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COMPACT COMPOSITION OPERATORS NOT IN THE SCHATTEN CLASSES

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To Professor J. M. Anderson on his retirement

Abstract. We provide a direct construction of the mapping φ given by Tom Carroll and Carl Cowen in 'Compact composition operators not in the Schatten classes' (1991, in J. Operator Theory 26).

1. Introduction

If φ is an analytic self-map of the unit disk $\mathbb{D} = \{z\colon |z| < 1\}$ in the complex plane and H is a Hilbert space of functions analytic in $\mathbb{D}$, we define the composition operator acting on $f \in H$ as $C_\varphi(f) = f \circ \varphi$. It was shown in [CC] that on various specific Hilbert spaces, including the Hardy space, there are composition operators that are compact but belong to none of the Schatten classes. This answers a question of Sarason in [C].

The construction of the function φ in [CC] involves the Königs function and is of particular interest given the Twisted Sector Theorem (see [PC] and [SSS] or Joel Shapiro's survey article [S]). However, the geometry of the mapping function itself is difficult to understand. It is the purpose of this paper to provide a direct construction of $\varphi(\mathbb{D})$ so that C_φ is compact but not in any Schatten class.

For $\alpha > -1$ we define D_α to be the weighted Dirichlet space of functions analytic in $\mathbb{D}$ for which the norm

$$\|f\|_\alpha^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha dA$$

is finite. Here dA denotes normalised Lebesgue area measure on $\mathbb{D}$. Note that the Dirichlet space is D_0, the Hardy space is D_1 and the Bergman space is D_2.

If φ maps $\mathbb{D}$ into $\mathbb{D}$, then φ is said to have a finite angular derivative at $\zeta \in \partial\mathbb{D}$ if there is an $\eta \in \partial\mathbb{D}$ for which the quotient

$$\frac{\varphi(z) - \eta}{z - \zeta}$$

has finite limit as $z \to \zeta$ non-tangentially. It is known that the above limit is finite if and only if

$$\liminf_{z\to\zeta} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty.$$

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This is the content of the Julia-Carathéodory theorem; see Theorem 1-5 in [A]. Points of $\partial\mathbb{D}$ for which φ has a finite angular derivative are well behaved in terms of the geometry of φ; for example, if φ has a finite angular derivative at ζ, then it is conformal there, that is, arcs in $\mathbb{D}$ ending at ζ and making an angle of β with $\partial\mathbb{D}$ are mapped onto arcs in $\varphi(\mathbb{D})$ ending at $\eta \in \partial\varphi(\mathbb{D})$ and making the same angle β with $\partial\varphi(\mathbb{D})$. It follows that if $\varphi(\mathbb{D})$ approaches $\zeta \in \partial\mathbb{D}$ in an angle less than π, then φ cannot have a finite angular derivative there. It was shown in [MacS] that, for φ univalent, C_φ is compact on D_α if and only if φ has no finite angular derivatives.

If T is a compact operator acting on a Hilbert space H, then $(T^*T)^{1/2}$ is a positive compact operator. It follows that the spectrum of $(T^*T)^{1/2}$ consists of countably many eigenvalues on $[0,\infty)$. If the sequence of eigenvalues of $(T^*T)^{1/2}$ lies in ℓ^p, then T is said to belong to the Schatten p-class. The Schatten p-classes should be seen as gradations of compactness for an operator. Each Schatten p-class is dense in the compact operators in the operator norm. For this reason it is of interest, given a certain class of operators, to ask whether or not there are compact operators not in any Schatten p-class. This was proved by Carroll and Cowen for composition operators acting on D_α for $\alpha > 0$ in [CC].

To prove the result in [CC] the authors use the fact that on certain invariant subspaces, the operator C_φ^* is similar to a weighted unilateral shift. Recall that a backward iteration sequence is a sequence of points z_n, $n = 1, 2, \ldots$, in $\mathbb{D}$ that satisfies $\varphi(z_{n+1}) = z_n$. Since $C_\varphi^* K_z = K_{\varphi(z)}$ for K_z the reproducing kernel for D_α, the authors of [CC] construct a φ and a backward iteration sequence r_n, $n = 1, 2, \ldots$, so that C_φ^* restricted to the invariant subspace

$$\bigvee_{n\geq 1} \frac{K_{r_n}}{\|K_{r_n}\|}$$

is similar to a weighted unilateral shift with weight sequence

$$(r_{n+1} - r_n)\frac{\|K_{r_n}\|}{\|K_{r_{n+1}}\|} \sim c\left(\frac{1 - r_{n+1}}{1 - r_n}\right)^{\alpha/2}.$$

The authors prove the result by constructing a φ with no finite angular derivatives and a backward iteration sequence r_n such that for all $p > 0$

$$\sum_{n=1}^{\infty}\left(\frac{1 - r_{n+1}}{1 - r_n}\right)^p = \infty.$$

This is the starting point of this paper. In [CC] the function φ is defined as $\varphi = \sigma^{-1}(e^{-1}\sigma(z))$ where σ is a univalent mapping of $\mathbb{D}$ onto a carefully defined domain Ω in $\mathbb{C}$. Since this construction gives only the geometry of the domain Ω, sometimes called the Königs domain of φ, it is of interest to give a construction of the mapping φ explicitly. The result presented in this paper does precisely that.

Theorem 1. *There is a univalent map* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *such that* $\overline{\varphi(\mathbb{D})}$ *intersects the unit circle only at the points* -1 *and* 1, *and a backward iteration sequence* r_n, $n = 1, 2, \ldots$, *such that*

$$\lim_{n\to\infty} \frac{1 - r_{n+1}}{1 - r_n} = 0,$$

$$\textit{for all } p > 0 \quad \sum_{n=1}^{\infty}\left(\frac{1 - r_{n+1}}{1 - r_n}\right)^p = \infty.$$

By the analysis in [CC] it follows that for the mapping φ in Theorem 1, C_φ is compact on D_α, $\alpha > 0$, but is not contained in any Schatten p-class for $p > 0$.

2. Proof of Theorem 1

We will construct the mapping in the infinite strip $\mathcal{S} = \{w = x + iy \colon |y| < \pi/2\}$. The unit disk is conformally equivalent to $\mathcal{S}$ by the transformation

$$w = \log \frac{1+z}{1-z}$$

where log denotes the principle branch of the logarithm so that $z = -1$ corresponds to $w = -\infty$ and $z = 1$ to $w = \infty$.

Let Γ^+ be the polygonal line that begins at the point $(0, 1/2)$ and passes through the points (X_n, X_{n+1}), $n = 1, 2, \ldots$, where

$$X_n = \prod_{k=1}^{n} \log(k+1).$$

Let $\Gamma = \Gamma^+ \cup \Gamma^-$ where Γ^- is the reflection of Γ^+ in the y-axis. We let $G(x)$ be the real-valued function with graph Γ.

Now let $F(x) = \log G(e^x)$ and $f(x) = \dfrac{\pi}{2} \dfrac{1}{F'(x)}$ where we define

$$F'(x) = \lim_{\epsilon \to 0+} F'(x - \epsilon).$$

Let $x > 0$ be arbitrary and suppose that n is chosen so that $\log X_n < x < \log X_{n+1}$. Then we have that $G'(e^x) = \Delta X_{n+1} / \Delta X_n$ where Δ is the forward difference operator, $\Delta X_n = X_{n+1} - X_n$. Therefore, if $\tau \in (0,1)$ is defined so that $e^x = X_n + \tau \Delta X_n$, then

$$\begin{aligned} F'(x) &= \frac{e^x G'(e^x)}{G(e^x)} = \left(\frac{X_n + \tau \Delta X_n}{X_{n+1} + \tau \Delta X_{n+1}} \right) \frac{\Delta X_{n+1}}{\Delta X_n} \\ &= \left(\tau + \frac{1}{\log(n+2) - 1} \right) \Big/ \left(\tau + \frac{1}{\log(n+3) - 1} \right). \end{aligned}$$

Hence $F'(x) > 1$ for all x and $\lim_{x \to \infty} F'(x) = 1$. Therefore the domain

$$\Omega = \{x + iy \colon |y| < f(x)\} \subset \mathcal{S}$$

is well defined and by the Riemann mapping theorem there is a conformal mapping ψ of $\mathcal{S}$ onto Ω, such that $\psi(0) = 0$, $\psi'(0) > 0$.

We will show that there is a backward iteration sequence s_n, $n = 1, 2 \ldots$, such that

$$\lim_{n \to \infty} s_{n+1} - s_n = \infty, \tag{1}$$

$$\forall\, p > 0 \quad \sum_{n=1}^{\infty} \exp -p(s_{n+1} - s_n) = \infty. \tag{2}$$

This is equivalent to the conclusion of the theorem with $\varphi(z) = \psi(w)$. By the symmetry of Ω, ψ is real-valued on the real axis and ∞ is a fixed point of ψ. Furthermore it follows from the Schwartz Lemma that for real $s > 0$, $\psi(s) < s$. We may therefore find a backward iteration sequence s_n such that $s_n \uparrow \infty$ as $n \to \infty$.

Let D be a domain in $\mathbb{C}$ and let Γ denote a family of arcs in D. A metric $\rho|dz|$ is admissible for Γ if it is non-negative in D and

$$\int_\gamma \rho(z)|dz| \geq 1 \quad \text{for all } \gamma \in \Gamma.$$

The modulus of Γ is defined as

$$\operatorname{mod} \Gamma = \inf_\rho \iint_D \rho(z)^2 dxdy.$$

The modulus is conformally invariant and its reciprocal is the extremal distance (see [A]).

A quadrilateral Q is a Jordan domain in $\mathbb{C}$ with disjoint arcs $\alpha_1, \alpha_2 \subset \partial Q$. The modulus of Q is defined as the modulus of Γ where Γ consists of all rectifiable arcs in Q that have one endpoint in α_1 and one in α_2. A special case is when $Q = \{x + iy\colon 0 < x < M,\ 0 < y < 1\}$ and α_i, $i = 1, 2$, are the horizontal sides of Q. In this case $\operatorname{mod} Q = M$ (see [A]).

We will need the following exact formula for the modulus.

Lemma 1. *Let $0 < a < b$ be arbitrary and $Q(a,b) = \Omega \cap \{x + iy \in \mathcal{S}\colon a < x < b\}$ with α_i, $i = 1, 2$, the two components of $Q(a,b) \cap \partial\Omega$. Then*

$$\operatorname{mod} Q(a,b) = \int_a^b \frac{dx}{2f(x)}.$$

Proof. Let Γ_0 consist of the arcs $\gamma_x = \{\Re z = x\} \cap \Omega$ for $a < x < b$ and define $\rho_0(w) = (2f(x))^{-1}$ where $w = x + iy$. For any $\gamma_x \in \Gamma_0$ we have that

$$\int_{\gamma_x} \rho_0(w)|dw| = 1.$$

If we now let h be any real-valued function defined on $Q(a,b)$ with

$$\int_{\gamma_x} h(w)|dw| = \int_{\gamma_x} h(w)dy \geq 0,$$

then

$$\iint_{Q(a,b)} h(w)\rho_0(w)dxdy \geq \frac{1}{2f(b)} \iint_{Q(a,b)} h(w)dydx \geq 0.$$

By Beurling's criterion (Theorem 4-4 in [A]), it follows that ρ_0 defines an extremal metric for $Q(a,b)$ and so

$$\operatorname{mod} Q(a,b) = \iint_{Q(a,b)} \frac{dxdy}{(2f(x))^2} = \int_a^b \frac{dx}{2f(x)}$$

as required. □

We prove (1) first. This will follow if we can show that ψ does not have an angular derivative at ∞. Results of Jenkins, Oikawa, Rodin and Warschawski (see Theorem 6 in [M]) and the references therein, imply that ψ has a finite angular derivative at ∞ if and only if

$$\int_0^\infty \left(\frac{1}{f(x)} - \frac{2}{\pi}\right) dx$$

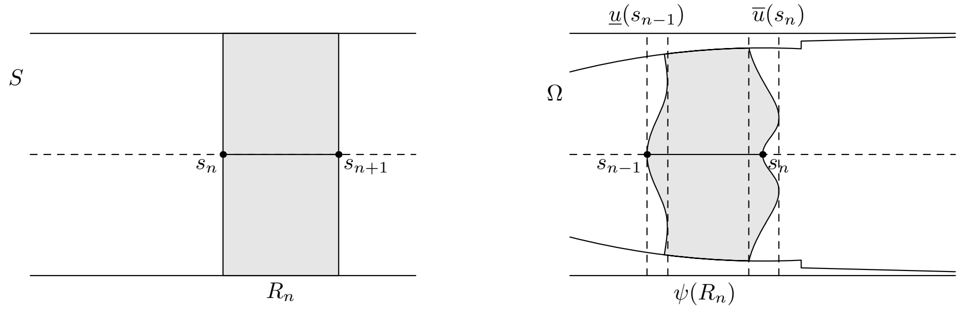

FIGURE 1. Proof of Lemma 2

exists. However a simple calculation shows that

$$\int_{\log X_n}^{\log X_{n+1}} (F'(x)-1)dx = \log\log(n+3) - \log\log(n+2)$$

and so

$$\begin{aligned}\int_0^\infty \left(\frac{1}{f(x)} - \frac{2}{\pi}\right) dx &= \frac{2}{\pi}\left(c + \lim_{n\to\infty}\sum_{k=1}^{n}\int_{\log X_k}^{\log X_{k+1}} (F'(x)-1)dx\right) \\ &= \frac{2}{\pi}\left(c + \lim_{n\to\infty}\log\log(n+3)\right) = \infty.\end{aligned}$$

Therefore ψ does not have a finite angular derivative at ∞ and $s_{n+1} - s_n \to \infty$ as $n \to \infty$.

We now turn our attention to proving (2).

Lemma 2. *The sequence s_n asymptotically satisfies the difference equation*

$$s_{n+1} - s_n = \frac{\pi}{2}\int_{s_{n-1}}^{s_n} \frac{dx}{f(x)}. \tag{3}$$

Proof. Let $R_n = [s_n, s_{n+1}] \times [-\pi/2, \pi/2] \subset \mathcal{S}$ and $\gamma_n = \mathcal{S} \cap \{x = s_n\}$. Note that $\psi(R_n)$ contains the interval $[s_{n-1}, s_n]$ and is symmetric about the real axis. Let $\underline{u}(s_n) = \inf\{x\colon x + iy \in \psi(\gamma_{n+1})\}$ and similarly $\overline{u}(s_n) = \sup\{x\colon x + iy \in \psi(\gamma_{n+1})\}$ (see Figure 1).

It then follows by the usual comparison principle of modules that

$$\operatorname{mod} Q(\overline{u}(s_{n-1}), \underline{u}(s_n)) \le \operatorname{mod}\psi(R_n) \le \operatorname{mod} Q(\underline{u}(s_{n-1}), \overline{u}(s_n)).$$

Therefore, by the conformal invariance of the modulus and Lemma 1 we have the double inequality

$$\int_{\overline{u}(s_{n-1})}^{\underline{u}(s_n)} \frac{dx}{2f(x)} \le \frac{s_{n+1} - s_n}{\pi} \le \int_{\underline{u}(s_{n-1})}^{\overline{u}(s_n)} \frac{dx}{2f(x)}.$$

We claim that $\overline{u} - \underline{u}$ is bounded by an absolute constant. To see this, map $Q(\overline{u}(s_n), \underline{u}(s_n))$ into $\mathcal{S}$ using ψ^{-1}. Then the image of the arcs $\{x = \overline{u}(s_n)\} \cap \Omega$ and $\{x = \underline{u}(s_n)\} \cap \Omega$ both touch the line $x = s_{n+1}$ and so by the Ahlfors distortion theorem (Theorem 4-8 in [A]),

$$\frac{\overline{u}(s_n) - \underline{u}(s_n)}{\pi} \le \Lambda(1) = \frac{1}{2},$$

for any n. Here $\Lambda(R)$ is the modulus of the Teichmüller annulus (see [A]). Therefore since $\underline{u}(s_n) < s_n < \overline{u}(s_n)$ we have

$$\begin{aligned}\left|\frac{s_{n+1}-s_n}{\pi} - \int_{s_{n-1}}^{s_n} \frac{dx}{2f(x)}\right| &\leq \int_{s_{n-1}-\pi/2}^{s_{n-1}} \frac{dx}{2f(x)} + \int_{s_n}^{s_n+\pi/2} \frac{dx}{2f(x)} \\ &= o(1)\int_{s_{n-1}}^{s_n} \frac{dx}{2f(x)}\end{aligned}$$

as $n \to \infty$. The last equality holds since $s_{n+1} - s_n \to \infty$ as $n \to \infty$. ☐

We therefore have that the sequence s_n is asymptotic to a sequence x_n that satisfies

$$x_{n+1} - x_n = \int_{x_{n-1}}^{x_n} F'(x)dx. \tag{4}$$

Lemma 3. *If* x_n, $n = 1, \ldots,$ *is a solution to equation* (4), *then*

$$x_{n+1} - x_n \sim \log\log(n+2)$$

as $n \to \infty$.

Proof. First note that (4) is equivalent to the equation $x_{n+1} = F(x_n) + c$ for a given constant $c > 0$. We may assume that $c = 0$, for if x_n is a solution of the difference equation $x_{n+1} = F(x_n)$, then $y_n = x_n + cn$ is a solution to $y_{n+1} = F(y_n) + c$ and $x_{n+1} - x_n \sim y_{n+1} - y_n$ as $n \to \infty$.

Using the definition of F we have

$$\exp x_{n+1} = G(\exp x_n).$$

It now follows by definition that the sequence X_n, $n = 1, 2, \ldots$, is a solution to (4) and so

$$\begin{aligned} x_{n+1} - x_n \sim X_{n+1} - X_n &= \log\left(\prod_{k=1}^{n+1} \log(k+1)\right) - \log\left(\prod_{k=1}^{n} \log(k+1)\right) \\ &= \log\log(n+2). \end{aligned}$$

☐

Now since the backward iteration sequence, s_n, satisfies (3) asymptotically it follows that

$$s_{n+1} - s_n \sim X_{n+1} - X_n = \log\log(n+2).$$

Therefore for any $p > 0$ there is an n_0 so that for all $n > n_0$,

$$\exp p(s_{n+1} - s_n) \leq c(\log(n))^p < n$$

and so

$$\sum_{n>n_0} \exp -p(s_{n+1} - s_n) > \sum_{n>n_0} \frac{1}{n} = \infty,$$

and the theorem is proved. ☐

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