recognized as a measurable function on R. We denote by $\|\cdot\|_\infty$ the essential supremum norm on $L^\infty(R)$.

We say that $\mu \in L^\infty(R)$ ($\cong L^\infty(D, \Gamma)$) *vanishes at infinity* on R when for any $\epsilon > 0$, there is a compact set $C \subset R$ such that $|\mu| < \epsilon$ a.e. on $R - C$. We denote by $L_0^\infty(R)$ (resp. $L_0^\infty(D, \Gamma)$) the closed subspace of $L^\infty(R)$ (resp. $L^\infty(D, \Gamma)$) consisting of all $\mu \in L^\infty(R)$ vanishing at infinity. A quasiconformal mapping f on R is said to be *asymptotically conformal* if its complex dilatation vanishes at infinity. We set

$\widehat{L}(R) := L^\infty(R)/L^\infty_0(R)$. Denote by $[\mu]_{\hat{L}}$ the equivalence class of $\mu \in L^\infty(R)$ in $\widehat{L}(R)$ and by $\|\cdot\|_{\hat{L}}$ the quotient norm.

Let $B_2(D,\Gamma)$ be a complex Banach space of holomorphic functions ϕ on D with $\sup\{\lambda_D(z)^{-2}|\phi(z)|, z \in D\} < \infty$ and satisfy the compatibility condition $\phi \circ \gamma \cdot (\gamma')^2 = \phi$ for all $\gamma \in \Gamma$, where λ_D is the hyperbolic metric on D. We say that $\phi \in B_2(D,\Gamma)$ *vanishes at infinity* if the associating $(-1,1)$-form $\lambda_D^{-2}\overline{\phi}$ vanishes at infinity on R, and denote by $B_2^0(D,\Gamma)$ the Banach space consisting of all $\phi \in B_2(D,\Gamma)$ vanishing at infinity. Set $\widehat{B}(\Gamma) := B_2(\mathbb{D}^-,\Gamma)/B_2^0(\mathbb{D}^-,\Gamma)$.

2.3. **Asymptotic Teichmüller spaces.** Let R be a Riemann surface. The *asymptotic Teichmüller space* $AT(R)$ of R is, by definition, the space of the equivalence classes of quasiconformal mappings f from R onto a variable Riemann surface $f(R)$. Two mappings f from R to R_0 and g from R to R_1 are *equivalent* if there is an asymptotically conformal mapping $h : R_0 \to R_1$ such that $h \circ f$ and g are quasiconformally isotopic rel $\partial^{id}R$. We denote by $[f]_{AT}$ the equivalence class of f in $AT(R)$. It is known that $AT(R)$ admits the natural structure of a complex Banach manifold (cf. [5]). The *Teichmüller space* $T(R)$ of R has the same definition with one exception. The mapping h has to be conformal. Since conformal mappings are asymptotically conformal, there is a canonical projection $T(R) \to AT(R)$.

2.4. **Bers embeddings and asymptotic Bers maps.** Let R be a Riemann surface and let Γ be the Fuchsian group acting on $\mathbb{D}$ uniformizing R. Let $M(R)$ be the open unit ball in $L^\infty(R)$ $(\cong L^\infty(\mathbb{D},\Gamma))$. Then there is a canonical projection $\Phi : M(R) \to T(R)$. Namely, $\Phi(\mu)$ is the equivalence class (in $T(R)$) of a quasiconformal mapping on R whose complex dilatation is μ. This projection is called the *Bers projection*.

Let $\mu \in M(\mathbb{D},\Gamma)$ and let $w^\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal mapping fixing 1, i and -1 whose dilatation coincides with either μ on $\mathbb{D}$, or 0 on $\mathbb{D}^- = \widehat{\mathbb{C}} - \overline{\mathbb{D}}$. Let $\mathcal{S}(\mu)$ denote the Schwarzian derivative of a conformal mapping $w^\mu|_{\mathbb{D}^-} : \mathbb{D}^- \to \widehat{\mathbb{C}}$. Then it follows from the compatibility of μ that $\mathcal{S}(\mu) \in B_2(\mathbb{D}^-,\Gamma)$. L. Bers showed that there is an embedding $\mathcal{B}_\Gamma : T(R) \to B_2(\mathbb{D}^-,\Gamma)$ satisfying $\mathcal{B}_\Gamma(\Phi(\mu)) = \mathcal{S}(\mu)$ for all $\mu \in M(R)$. This embedding is called the *Bers embedding* of $T(R)$.

Let $\widehat{M}(R)$ be the unit ball in $\widehat{L}(R)$. In [5], C. Earle, F. Gardiner and N. Lakic proved the existence of a holomorphic splitting submersion $\widehat{\Phi}_R : \widehat{M}(R) \to AT(R)$ and a holomorphic mapping $\widehat{\mathcal{B}}_\Gamma : AT(R) \to \widehat{B}(\Gamma)$ with the following diagram commutative:

$$\begin{array}{ccccc} M(R) & \xrightarrow{\Phi} & T(R) & \xrightarrow{\mathcal{B}_\Gamma} & B_2(\mathbb{D}^-,\Gamma) \\ {\scriptstyle P_R}\big\downarrow & & \big\downarrow & & \big\downarrow \\ \widehat{M}(R) & \xrightarrow{\widehat{\Phi}_R} & AT(R) & \xrightarrow{\widehat{\mathcal{B}}_\Gamma} & \widehat{B}(\Gamma), \end{array}$$

where the vertical directions are canonical projections. It follows from the implicit function theorem that $\widehat{\Phi}_R$ admits a local holomorphic section at any point of $AT(R)$ (cf. Theorem 2.11 of [5] and p. 89 of [13]).

In [8], C. Earle, V. Markovic and D. Saric obtained that the mapping $\widehat{\mathcal{B}}_\Gamma$ is actually injective and its image is a bounded domain in $\widehat{B}(\Gamma)$. The mapping $\widehat{\mathcal{B}}_\Gamma$ is called the *asymptotic Bers map* from $AT(R)$ to $\widehat{B}(\Gamma)$.

Lemma 2.1. *The image $\widehat{\mathcal{B}}_\Gamma(AT(R))$ contains the $1/2$-ball with center 0, and is contained in the $3/2$-ball with center 0 in $\widehat{B}(\Gamma)$.*

This is derived from Lemma 2.2 below and the fact that the image of the Teichmüller space under the Bers' embedding contains the 1/2-ball with center 0, and is contained in the 3/2-ball with center 0 (cf. [1], [12], [13]).

Lemma 2.2. *Let $B = (B, \|\cdot\|_B)$ be a normed space and let B_0 be a subspace of B. Denote by $\pi : B \to B/B_0$ the quotient map. Suppose that a subset D in B contains the r_1-ball with center 0 and is contained in the r_2-ball with center 0. Then $\pi(D)$ satisfies the same conditions with respect to the quotient norm $\|\cdot\|_{B/B_0}$ in B/B_0.*

Proof. Since the quotient map is a norm-decreasing map, $\pi(D)$ is contained in the r_2-ball with center 0 in B/B_0. Let $[b] \in B/B_0$ with $\|[b]\|_{B/B_0} < r_1$. From the definition of the quotient norm, there is a representative $b \in [b]$ with $\|b\|_B < r_1$, which implies that $[x] \in \pi(D)$. □

3. Invariant pseudodistances

In this section, we give the definition and basic properties of invariant distances on complex Banach manifolds.

3.1. **Invariant distances.** An *invariant distance* d_X on a complex Banach manifold X is a pseudodistance which satisfies the following inequalities:

$$d_{\mathbb{D}}(f(x_1), f(x_2)) \le d_X(x_1, x_2) \quad \text{and} \quad d_X(g(z_1), g(z_2)) \le d_{\mathbb{D}}(z_1, z_2)$$

for every $x_1, x_2 \in X$, $z_1, z_2 \in \mathbb{D}$, and holomorphic functions $f : X \to \mathbb{D}$ and $g : \mathbb{D} \to X$, where $d_{\mathbb{D}}$ is the Poincaré distance on $\mathbb{D}$.

3.2. **Inner distances.** Let (X, d) be a metric space. A path $\gamma : [a, b] \to X$ is said to be *rectifiable* with respect to d if d-length

$$\mathrm{Length}_d(\gamma) := \sup\left\{\sum_{k=0}^{m} d(\gamma(t_k), \gamma(t_{k+1})) \mid a = t_0 < t_1 < t_2 < \cdots < t_m = b\right\}$$

of γ is finite. We say that d is an *inner distance* if

$$d(x_1, x_2) = \inf_{\gamma} \mathrm{Length}_d(\gamma)$$

holds for every $x_1, x_2 \in X$, where the infimum is taken over all rectifiable paths connecting x_1 and x_2 with respect to d. The following proposition follows from the definition.

Proposition 3.1. *Let d_1 and d_2 be distances on X. Suppose that there exist constants D_1 and D_2 such that*

$$\frac{1}{D_1} d_1(x_1, x_2) \le d_2(x_1, x_2) \le D_1 d_1(x_1, x_2)$$

for all $x_1, x_2 \in X$ with $d_1(x_1, x_2) \le D_2$. Then, for any rectifiable path γ with respect to d_1, an inequality

$$\frac{1}{D_1}\mathrm{Length}_{d_1}(\gamma) \le \mathrm{Length}_{d_2}(\gamma) \le D_1 \mathrm{Length}_{d_1}(\gamma)$$

holds. In particular, if both d_1 and d_2 are inner, then

$$\frac{1}{D_1} d_1(x_1, x_2) \le d_2(x_1, x_2) \le D_1 d_1(x_1, x_2)$$

for all $x_1, x_2 \in X$.

3.3. **Carathéodory and Kobayashi distances.** Let X be a complex Banach manifold. The *Carathéodory pseudodistance* c_X on X is a pseudodistance defined by

$$c_X(x_1, x_2) = \sup\{d_{\mathbb{D}}(f(x_1), f(x_2)) \mid f : X \to \mathbb{D} \text{ holomorphic}\}$$

for $x_1, x_2 \in X$. The *Kobayashi pseudodistance* k_X on X is also a pseudodistance on X which is defined to be

$$k_X(x_1, x_2) = \inf \sum_{k=1}^{n} d_{\mathbb{D}}(a_k, b_k),$$

where the infimum is taken over $n \in \mathbb{N}$ and all families of pairs of points $\{(a_k, b_k)\}_{k=1}^n$ in $\mathbb{D}$ and holomorphic functions $\{f_k\}_{k=1}^n$ with $f_1(a_1) = x_1$, $f_n(b_n) = x_2$, and $f_k(b_k) = f_{k+1}(a_{k+1})$ for $k = 1, \cdots, n-1$. It is known that the kobayashi distance is inner (cf. [9]).

The following properties are well known (cf. [3]).

Lemma 3.1. *Let X be a complex Banach manifold. Then:*

(a) *if X admits a holomorphic embedding into a Banach space whose image is bounded, c_X is a distance,*
(b) *when $X = \mathbb{D}$, both $c_{\mathbb{D}}$ and $k_{\mathbb{D}}$ coincide with the Poincaré distance on $\mathbb{D}$,*
(c) *let d_X be an invariant distance on X. Then*

$$c_X(x_1, x_2) \le d_X(x_1, x_2) \le k_X(x_1, x_2)$$

for every $x_1, x_2 \in X$,
(d) *c_X and k_X have the distance-decreasing property, that is,*

$$c_{X_2}(F(x_1), F(x_2)) \le c_{X_1}(x_1, x_2) \quad \textit{and}$$
$$k_{X_2}(F(x_1), F(x_2)) \le k_{X_1}(x_1, x_2)$$

for $x_1, x_2 \in X$ and a holomorphic function $F : X_1 \to X_2$ between Banach manifolds. Especially, c_X and k_X are invariant distances, and
(e) *if X is the ball with radius r and center the origin in a Banach space $(B, \|\cdot\|_B)$, then any invariant distance on X coincides with the Carathéodory distance and*

$$\|b\|_B = r \tanh(c_X(0, b)/2) = r \tanh(k_X(0, b)/2)$$

for $b \in X$.

3.4. **A lemma.** This section gives a criterion for the completeness of invariant distances on complex Banach manifolds.

Lemma 3.2. *Let X be a complex Banach manifold. Suppose that there exist $\delta_1, \delta_2 > 0$ and a family of Banach spaces $\{B_x\}_{x \in X}$ such that, for any point $x \in X$, X admits a holomorphic embedding $F_x : X \to B_x$ satisfying that $F_x(x) = 0$ and*

$$\{b \in B_x \mid \|b\|_{B_x} < \delta_1\} \subset F_x(X) \subset \{b \in B_x \mid \|b\|_{B_x} < \delta_2\}, \tag{3.1}$$

where $\|\cdot\|_{B_x}$ is the norm on B_x. Then,

(1) *c_X and k_X are locally biLipschitz equivalent with uniform constants, that is, there exist constants $D_1 \geq 1$ and $D_2 > 0$ depending only on δ_1 and δ_2 such that*
$$c_X(x,y) \leq k_X(x,y) \leq D_1 c_X(x,y)$$
for $x, y \in X$ with $c_X(x,y) \leq D_2$, and

(2) *any invariant distance on X is a complete distance.*

Proof. (1) For $x \in X$, we set $U_x = F_x^{-1}(\{\|b\|_{B_x} < \delta_1/2\})$. Then we can see that c_X and k_X are locally comparable with a uniform constant. Indeed, since F_x is a holomorphically embedding with $F_x(x) = 0$, by (e) in Lemma 3.1,

$$\tanh^{-1}(\|F_x(y)\|_{B_x}/\delta_2) \leq c_X(x,y) \leq k_X(x,y) \leq \tanh^{-1}(\|F_x(y)\|_{B_x}/\delta_1) \tag{3.2}$$

for $y \in U_x$. Since the left-hand side of (3.2) is comparable with the right-hand side on U_x, and U_x contains the D_2-ball with respect to c_X for some $D_2 = D_2(\delta_1, \delta_2) > 0$ (by (3.2) again), we conclude the assertion.

(2) We only show the case of the Carathéodory distance. The other case is derived from this case because of (b) in Lemma 3.1 and the local comparability in (1).

Let $\{x_i\}_{i=1}^{\infty}$ be a Cauchy sequence with respect to (X, c_X). We may assume that $c_X(x_i, x_j) \leq 2\tanh^{-1}(\delta_1/3\delta_2)$ for all $i, j \geq 1$. For the sake of simplicity, we set $F = F_{x_1}$ and $B = B_{x_1}$. Since $F(X) \subset \{b \in B \mid \|b\|_B < \delta_2\}$ and $F(x_1) = 0$, by (d) and (e) in Lemma 3.1, we have
$$\|F(x_i)\|_B \leq \delta_2 \tanh(c_X(x_1, x_j)/2) \leq \delta_1/3.$$
This means that $F(X)$ is contained in the ball $\{b \in B \mid \|b - F(x_i)\|_B \leq \delta_2 + \delta_1/3\}$ for all $i \geq 1$. Therefore, by (e) in Lemma 3.1 again, we get
$$\|F(x_i) - F(x_j)\|_B \leq (\delta_2 + \delta_1/3)\tanh(c_X(x_i, x_j)/2).$$
This implies that $\{F(x_i)\}_{i=1}^{\infty}$ is a Cauchy sequence with respect to $(B, \|\cdot\|_B)$. Thus, $F(x_i)$ converges to some $b_\infty \in B$ with $\|b_\infty\|_B \leq \delta_1/3$. Therefore, $b_\infty \in F(X)$, and hence, x_i converges to $F^{-1}(b_\infty) \in X$. □

4. Proof of the theorems

We begin the proof of our theorems by stating the following proposition.

Proposition 4.1 (Allowable bijections)**.** *Let R and S be Riemann surfaces and let f be a quasiconformal mapping from R to S. Then the map $[f]^* : AT(S) \to AT(R)$ defined by $[f]^*([g]_{AT}) = [g \circ f]_{AT}$ is well defined and biholomorphic.*

A biholomorphism $[f]^*$ in Proposition 4.1 is called an *allowable bijection* induced by a quasiconformal mapping $f : R \to S$ (cf. §2.3.1 of [13]).

Actually, Proposition 4.1 was already observed by C. Earle, F. Gardiner and N. Lakic in their series of works (cf. [5], [6] and [7]). In their works, the proposition was obtained as a consequence of general principles for holomorphic mappings on domains in Banach spaces. Here, for the convenience of readers, we will try to give a direct proof of Proposition 4.1.[1]

Before proving Proposition 4.1, we discuss how Proposition 4.1 is applied to obtain Theorems 1 and 2. Indeed, by Lemma 3.2, it suffices to construct collections

[1]In [6], they called $[f]^*$ a geometric isomorphism.

of Banach spaces $\{B_x\}_{x \in AT(R)}$ and holomorphic embeddings $\{F_x\}_{x \in AT(R)}$ with $F_x : AT(R) \to B_x$ for $x \in AT(R)$ such that $F_x(x) = 0$ and

$$\{b \in B_x \mid \|b\|_{B_x} < \delta_1\} \subset F_x(AT(R)) \subset \{b \in B_x \mid \|b\|_{B_x} < \delta_2\}$$

for some constants δ_1 and δ_2 independent of the choice of points in $AT(R)$.

Let $x = [f]_{AT} \in AT(R)$ and let Γ_f be the Fuchsian group acting on $\mathbb{D}$ which uniformizes $f(R)$. Denote by $\widehat{\mathcal{B}}_{\Gamma_f} : AT(f(R)) \to \widehat{B}(\Gamma_f)$ the asymptotic Bers map as in §2.4. Then, by Lemma 2.1 and Proposition 4.1, one see that $F_x := \widehat{\mathcal{B}}_{\Gamma_f} \circ ([f]^*)^{-1}$ and $B_x := \widehat{B}(\Gamma_f)$ satisfies the desired properties.

Proof of Proposition 4.1. We first check the well-definedness and the bijectivity of allowable bijections. Let g_1 and g_2 be quasiconformal mappings on S with $[g_1]_{AT} = [g_2]_{AT}$. By definition, there is an asymptotically conformal mapping $h : g_1(S) \to g_2(S)$ which is quasiconformally isotopic to $g_2 \circ g_1^{-1}$ rel $\partial^{id} g_1(S)$. Then $g_i \circ f$ is a quasiconformal mapping from R to $g_i \circ f(R) = g_i(S)$, and $(g_2 \circ f) \circ (g_1 \circ f)^{-1} = g_2 \circ g_1^{-1}$ is quasiconformally isotopic to an asymptotically conformal mapping h rel $\partial^{id}(g_1 \circ f(R))$. This means that $[g_1 \circ f]_{AT} = [g_2 \circ f]_{AT}$ in $AT(R)$. Since $[f]^* \circ [f^{-1}]^* = id_{AT(R)}$ and $[f^{-1}]^* \circ [f]^* = id_{AT(S)}$, $[f]^*$ is bijective. $\square$

We next show that $[f]^*$ is holomorphic. Since the projection $\widehat{\Phi}_R : \widehat{M}(R) \to AT(R)$ is a holomorphic split submersion, this is deduced from the following claim.

Claim 1. *f induces a biholomorphic mapping $[f]^{\hat{*}} : \widehat{M}(S) \to \widehat{M}(R)$ which commutes the following diagram:*

$$\begin{array}{ccc} \widehat{M}(S) & \xrightarrow{[f]^{\hat{*}}} & \widehat{M}(R) \\ {\scriptstyle \widehat{\Phi}_S}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\Phi}_R} \\ AT(S) & \xrightarrow{[f]^*} & AT(R). \end{array}$$

Proof of Claim 1. Consider a biholomorphic mapping $G : M(S) \to M(R)$ defined by

$$G(\nu) = \frac{\mu_f + f^*(\nu)}{1 + f^*(\nu)\overline{\mu_f}},$$

where μ_f is the complex coefficient of f and $f^*(\nu)$ is a Beltrami differential on R defined by the pull-back formula

$$f^*(\nu) = (\nu \circ f)\overline{f_z}/f_z.$$

By definition, $G(\nu)$ is the complex coefficient of the ν-quasiconformal mapping and f. Therefore, G satisfies the following commutative diagram:

$$\begin{array}{ccc} M(S) & \xrightarrow{G} & M(R) \\ {\scriptstyle \widehat{\Phi}_S \circ P_S}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\Phi}_R \circ P_R} \\ AT(S) & \xrightarrow{[f]^*} & AT(R). \end{array}$$

Let ν_1 and ν_2 be Beltrami coefficients with $\nu_1 - \nu_2 \in L_0^\infty(S)$. Then

$$G(\nu_1) - G(\nu_2) = \frac{(1 - |\mu_f|^2) f^*(\nu_1 - \nu_2)}{(1 + f^*(\nu_1)\overline{\mu_f})(1 + f^*(\nu_2)\overline{\mu_f})} \in L_0^\infty(R), \tag{4.1}$$

since $f^*(\nu_1-\nu_2)\in L_0^\infty(R)$. Therefore, G descends to a bijective mapping $[f]^{\hat{*}}:\widehat{M}(S)\to\widehat{M}(R)$. We now check that $[f]^{\hat{*}}$ is holomorphic. The continuity follows from (4.1) since $\|f^*(\nu_i)\|_\infty=\|\nu_i\|_\infty<1$ and

$$\left\|[f]^{\hat{*}}([\nu_1]_{\hat{L}})-[f]^{\hat{*}}([\nu_2]_{\hat{L}})\right\|_{\hat{L}}\leq\frac{1}{(1-\|\mu_f\|_\infty)^2}\left\|[\nu_1]_{\hat{L}}-[\nu_1]_{\hat{L}}\right\|_{\hat{L}}.$$

Let $\nu_1,\nu_2\in M(S)$ and $\tau_1,\tau_2\in L^\infty(S)$ with $\nu_1-\nu_2,\tau_1-\tau_2\in L_0^\infty(S)$. By (4.1), we have

$$\frac{1}{\epsilon}(G(\nu_i+\epsilon\tau_i)-G(\nu_i))=\frac{(1-|\mu_f|^2)f^*(\tau_i)}{(1+f^*(\nu_i)\overline{\mu_f})(1+f^*(\nu_i+\epsilon\tau_i)\overline{\mu_f})}$$

for $i=1,2$. Hence,

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{1}{\epsilon}(G(\nu_1+\epsilon\tau_1)-G(\nu_1))-\lim_{\epsilon\to 0}\frac{1}{\epsilon}(G(\nu_2+\epsilon\tau_2)-G(\nu_2))\\&\quad=\frac{(1-|\mu_f|^2)f^*(\tau_1)}{(1+f^*(\nu_1)\overline{\mu_f})^2}-\frac{(1-|\mu_f|^2)f^*(\tau_2)}{(1+f^*(\nu_2)\overline{\mu_f})^2}=\frac{(1-|\mu_f|^2)\mathcal{M}}{(1+f^*(\nu_1)\overline{\mu_f})^2(1+f^*(\nu_2)\overline{\mu_f})^2},\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}&=f^*(\tau_1)-f^*(\tau_2)+2(f^*(\tau_2)f^*(\nu_1)-f^*(\tau_1)f^*(\nu_2))\overline{\mu_f}\\&\qquad+(f^*(\tau_2)f^*(\nu_1)^2-f^*(\tau_1)f^*(\nu_2)^2)\overline{\mu_f}^2\\&=\{f^*(\tau_1)-f^*(\tau_2)\}\\&\qquad+2\{(f^*(\tau_2)-f^*(\tau_1))f^*(\nu_1)+f^*(\tau_1)(f^*(\nu_1)-f^*(\nu_2))\}\overline{\mu_f}\\&\qquad\quad+\{f^*(\tau_2)(f^*(\nu_1)^2-f^*(\nu_2)^2)+(f^*(\tau_2)-f^*(\tau_1))f^*(\nu_2)^2\}\overline{\mu_f}^2\\&\in L_0^\infty(R).\end{aligned}$$

Therefore, for $p=[\nu]_{\hat{L}}\in\widehat{M}(S)$ and $v=[\tau]_{\hat{L}}\in\widehat{L}(S)=T_p\widehat{M}(S)$, the complex Gateaux derivative

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left([f]^{\hat{*}}(p+\epsilon v)-[f]^{\hat{*}}(p)\right)$$

exists and is equal to

$$\left[\lim_{\epsilon\to 0}\frac{1}{\epsilon}(G(\nu+\epsilon\tau)-G(\nu))\right]_{\hat{L}}=\left[\frac{(1-|\mu_f|^2)}{(1+f^*(\nu)\overline{\mu_f})^2}f^*(\tau)\right]_{\hat{L}},$$

which leads what we desired (cf. §1.6.1 of [13]). □

The following corollaries follow from Proposition 4.1.

Corollary 3. *The complex structure on $AT(R)$ is canonical in the sense that it is independent of the choice of the base surface. Namely, $AT(R)$ and $AT(S)$ are biholomorphic when R and S are quasiconformally equivalent.*

Corollary 4. $\mathrm{Aut}(AT(\mathbb{D}))$ *acts transitively on* $AT(\mathbb{D})$.

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