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STRONG COMPACTNESS AND A PARTITION PROPERTY

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ABSTRACT. We show that if $\mathrm{Part}(\kappa, \lambda)$ holds for every $\lambda \geq \kappa$, then κ is strongly compact.

Let κ be a regular infinite cardinal, and let $\lambda \geq \kappa$ be a cardinal. $P_\kappa(\lambda)$ denotes the set of all subsets of λ of size less than κ. $\mathrm{Part}(\kappa, \lambda)$ means that for every $F\colon P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$, there is a cofinal subset A of $(P_\kappa(\lambda), \subseteq)$ such that F is constant on the set $\{(a, b) \in A \times A\colon a \subset b\}$. This definition is due to Jech [4]. Jech and Shelah [5] established that $\mathrm{Part}(\kappa, \kappa^+)$ holds for $\kappa = \omega$. We proved in [10] that if κ is almost $\lambda^{<\kappa}$-ineffable, then $\mathrm{Part}(\kappa, \lambda)$ holds. It is also known ([5], [11], [12], [8]) that if κ is mildly $\lambda^{<\kappa}$-ineffable and $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda^{<\kappa}}) > \lambda^{<\kappa}$, then $\mathrm{Part}(\kappa, \lambda)$ holds.

Let $\mu \geq \kappa$ be a cardinal. We will show that if $\mathrm{Part}(\kappa, 2^{2^{\mu^{<\kappa}}})$ holds, then κ is μ-compact. First we recall a few definitions.

Given a cardinal $\nu \geq \kappa$, $I_{\kappa,\nu}$ denotes the set of all $A \subseteq P_\kappa(\nu)$ such that $\{a \in A\colon b \subseteq a\} = \emptyset$ for some $b \in P_k(\nu)$. By an *ideal* on $P_\kappa(\nu)$ we mean a subset K of $P(P_\kappa(\nu))$ such that (i) $I_{\kappa,\nu} \subseteq K$, (ii) $P_\kappa(\nu) \notin K$, (iii) $P(A) \subseteq K$ for every $A \in K$, and (iv) $\bigcup T \in K$ for every subset T of K of size less than κ. We let $K^+ = P(P_\kappa(\nu)) \backslash K$. For $A \in K^+$, we let $K|A = \{B \subseteq P_\kappa(\nu)\colon B \cap A \in K\}$. For a cardinal $\tau \geq 2$, K is *τ-saturated* if there is no size τ subset T of K^+ with the property that $A \cap B \in K$ for any two distinct members A, B of T. K is *nowhere τ-saturated* if for every $A \in K^+$, $K|A$ is not τ-saturated. K is *prime* if it is 2-saturated. We say that κ is *ν-compact* if there exists a prime ideal on $P_\kappa(\nu)$.

The following is due to Jech [3].

Lemma 1. *If $\mathrm{Part}(\kappa, \lambda)$ holds, then κ is weakly compact.*

The following is due to Levy and Silver (see [6], Proposition 16.4(b)).

Lemma 2. *Let J be an ideal on $P_\kappa(\mu)$. If J is κ-saturated and κ is weakly compact, then $J|A$ is prime for some $A \in J^+$.*

Let $\mathcal{J}_{\kappa,\mu}$ denote the collection of all nowhere κ-saturated ideals on $P_\kappa(\mu)$.

Note that an ideal J on $P_\kappa(\mu)$ belongs to $\mathcal{J}_{\kappa,\mu}$ if and only if for every $A \in J^+$, there is a partition $\langle D_\xi\colon \xi < \kappa\rangle$ of A such that $D_\xi \in J^+$ for every $\xi < \kappa$.

The following is essentially due to Taylor (see [13], Theorem 2.2).

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Lemma 3. *Let X be a subset of $\mathcal{J}_{\kappa,\mu}$ with $0 < |X| < \kappa$. Then there is a partition $\langle D_i \colon i < 2\rangle$ of $P_\kappa(\mu)$ such that $D_0, D_1 \in \bigcap_{J\in X} J^+$.*

Proof. Set $\tau = |X|$ and let $X = \{J_\alpha \colon \alpha < \tau\}$. For $\alpha < \tau$, select a partition $\langle A^\alpha_\xi \colon \xi < \kappa\rangle$ of $P_\kappa(\mu)$ so that $A^\alpha_\xi \in J^+_\alpha$ for all $\xi < \kappa$. For $\alpha, \beta < \tau$, set $x^\alpha_\beta = \{\xi < \kappa \colon A^\beta_\xi \in J^+_\alpha\}$. Define $h \colon \tau \to \tau$ by $h(\alpha) =$ the least $\beta < \tau$ such that $|x^\alpha_\beta| = \kappa$. There is $\eta \in \kappa$ such that $\bigcup_{\beta<h(\alpha)} x^\alpha_\beta \subseteq \eta$ for every $\alpha < \tau$. Now pick a one-to-one $k \in \prod_{\alpha<\tau}\{A^{h(\alpha)}_\xi \colon \xi \in x^\alpha_{h(\alpha)} \setminus \eta\}$. For $\alpha < \tau$, set

$$B_\alpha = k(\alpha) \setminus \bigcup\{k(\beta) \colon \beta < \tau \text{ and } h(\beta) < h(\alpha)\}$$

and select a partition $\langle C^\alpha_i \colon i < 2\rangle$ of B_α so that $C^\alpha_0, C^\alpha_1 \in J^+_\alpha$. Finally, pick a partition $\langle D_i \colon i < 2\rangle$ of $P_\kappa(\mu)$ so that $\bigcup_{\alpha<\tau} C^\alpha_i \subseteq D_i$ for $i = 0, 1$. It is immediate that $D_0, D_1 \in \bigcap_{\alpha<\tau} J^+_\alpha$. □

Throughout the remainder of this paper it is assumed that $\mu \leq \lambda$.

For an ideal H on $P_\kappa(\lambda)$, let $H \restriction P_\kappa(\mu)$ denote the set of all $B \subseteq P_\kappa(\mu)$ such that $\{a \in P_\kappa(\lambda) : a \cap \mu \in B\} \in H$.

It is easy to check that $H \restriction P_\kappa(\mu)$ is an ideal on $P_\kappa(\mu)$. Note that if $A \in H^+$, then $(H|A) \restriction P_\kappa(\mu)$ is the set of all $B \subseteq P_\kappa(\mu)$ such that $\{a \in A \colon a \cap \mu \in B\} \in H$.

Theorem 4. *If* $\mathrm{Part}(\kappa, \lambda)$ *holds and* $\lambda \geq 2^{2^{\mu^{<\kappa}}}$, *then κ is μ-compact.*

Proof. Let $\langle J_\alpha \colon \alpha < \lambda\rangle$ enumerate (possibly with repetition) all members of $\mathcal{J}_{\kappa,\mu}$. Lemma 3 gives us a partition $\langle D^a_i \colon i < 2\rangle$ of $P_\kappa(\mu)$ for each $a \in P_\kappa(\lambda)$ such that $D^a_0, D^a_1 \in J^+_\alpha$ for every $\alpha \in a$. Define $F \colon P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$ by $F(a, b) = 0$ if and only if $b \cap \mu \in D^a_0$. Let $A \in I^+_{\kappa,\lambda}$ and $j < 2$ be such that F takes the constant value j on the set $\{(a, b) \in A \times A \colon a \subset b\}$. We claim that $(I_{\kappa,\lambda}|A) \restriction P_\kappa(\mu) \notin \mathcal{J}_{\kappa,\mu}$. Suppose otherwise. Then $(I_{\kappa,\lambda}|A) \restriction P_\kappa(\mu) = J_\alpha$ for some $\alpha < \lambda$. Since $A \in I^+_{\kappa,\lambda}$, we have an $a \in A$ with $\alpha \in a$. Now $b \cap \mu \in D^a_j$ for every $b \in A$ with $a \subset b$, so $\{b \in A \colon b \cap \mu \notin D^a_j\} \in I_{\kappa,\lambda}$. On the other hand, $D^a_{1-j} \in J^+_\alpha$ by $\alpha \in a$. Hence $\{b \in A \colon b \cap \mu \notin D^a_j\} \in I^+_{\kappa,\lambda}$. Together with Lemmas 1 and 2, this contradiction completes the proof. □

With a little work we can get a more general version of Theorem 4. We start with some definitions.

Let H be an ideal on $P_\kappa(\lambda)$, let Z be a subset of $P(P_\kappa(\lambda))$, and let $\rho \geq 2$ be a cardinal. $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{H} [Z]^2_\rho$ means that for every $F \colon P_\kappa(\lambda) \times P_\kappa(\mu) \to \rho$, one can find $A \in Z$ and $\xi \in \rho$, so that for every $\alpha \in A$, $\{b \in A \colon F(a, b \cap \mu) = \xi\} \in H$. For $H = I_{\kappa,\lambda}$, we write $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{w} [Z]^2_\rho$ instead of $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{H} [Z]^2_\rho$ (the w stands for "weak").

Note that $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{w} [I^+_{\kappa,\lambda}]^2_2$ follows from $\mathrm{Part}(\kappa, \lambda)$.

We let $u(\kappa, \lambda)$ denote the least size of any member of $I^+_{\kappa,\lambda}$. It is well known (see e.g. [1]) that if $\lambda < \kappa^{+\omega}$, then $u(\kappa, \lambda) = \lambda$.

Note that $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\lambda)]{w} [I^+_{k,\lambda}]^2_{\sigma^+}$ holds, where $\sigma = u(\kappa, \lambda)$.

The following is a straightforward strengthening of Lemma 3.

Lemma 5. *Let X be a subset of $\mathcal{J}_{\kappa,\mu}$ with $0 < |X| < \kappa$. Then there is a partition $\langle D_\gamma \colon \gamma < \kappa\rangle$ of $P_\kappa(\mu)$ such that $D_\gamma \in \bigcap_{J\in X} J^+$ for every $\gamma < \kappa$.*

Proposition 6. *Suppose that* (a) *there exists $Z \subseteq P_\kappa(\lambda)$ such that $|Z| = |\mathcal{J}_{\kappa,\mu}|$ and $|Z \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$,* (b) *$Y \subseteq P_\kappa(\lambda)$ is such that $|Y| \geq 2$ and $|Y \cap P(a)| \leq \kappa$ for all $a \in P_\kappa(\lambda)$, and* (c) *H is an ideal on $P_\kappa(\lambda)$ such that $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{H} [H^+]^2_{|Y|}$. Then there is $B \in H^+$ such that $(H|B) \restriction P_\kappa(\mu)$ is κ-saturated.*

Proof. Select a bijection $u : \mathcal{J}_{\kappa,\mu} \to Z$. Define a partial function F from $P_\kappa(\lambda) \times P_\kappa(\mu)$ to Y as follows. Let $a \in P_\kappa(\lambda)$ with $Z \cap P(a) \neq \emptyset$. By Lemma 5 there is a partition $\langle C^a_d \colon d \in Y \cap P(a)\rangle$ of $P_\kappa(\mu)$ such that for every $d \in Y \cap P(a)$, $C^a_d \in \bigcap_{J\in u^{-1}(Z\cap P(a))} J^+$. Put $F(a,e) = d$ whenever $e \in C^a_d$.

Pick $A \in H^+$ and $d \in Y$ so that for every $a \in A$, $D_a \in H$, where $D_a = \{b \in A \colon F(a, b\cap\mu) = d\}$. Assume to the contrary that $J \in \mathcal{J}_{\kappa,\mu}$, where $J = (H|A) \restriction P_\kappa(\mu)$. Pick $a \in A$ with $d \cup u(J) \subseteq a$. Then setting $S = \{b \in A \colon b \cap \mu \in C^a_d\}$, we obtain $S \in H^+$ and $S \subseteq D_a$. Contradiction! Thus we can find $E \in J^+$ so that $J|E$ is κ-saturated. Set $B = \{c \in A \colon c \cap \mu \in E\}$. Obviously, $B \in H^+$. It remains to observe that $(H|B) \restriction P_\kappa(\mu) = J|E$. Thus B is as desired. □

Let $\mathcal{T}_{\kappa,\lambda}$ be the set of all cardinals τ such that there is a size τ subset T of $P_\kappa(\lambda)$ with the property that $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$.

It is simple to see that (a) $\lambda \in \mathcal{T}_{\kappa,\lambda}$, and (b) if κ is inaccessible, then $\lambda^{<\kappa} \in \mathcal{T}_{\kappa,\lambda}$. For more on $\mathcal{T}_{\kappa,\lambda}$ see [8].

Corollary 7. *Suppose that κ is weakly compact, $2^{2^{\mu^{<\kappa}}} \leq \lambda^{<\kappa}$ and H is an ideal on $P_\kappa(\lambda)$ such that $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{H} [H^+]^2_{\lambda^{<\kappa}}$ holds. Then there is $C \in H^+$ such that $(H|C) \restriction P_\kappa(\mu)$ is prime.*

It also follows from Proposition 6 that if $2^{2^{\mu^{<\kappa}}} \leq \lambda^{<\kappa}$ and H is an ideal on $P_\kappa(\lambda)$ such that $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{H} [H^+]^2_2$, then there is $C \in H^+$ such that $(H|C) \restriction P_\kappa(\mu)$ is prime. Now it is immediate that $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{H} [H^+]^2_2$ if and only if for every $F \colon P_\kappa(\lambda) \times P_\kappa(\mu) \to 2$, one can find $A \in H^+$ and $f \colon P_\kappa(\lambda) \to 2$ so that for every $a \in A$, $\{b \in A \colon F(a, b\cap\mu) = f(a)\} \in H$. So it is interesting to note the following result which is essentially due to Di Prisco and Zwicker (see Lemma 3 in [3]).

Proposition 8. *Suppose that $2^{\mu^{<\kappa}} \leq \lambda^{<\kappa}$ and H is an ideal on $P_\kappa(\lambda)$ such that for every $F \colon P_\kappa(\lambda) \times P_\kappa(\mu) \to 2$, one can find $A \in H^+$ and $f \colon P_\kappa(\lambda) \to 2$ so that for every $a \in P_\kappa(\lambda)$, $\{b \in A \colon F(a, b\cap\mu) = f(a) \in H\}$. Then there is $C \in H^+$ such that $(H|C) \restriction P_\kappa(\mu)$ is prime.*

Proof. Let $\langle S_a \colon a \in P_\kappa(\lambda)\rangle$ enumerate (possibly with repetition) all subsets of $P_\kappa(\mu)$. Define $F \colon P_\kappa(\lambda) \times P_\kappa(\mu) \to 2$ by $F(a,e) = 0$ if and only if $e \notin S_a$. Select $C \in H^+$ and $f \colon P_\kappa(\lambda) \to 2$ so that for every $a \in P_\kappa(\lambda)$, $\{b \in C \colon F(a, b\cap\mu) = f(a)\} \in H$. Note that if $S_{a'}$ is the complement of S_a, then $f(a') = 1 - f(a)$. Moreover, $f(a) = 1$ if and only if $S_a \in (H|C) \restriction P_\kappa(\mu)$. It clearly follows that $(H|C) \restriction P_\kappa(\mu)$ is a prime ideal. □

Proposition 6 has partial converses. Let us first consider the case when there exists an ideal on $P_\kappa(\mu)$ that is $<\kappa$-saturated.

Proposition 9. *Let τ be a cardinal with $2 \leq \tau < \kappa$, and let H be an ideal on $P_\kappa(\lambda)$ such that $H \restriction P_\kappa(\mu)$ is τ-saturated. Then $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{H} [H^+]^2_\tau$ holds.*

Proof. Given $F\colon P_\kappa(\lambda) \times P_\kappa(\mu) \to \tau$, define $g\colon P_\kappa(\lambda) \to \tau$ so that for every $a \in P_\kappa(\lambda)$,

$$\{e \in P_\kappa(\mu)\colon F(a,e) = g(a)\} \in H \restriction P_\kappa(\mu).$$

We can find $A \in H^+$ and $\xi \in \tau$ so that g takes the constant value ξ on A. Then for every $a \in A$, $\{b \in P_\kappa(\lambda)\colon F(a, b \cap \mu) = \xi\} \in H$. □

For more on the partition property $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\kappa)]{H} [H^+]^2_\rho$ see [9].

For $A \subseteq P_\kappa(\lambda)$, let $[A]^2 = \{(a,b) \in A \times A\colon a \subset b\}$. Given a cardinal $\rho \geq 2$, $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{\subset} [I^+_{\kappa,\lambda}]^2_\rho$ means that for every $F\colon P_\kappa(\lambda) \times P_\kappa(\mu) \to \rho$, one can find $A \in I^+_{\kappa,\lambda}$ and $\xi \in \rho$ so that $F(a, b \cap \mu) \neq \xi$ for every $(a,b) \in [A]^2$.

Note that $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{\subset} [I^+_{\kappa,\lambda}]^2_\rho$ implies $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{w} [I^+_{\kappa,\lambda}]^2_\rho$.

Lemma 10. *Suppose $\mu < \lambda$ and $u(\kappa,\lambda) \in \mathcal{T}_{\kappa,\lambda}$. Let $Q \subseteq P_\kappa(\mu)$, and let $\nu \geq 2$ be a cardinal such that given a partition $\langle S^\xi_\alpha\colon \alpha < \nu\rangle$ of Q for each $\xi < \lambda^{<\kappa}$, there exist $h\colon \lambda^{<\kappa} \to \nu$ with the property that $\bigcap_{\xi \in x}(Q \setminus S^\xi_{h(\xi)}) \in I^+_{\kappa,\mu}$ for every nonempty $x \in P_\kappa(\lambda^{<\kappa})$. Then for each $F\colon P_\kappa(\lambda) \times P_\kappa(\mu) \to \nu$, one can find $A \in I^+_{\kappa,\lambda}$ and $h\colon A \to \nu$ such that $\{a \cap \mu\colon a \in A\} \subseteq Q$, and* (b) *$F(a, b \cap \mu) \neq h(a)$ for every $(a,b) \in [A]^2$.*

Proof. Select $C \in I^+_{\kappa,\lambda}$ so that $|C| = u(\kappa,\lambda)$, and $D \subseteq P_\kappa(\lambda \setminus \mu)$ so that $|D| = u(\kappa,\lambda)$ and $|D \cap P(a)| < \kappa$ for all $a \in P_\kappa(\lambda)$. Pick bijections $c\colon u(k,\lambda) \to C$ and $d\colon u(\kappa,\lambda) \to D$. Now fix $F\colon P_\kappa(\lambda) \times P_\kappa(\mu) \to \nu$. For $a \in P_\kappa(\lambda)$ and $i \in \nu$, set $E^i_a = \{e \in Q\colon F(a,e) \neq i\}$. Select $h\colon P_\kappa(\lambda) \to \nu$ so that $\bigcap_{a \in x} E^{h(a)}_a \in I^+_{\kappa,\mu}$ for every $x \in P_\kappa(P_\kappa(\lambda)) \setminus \{\emptyset\}$. Proceeding by induction, we define $a_\gamma \in P_\kappa(\lambda)$ for $\gamma < u(k,\lambda)$ so that:

(i) $c(\gamma) \cup d(\gamma) \subseteq a_\gamma$.
(ii) $a_\gamma \not\subseteq a_\beta$ whenever $\beta < \gamma$.
(iii) $a_\gamma \cap \mu \in E^{h(a_\beta)}_{a_\beta}$ whenever $\beta < \gamma$ and $d(\beta) \subseteq a_\gamma$.
(iv) $a_0 \cap \mu \in Q$.

The definition of a_0 is easy. Now assume $\gamma > 0$ and we have already constructed a_β for $\beta < \gamma$. Pick $b_\gamma \in P_\kappa(\lambda)$ so that $c(\gamma) \cup d(\gamma) \subseteq b_\gamma$ and $b_\gamma \not\subseteq a_\beta$ for all $\beta < \gamma$. Put $y = \{\beta < \gamma\colon d(\beta) \subseteq b_\gamma\}$ and select $e_\gamma \in \bigcap_{\beta \in y} E^{h(a_\beta)}_{a_\beta}$ so that $b_\gamma \cap \mu \subseteq e_\gamma$. Finally, let $a_\gamma = e_\gamma \cup (b_\gamma \setminus \mu)$.

Set $A = \{a_\gamma\colon \gamma < u(\kappa,\lambda)\}$. Clearly, $A \in I^+_{\kappa,\lambda}$ and $\{a \cap \mu\colon a \in A\} \subseteq Q$. Given $(a,b) \in [A]^2$, let $a = a_\beta$ and $b = a_\gamma$, where $\beta, \gamma \in u(\kappa,\lambda)$. We have $\beta < \gamma$ and $d_\beta \subseteq a_\beta$, so $a_\gamma \cap \mu \in E^{h(a_\beta)}_{a_\beta}$. Hence, $F(a, b \cap \mu) \neq h(a)$. □

The following is a simple consequence of Lemma 10.

Proposition 11.

(i) *Suppose* $\kappa < \lambda$ *and* $u(\kappa, \lambda) \in \mathcal{T}_{\kappa,\lambda}$. *Let* ν *be a cardinal such that* $2 \le \nu < \kappa$ *and there exists a* ν*-saturated ideal on* κ. *Then* $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\kappa)]{\subset} [I^+_{\kappa,\lambda}]^2_\nu$ *holds.*

(ii) *Suppose* $\mu < \lambda$ *and* $u(\kappa, \lambda) \in \mathcal{T}_{\kappa,\lambda}$. *Let* ν *be a cardinal such that* $2 \le \nu < \kappa$ *and there exists a* ν*-saturated ideal on* $P_\kappa(\mu)$. *Then* $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{\subset} [I^+_{\kappa,\lambda}]^2_\nu$ *holds.*

Corollary 12. *Suppose* $\mu < \lambda$ *and* κ *is* μ*-compact. Then* $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{\subset} [I^+_{\kappa,\lambda}]^2_2$ *holds.*

Given a cardinal $\nu \ge \kappa$, κ is said to be *mildly* ν*-ineffable* if given $t_a \colon a \to 2$ for $a \in P_\kappa(\nu)$, there is $g \colon \nu \to 2$ such that for every $a \in P_\kappa(\nu)$, $\{b \in P_\kappa(\nu) \colon a \subseteq b$ and $t_b \restriction a = g \restriction a\} \in I^+_{\kappa,\nu}$.

It is simple to see that if κ is ν-compact, then κ is mildly ν-ineffable. Carr [2] proved that if κ is mildly ν-ineffable, then κ is weakly compact.

Proposition 13. *Suppose* $\mu < \lambda$ *and* κ *is mildly* $\lambda^{<\kappa}$*-ineffable. Then* $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\mu)]{\subset} [I^+_{\kappa,\lambda}]^2_2$.

Proof. This follows from Lemma 10 and the fact [8] that if κ is mildly $\lambda^{<\kappa}$-ineffable and $\langle A^\xi_i \colon i < 2\rangle$ is a partition of $P_\kappa(\lambda)$ for each $\xi < \lambda^{<\kappa}$, then there is $h \colon \lambda^{<\kappa} \to 2$ such that $\bigcap_{\xi \in x} A^\xi_{h(\xi)} \in I^+_{\kappa,\lambda}$ for every nonempty $x \in P_\kappa(\lambda^{<\kappa})$. □

If $2^{2^\kappa} \le \lambda^{<\kappa}$ and $\{P_\kappa(\lambda)\} \xrightarrow[P_\kappa(\kappa)]{\subset} [I^+_{\kappa,\lambda}]^2_2$ holds, then by Corollary 7 κ must be measurable. On the other hand, a result of Levinski can be used to establish the consistency of $\{P_\kappa(\kappa^+)\} \xrightarrow[P_\kappa(\kappa)]{\subset} [I^+_{\kappa,\kappa^+}]^2_2$ and "κ is not measurable":

Proposition 14. *It is consistent relative to the existence of a measurable cardinal that there is a nonmeasurable regular infinite cardinal* χ *such that* $\{P_\chi(\chi^+)\} \xrightarrow[P_\chi(\chi)]{\subset} [I^+_{\chi,\chi^+}]^2_2$ *holds.*

Proof. Starting from a model where GCH holds and χ is a measurable uncountable cardinal, Levinski [7] constructs a generic extension that satisfies: (i) χ is completely ineffable, (ii) χ is the least infinite cardinal ρ such that $2^\rho > \rho^+$, and (iii) given a partition $\langle S^\xi_i \colon i < 2\rangle$ of χ for each $\xi < \chi^+$, there is $h \colon \chi^+ \to 2$ such that $|\bigcap_{\xi \in x} S^\xi_{h(\xi)}| = \kappa$ for every nonempty $x \in P_\chi(\chi^+)$. By Lemma 10, $\{P_\chi(\chi^+)\} \xrightarrow[P_\chi(\chi)]{\subset} [I^+_{\chi,\chi^+}]^2_2$ holds in this generic extension. □

Concerning the consistency of $\mathrm{Part}(\kappa, \kappa^+)$ at a nonmeasurable cardinal, see [10].

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