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FAILURE OF KRULL-SCHMIDT FOR INVERTIBLE LATTICES OVER A DISCRETE VALUATION RING

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ABSTRACT. Let p be a prime greater than 3, and let N be the semi-direct product of a group H of order p by a cyclic C group of order $p-1$, which acts faithfully on H. Let R be the localization of Z at p. We show that the Krull-Schmidt Theorem fails for the category of invertible RN-lattices.

1. Introduction

Let G be a finite group and let RG be the group ring of G with coefficients in a Dedekind domain R. An RG-lattice M is defined to be a finitely generated R-torsion-free RG-module. M is said to be a permutation lattice if it is R-free and has an R-basis permuted by G. M is said to be an invertible or a permutation projective lattice, if it is a direct summand of a permutation lattice. This note was motivated by a question of A. Merkurjev about the existence of a category of invertible lattices over a discrete valuation ring, for which the Krull-Schmidt Theorem failed. The question arose in the study of the problem of the uniqueness of a direct sum decomposition of the motive of a projective homogeneous variety into indecomposable objects in the category of Chow motives. This category contains a subcategory equivalent to the category of invertible lattices for a certain finite group. Failure of Krull-Schmidt for this subcategory implies failure of uniqueness of direct sum decompositions for the motives; see [4].

Let p be a prime greater than 3, and let N be the semi-direct product of a group H of order p by a cyclic group C of order $p-1$, where C acts faithfully on H by conjugation. Let R denote the localization of the ring of integers, Z, at the prime p. We show that the Krull-Schmidt Theorem fails for the category of invertible RN-lattices.

2. Invertible lattices

Let $N = H \rtimes C$ be as defined in the introduction. The ZN-lattice ZN/H is isomorphic to ZC, and ZH has the structure of a ZN-lattice via the isomorphism

$$ZN/C \cong ZH.$$

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Let I_H be the augmentation ideal of ZH. We have the following ZN-exact sequences:

$$0 \to I_H \to ZH \to Z \to 0.$$

Tensoring by I_H over Z and setting $V = I_H \otimes I_H$ we obtain

$$0 \to V \to ZH \otimes I_H \to I_H \to 0. \tag{2.1}$$

One checks directly that $Res_C^N I_H \cong ZC$, and so the following isomorphisms are given by Frobenius reciprocity:

$$ZH \otimes I_H \cong ZN \otimes_{ZC} Res_C^N I_H \cong ZN \otimes_{ZC} ZC \cong ZN.$$

Therefore, (2.1) becomes

$$0 \to V \to ZN \to I_H \to 0. \tag{2.2}$$

For any ZN-lattice M, we let $M^* = Hom(M, Z)$. Let q be a prime different from p, and let C_q be a q-Sylow subgroup of N. Without loss of generality, we may assume that C_q is contained in C. Thus $H^1(C_q, I_H) \cong H^1(C_q, ZC) = 0$. We also have $H^1(H, I_H) \cong Z/pZ$, and hence $H^1(H, I_H^*) \cong Z/pZ$. Let α generate $H^1(H, I_H^*)$. By [6, Proposition 12.5] there exists a ZN-lattice W and an exact sequence

$$0 \to I_H^* \to W^* \to ZN/H \to 0$$

such that the image of α in $H^1(H,W^*)$ is 0, and hence W^* is H^1-trivial since $H^1(N, I_H^*)$ injects into $H^1(H, I_H^*)$. Since N is meta-cyclic this implies that W^* is invertible by [3, Proposition 2, section 1]. Dualizing the above sequence we obtain

$$0 \to ZN/H \to W \to I_H \to 0 \tag{2.3}$$

with W invertible.

Lemma 2.1. *There is an isomorphism of ZN-lattices*

$$V \oplus W \cong ZN \oplus ZN/H.$$

Proof. We form the following pullback diagram with sequences (2.2) and (2.3):

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & V & \to & ZN & \to & I_H & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & V & \to & M & \to & W & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & ZN/H & \to & ZN/H & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We have $(I_H)^K = 0$ for all subgroups K of N so that V is H^1-trivial. Since N is meta-cyclic, this implies that V is invertible by [3, Proposition 2, section 1] By [3, Lemma 9, section 1], the middle horizontal and vertical sequences split, which proves the lemma. □

Notation 2.2. For a ZN-lattice M, let M_p denote its localization at the prime p, and let $\hat{M}$ denote its p-adic completion.

Remark 2.3. We have

$$\hat{Z}N/H \cong \hat{Z}[x]/(x^{p-1}-1) \cong \bigoplus_{k=1}^{p-1} \hat{Z}[x]/(x-\theta^k) \cong \bigoplus_{k=1}^{p-1} Z_k$$

where $Z_k \cong \hat{Z}[x]/(x-\theta^k)$ and θ is a primitive $(p-1)^{st}$ root of 1 in $\hat{Z}$. So Z_k is a $\hat{Z}N$-module of $\hat{Z}$-rank 1 with trivial H-action and such that if c generates C, then $c.1 = \theta^k$. Therefore,

$$\hat{Z}N \cong \hat{Z}H \otimes \hat{Z}C \cong \bigoplus_{k=1}^{p-1} \hat{Z}H \otimes Z_k.$$

Note that for each k, $\hat{Z}H \otimes Z_k$ is indecomposable since $Res_H^N \hat{Z}H \otimes Z_k \cong \hat{Z}H$, and $\hat{Z}H$ is an indecomposable $\hat{Z}H$-module by [2, Corollary 19.24].

Theorem 2.4. *Let R denote the localization of Z at the prime p. Then the Krull-Schmidt Theorem fails for invertible RN-lattices.*

Proof. By [1, Theorem 2.3] we have

$$\hat{V} \cong (\bigoplus_{k=2}^{p-1} \hat{Z}H \otimes Z_k) \oplus Z_1$$

Therefore by Lemma 2.1 and Remark 2.3

$$\hat{W} \cong (\bigoplus_{k=2}^{p-1} Z_k) \oplus \hat{Z}H \otimes Z_1.$$

Let Q be the field of rational numbers. For each k dividing $p-1$, let ω_k be a primitive k^{th} root of 1 over Q. Then

$$QN/H \cong \bigoplus_{k|p-1} Q(\omega_k)$$

and the $Q(\omega_k)$ are the irreducible components of QN/H. Now ZN/H is isomorphic to ZC as a ZN-module, and since RC is a maximal R-order in QC we have

$$RN/H \cong \bigoplus_{k|p-1} R[\omega_k]$$

by [2, Proposition 31.2]. Therefore,

$$RN \cong \bigoplus_{k|p-1} RH \otimes R[\omega_k].$$

Now for each k we have

$$\hat{Z}[\omega_k] \cong \hat{Z}[x]/\phi_k(x)$$

where $\phi_k(x)$ is the k^{th} cyclotomic polynomial. As above we let $\theta \in Q_p$ be a primitive $(p-1)^{st}$ root of 1 over Q, where Q_p is the completion of Q at the prime p. Set $\omega_k = \theta^{(p-1)/k}$ and let $J_k = \{i \in Z : 1 \leq i < k, (i,k) = 1\}$. Then $\phi_k(x) = \prod_{j \in J_k}(x - \omega_k^j)$. Therefore,

$$\hat{Z}[\omega_k] = \bigoplus_{j \in J_k} Z_j.$$

Consequently,

$$\hat{V} \cong (\bigoplus_{k|p-1, k\neq p-1} \hat{Z}H \otimes \hat{Z}[\omega_k]) \oplus (\bigoplus_{k\in J_{p-1}, k\neq 1} \hat{Z}H \otimes Z_k) \oplus Z_1$$

and

$$\hat{W} \cong (\bigoplus_{k|p-1, k\neq p-1} \hat{Z}[\omega_k]) \oplus (\bigoplus_{k\in J_{p-1}, k\neq 1} Z_k) \oplus \hat{Z}H \otimes Z_1.$$

To simplify notation set

$$M = \bigoplus_{k|p-1, k\neq p-1} RH \otimes R[\omega_k] \quad \text{and} \quad M' = \bigoplus_{k|p-1, k\neq p-1} R[\omega_k].$$

Since $\hat{Z}$ is a faithfully flat R-module by [5, Corollary 2.2] we have $\hat{V}/\hat{M} \cong \widehat{V_p/M}$. Similarly, $\hat{W}/\hat{M'} \cong \widehat{W_p/M'}$. Therefore, the RN-lattices $S = V_p/M$ and $S' = W_p/M'$ have the property that

$$\hat{S} = (\bigoplus_{(k\in J_{p-1}, k\neq 1} Z_k \otimes \hat{Z}H) \oplus Z_1$$

and

$$\hat{S}' = (\bigoplus_{k\in J_{p-1}, k\neq 1} Z_k) \oplus \hat{Z}H \otimes Z_1.$$

We have $\hat{V} = \hat{M} \oplus \hat{S}$ and $\hat{W} = \hat{M}' \oplus \hat{S}'$, which implies that $V_p = M \oplus S$ and $W_p = M' \oplus S'$, [2, Proposition 30.17]. Since $V_p \oplus W_p = RN/H \oplus RN$ by Lemma 2.1, S and S' are invertible RN-lattices. Since

$$\hat{Z}[w_{p-1}] = \bigoplus_{k\in J_{p-1}} Z_k,$$

we get

$$\hat{S} \oplus \hat{S}' \cong \hat{Z}H \otimes \hat{Z}[\omega_{p-1}] \oplus \hat{Z}[\omega_{p-1}],$$

and so by [2, Proposition 30.17],

$$S \oplus S' \cong RH \otimes R[\omega_{p-1}] \oplus R[\omega_{p-1}].$$

Now $R[\omega_{p-1}]$ is indecomposable since $Q(\omega_{p-1})$ is irreducible, but it is not a direct summand of either S or S', for if it were, then $\hat{Z}[\omega_{p-1}]$ would be a direct summand of $\hat{S}$ or $\hat{S'}$ which is a contradiction.

Note that the condition, that the prime p be greater than 3, is necessary for if $p = 3$, then $S = Z[\omega_2]$ and if $p = 2$, then $S = R = Z[\omega_1]$. □

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References

[1] E. Beneish, Monomial actions of the symmetric group, J. of Alg. 265 (2003) 405-419. MR1987009 (2004b:20008)

[2] C. Curtis and I. Reiner, Methods of Representation Theory, Vol. 1, Springer-Verlag, 1982. MR0632548 (82i:20001)

[3] J.-L. Colliot-Thelene and J.-P. Sansuc, La R-equivalence sur les tores, Ann. Sci. Ecole Normale Sup. (4) 10 (1977), 175-230. MR0450280 (56:8576)

[4] V. Chernousov and A. Merkurjev, Motivic decomposition of projective homogeneous varities and the Krull-Schmidt Theorem, preprint.

[5] I. Reiner, Maximal Orders, Academic Press, 1975.

[6] D. Saltman, Lectures on Division Algebras, CMBS, Regional Conference series in Mathematics, number 94, 1998. MR1692654 (2000f:16023)

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