STRONG COMPACTNESS AND A PARTITION PROPERTY

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Abstract. We show that if Part($\kappa, \lambda$) holds for every $\lambda \geq \kappa$, then $\kappa$ is strongly compact.

Let $\kappa$ be a regular infinite cardinal, and let $\lambda \geq \kappa$ be a cardinal. $P_\kappa(\lambda)$ denotes the set of all subsets of $\lambda$ of size less than $\kappa$. Part($\kappa, \lambda$) means that for every $F: P_\kappa(\lambda) \times P_\kappa(\lambda) \rightarrow 2$, there is a cofinal subset $A$ of $(P_\kappa(\lambda), \subseteq)$ such that $F$ is constant on the set $\{(a, b) \in A \times A: a \subset b\}$. This definition is due to Jech [4]. Jech and Shelah [5] established that Part($\kappa, \kappa^+$) holds for $\kappa = \omega$. We proved in [10] that if $\kappa$ is almost $\lambda^{<\kappa}$-ineffable, then Part($\kappa, \lambda$) holds. It is also known ([5], [11], [12], [8]) that if $\kappa$ is mildly $\lambda^{<\kappa}$-ineffable and $\text{cov}(M_{\kappa, \lambda^{<\kappa}}) > \lambda^{<\kappa}$, then Part($\kappa, \lambda$) holds.

Let $\mu \geq \kappa$ be a cardinal. We will show that if Part($\kappa, 2^{2\mu^{<\kappa}}$) holds, then $\kappa$ is $\mu$-compact. First we recall a few definitions.

Given a cardinal $\nu \geq \kappa$, $I_{\kappa, \nu}$ denotes the set of all $A \subseteq P_\kappa(\nu)$ such that $\{a \in A: b \subseteq a\} = \emptyset$ for some $b \in P_\kappa(\nu)$. By an ideal on $P_\kappa(\nu)$ we mean a subset $K$ of $P(P_\kappa(\nu))$ such that (i) $I_{\kappa, \nu} \subseteq K$, (ii) $P_\kappa(\nu) \notin K$, (iii) $P(A) \subseteq K$ for every $A \in K$, and (iv) $\bigcup T \in K$ for every subset $T$ of $K$ of size less than $\kappa$. We let $K^+ = P(P_\kappa(\nu)) \setminus K$. For $A \in K^+$, we let $K|A = \{B \subseteq P_\kappa(\nu): B \cap A \in K\}$. For a cardinal $\tau \geq 2$, $K$ is $\tau$-saturated if there is no size $\tau$ subset $T$ of $K^+$ with the property that $A \cap B \in K$ for any two distinct members $A, B$ of $T$. $K$ is nowhere $\tau$-saturated if for every $A \in K^+$, $K|A$ is not $\tau$-saturated. $K$ is prime if it is 2-saturated. We say that $\kappa$ is $\nu$-compact if there exists a prime ideal on $P_\kappa(\nu)$.

The following is due to Jech [3].

Lemma 1. If Part($\kappa, \lambda$) holds, then $\kappa$ is weakly compact.

The following is due to Levy and Silver (see [6], Proposition 16.4(b)).

Lemma 2. Let $J$ be an ideal on $P_\kappa(\mu)$. If $J$ is $\kappa$-saturated and $\kappa$ is weakly compact, then $J|A$ is prime for some $A \in J^+$.

Let $J_{\kappa, \mu}$ denote the collection of all nowhere $\kappa$-saturated ideals on $P_\kappa(\mu)$.

Note that an ideal $J$ on $P_\kappa(\mu)$ belongs to $J_{\kappa, \mu}$ if and only if for every $A \in J^+$, there is a partition $\langle D_\xi: \xi < \kappa\rangle$ of $A$ such that $D_\xi \in J^+$ for every $\xi < \kappa$.

The following is essentially due to Taylor (see [13], Theorem 2.2).
Lemma 3. Let $X$ be a subset of $\mathcal{J}_{\kappa,\mu}$ with $0 < |X| < \kappa$. Then there is a partition $\langle D_i : i < 2 \rangle$ of $P_\kappa(\mu)$ such that $D_0, D_1 \in \bigcap_{J \in X} J^+$. 

Proof. Set $\tau = |X|$ and let $X = \{ J_\alpha : \alpha < \tau \}$. For $\alpha < \tau$, select a partition $\langle A^\alpha_\xi : \xi < \kappa \rangle$ of $P_\kappa(\mu)$ so that $A^\alpha_\xi \in J^+_\alpha$ for all $\xi < \kappa$. For $\alpha, \beta < \tau$, set $x^\alpha_\beta = \{ \xi < \kappa : A^\alpha_\xi \in J^+ \}$. Define $h : \tau \to \tau$ by $h(\alpha) = \alpha$ the least $\beta < \tau$ such that $|x^\alpha_\beta| = \kappa$. Then $h(\alpha) = \alpha$ for all $\alpha < \tau$. Now pick a one-to-one set $\lambda < \kappa$.

Throughout the remainder of this paper it is assumed that $\mu < \kappa$.

Lemma 4. If $\text{Part}(\kappa, \lambda)$ holds and $\lambda \geq 2^{\omega < \kappa}$, then $\kappa$ is $\mu$-compact.

Proof. Let $\langle J_\alpha : \alpha < \kappa \rangle$ enumerate (possibly with repetition) all members of $\mathcal{J}_{\kappa,\mu}$. Lemma 8 gives us a partition $\langle D_\alpha^i : i < 2 \rangle$ of $P_\kappa(\mu)$ for each $\alpha \in P_\kappa(\lambda)$ such that $D_{0,\alpha}^i, D_{1,\alpha}^i \in J^+_{\alpha}$ for every $\alpha \in a$. Define $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$ by $F(a, b) = 0$ if and only if $b \cap \mu \in D_{0,\alpha}^i$. Let $A \in I^+_{\kappa,\lambda}$ and $j < 2$ be such that $F$ takes the constant value $j$ on the set $\{ (a, b) \in A \times A : a \cup b \in B \}$. We claim that $I_{\kappa,\lambda} | A \upharpoonright P_\kappa(\mu) \notin \mathcal{J}_{\kappa,\mu}$. Suppose otherwise. Then $I_{\kappa,\lambda} | A \upharpoonright P_\kappa(\mu) = J_\alpha$ for some $\alpha < \kappa$. Since $A \in I^+_{\kappa,\lambda}$, we have an $\alpha \in A$ with $\alpha \in a$. Now $b \cap \mu \in D_{j,\alpha}^i$ for every $b \in A$ with $a \cup b$, so $\{ b \in A : b \cap \mu \notin D_{j,\alpha}^i \} \in I_{\kappa,\lambda}$. On the other hand, $D_{1-i,\alpha}^j \in J^+_{\alpha}$ by $\alpha \in a$. Hence $\{ b \in A : b \cap \mu \notin D_{1-i,\alpha}^j \} \in I_{\kappa,\lambda}$. Together with Lemmas 1 and 2 this contradiction completes the proof.

With a little work we can get a more general version of Theorem 4. We start with some definitions.

Let $H$ be an ideal on $P_\kappa(\lambda)$, let $Z$ be a subset of $P(P_\kappa(\lambda))$, and let $\rho \geq 2$ be a cardinal. $\{ P_\kappa(\lambda) \} \xrightarrow{w} P_\kappa(\mu) \xrightarrow{Z}_{\rho}$ means that for every $F : P_\kappa(\lambda) \times P_\kappa(\mu) \to \rho$, one can find $A \in Z$ and $\xi \in \rho$, so that for every $\alpha \in A$, $\{ b \in A : F(a, b \cap \mu) = \xi \} \in H$. For $H = I_{\kappa,\lambda}$, we write $\{ P_\kappa(\lambda) \} \xrightarrow{w} P_\kappa(\mu) \xrightarrow{w}[Z]_\rho^2$ instead of $\{ P_\kappa(\lambda) \} \xrightarrow{w} P_\kappa(\mu) \xrightarrow{w}[Z]_\rho^2$ (the $w$ stands for “weak”).

Note that $\{ P_\kappa(\lambda) \} \xrightarrow{w} P_\kappa(\mu) \xrightarrow{w}[I^+_{\kappa,\lambda}]_2$ follows from Part$(\kappa, \lambda)$.

We let $u(\kappa, \lambda)$ denote the least size of any member of $I^+_{\kappa,\lambda}$. It is well known (see e.g. [1]) that if $\lambda < \kappa + \omega$, then $u(\kappa, \lambda) = \lambda$.

Note that $\{ P_\kappa(\lambda) \} \xrightarrow{w} P_\kappa(\mu) \xrightarrow{w}[I^+_{\kappa,\lambda}]_{\sigma+}$ holds, where $\sigma = u(\kappa, \lambda)$.

The following is a straightforward strengthening of Lemma 8.
Lemma 5. Let $X$ be a subset of $\mathcal{J}_{\kappa, \mu}$ with $0 < |X| < \kappa$. Then there is a partition 
\[ \langle D_\gamma : \gamma < \kappa \rangle \] of $P_\kappa(\mu)$ such that $D_\gamma \in \bigcap_{J \in X} J^+$ for every $\gamma < \kappa$.

Proposition 6. Suppose that (a) there exists $Z \subseteq P_\kappa(\lambda)$ such that $|Z| = |\mathcal{J}_{\kappa, \mu}|$ and $|Z \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$, (b) $Y \subseteq P_\kappa(\lambda)$ is such that $|Y| \geq 2$ and $|Y \cap P(a)| \leq \kappa$ for all $a \in P_\kappa(\lambda)$, and (c) $H$ is an ideal on $P_\kappa(\lambda)$ such that $\{P_\kappa(\lambda)\} \xrightarrow{H}{P_\kappa(\mu)} [H^+]^2_{|Y|}$. Then there is $B \in H^+$ such that $(H|B) \upharpoonright P_\kappa(\mu)$ is $\kappa$-saturated.

Proof. Select a bijection $u : \mathcal{J}_{\kappa, \mu} \to Z$. Define a partial function $F$ from $P_\kappa(\lambda) \times P_\kappa(\mu)$ to $Y$ as follows. Let $a \in P_\kappa(\lambda)$ with $Z \cap P(a) \neq \emptyset$. By Lemma 5 there is a partition $\langle C^a_d : d \in Y \cap P(a) \rangle$ of $P_\kappa(\mu)$ such that for every $d \in Y \cap P(a)$, $C^a_d \in \bigcap_{J \in u^{-1}(Z \cap P(a))} J^+$. Put $F(a, c) = d$ whenever $c \in C^a_d$.

Pick $A \in H^+$ and $d \in Y$ so that for every $a \in A$, $D_a \in H$, where $D_a = \{b \in A : F(a, b \cap \mu) = d\}$. Assume to the contrary that $J \in \mathcal{J}_{\kappa, \mu}$, where $J = (H|A) \upharpoonright P_\kappa(\mu)$. Pick $a \in A$ with $d \cup u(J) \subseteq a$. Then setting $S = \{b \in A : b \cap \mu \in C^a_d\}$, we obtain $S \in H^+$ and $S \subseteq D_a$. Contradiction! Thus we can find $E \in J^+$ so that $J|E$ is $\kappa$-saturated. Set $B = \{c \in A : c \cap \mu \in E\}$. Obviously, $B \in H^+$. It remains to observe that $(H|B) \upharpoonright P_\kappa(\mu) = J|E$. Thus $B$ is as desired. \hfill \Box

Let $\mathcal{T}_{\kappa, \lambda}$ be the set of all cardinals $\tau$ such that there is a size $\tau$ subset $T$ of $P_\kappa(\lambda)$ with the property that $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$.

It is simple to see that (a) $\lambda \in \mathcal{T}_{\kappa, \lambda}$, and (b) if $\kappa$ is inaccessible, then $\lambda^{<\kappa} \in \mathcal{T}_{\kappa, \lambda}$. For more on $\mathcal{T}_{\kappa, \lambda}$ see [8].

Corollary 7. Suppose that $\kappa$ is weakly compact, $2^{2^{\kappa^{<\kappa}}} \leq \lambda^{<\kappa}$ and $H$ is an ideal on $P_\kappa(\lambda)$ such that $\{P_\kappa(\lambda)\} \xrightarrow{H}{P_\kappa(\mu)} [H^+]_{\lambda^{<\kappa}}^2$. Then there is $C \in H^+$ such that $(H|C) \upharpoonright P_\kappa(\mu)$ is prime.

It also follows from Proposition 8 that if $2^{2^{\kappa^{<\kappa}}} \leq \lambda^{<\kappa}$ and $H$ is an ideal on $P_\kappa(\lambda)$ such that $\{P_\kappa(\lambda)\} \xrightarrow{H}{P_\kappa(\mu)} [H^+]^2$, then there is $C \in H^+$ such that $(H|C) \upharpoonright P_\kappa(\mu)$ is prime. Now it is immediate that $\{P_\kappa(\lambda)\} \xrightarrow{H}{P_\kappa(\mu)} [H^+]^2$ if and only if for every $F : P_\kappa(\lambda) \times P_\kappa(\mu) \to 2$, one can find $A \in H^+$ and $f : P_\kappa(\lambda) \to 2$ so that for every $a \in A$, $\{b \in A : F(a, b \cap \mu) = f(a)\} \in H$. So it is interesting to note the following result which is essentially due to Di Prisco and Zwick (see Lemma 5 in [8]).

Proposition 8. Suppose that $2^{2^{\kappa^{<\kappa}}} \leq \lambda^{<\kappa}$ and $H$ is an ideal on $P_\kappa(\lambda)$ such that for every $F : P_\kappa(\lambda) \times P_\kappa(\mu) \to 2$, one can find $A \in H^+$ and $f : P_\kappa(\lambda) \to 2$ so that for every $a \in P_\kappa(\lambda)$, $\{b \in A : F(a, b \cap \mu) = f(a)\} \in H$. Then there is $C \in H^+$ such that $(H|C) \upharpoonright P_\kappa(\mu)$ is prime.

Proof. Let $\langle S_a : a \in P_\kappa(\lambda) \rangle$ enumerate (possibly with repetition) all subsets of $P_\kappa(\mu)$. Define $F : P_\kappa(\lambda) \times P_\kappa(\mu) \to 2$ by $F(a, e) = 0$ if and only if $e \not\in S_a$. Select $C \in H^+$ and $f : P_\kappa(\lambda) \to 2$ so that for every $a \in P_\kappa(\lambda)$, $\{b \in C : F(a, b \cap \mu) = f(a)\} \in H$. Note that if $S_a$ is the complement of $S_a$, then $f(a^c) = 1 - f(a)$. Moreover, $f(a) = 1$ if and only if $S_a \in (H|C) \upharpoonright P_\kappa(\mu)$. It clearly follows that $(H|C) \upharpoonright P_\kappa(\mu)$ is a prime ideal. \hfill \Box
Proposition 9 has partial converses. Let us first consider the case when there exists an ideal on $P_\kappa(\mu)$ that is $<\kappa$-saturated.

**Proposition 9.** Let $\tau$ be a cardinal with $2 \leq \tau < \kappa$, and let $H$ be an ideal on $P_\kappa(\lambda)$ such that $H \upharpoonright P_\kappa(\mu)$ is $\tau$-saturated. Then $\{P_\kappa(\lambda)\} \xrightarrow{H}{P_\kappa(\mu)} [H^+]^2$ holds.

**Proof.** Given $F: P_\kappa(\lambda) \times P_\kappa(\mu) \to \tau$, define $g: P_\kappa(\lambda) \to \tau$ so that for every $a \in P_\kappa(\lambda)$,

$$\{e \in P_\kappa(\mu): F(a,e) = g(a)\} \in H \upharpoonright P_\kappa(\mu).$$

We can find $A \subseteq H^+$ and $\xi \in \tau$ so that $g$ takes the constant value $\xi$ on $A$. Then for every $a \in A$, $\{b \in P_\kappa(\lambda): F(a,b \cap \mu) = \xi\} \in H$.

For more on the partition property $\{P_\kappa(\lambda)\} \xrightarrow{H}{P_\kappa(\mu)} [H^+]^2$ see [9].

For $A \subseteq P_\kappa(\lambda)$, let $[A]^2 = \{(a,b) \in A \times A: a \subset b\}$. Given a cardinal $\rho \geq 2$, $\{P_\kappa(\lambda)\} \xrightarrow{P_\kappa(\mu)} [I_{\kappa,\lambda}^+]^2$ means that for every $F: P_\kappa(\lambda) \times P_\kappa(\mu) \to \rho$, one can find $A \subseteq I_{\kappa,\lambda}^+$ and $\xi \in \rho$ so that $F(a,b \cap \mu) \neq \xi$ for every $(a,b) \in [A]^2$.

Note that $\{P_\kappa(\lambda)\} \xrightarrow{P_\kappa(\mu)} [I_{\kappa,\lambda}^+]^2$ implies $\{P_\kappa(\lambda)\} \xrightarrow{P_\kappa(\mu)} [I_{\kappa,\lambda}^+]^2$.

**Lemma 10.** Suppose $\mu < \lambda$ and $u(\kappa,\lambda) \in T_{\kappa,\lambda}$. Let $Q \subseteq P_\kappa(\mu)$, and let $\nu \geq 2$ be a cardinal such that given a partition $\{S^x_\alpha: \alpha < \kappa\}$ of $Q$ for each $\xi < \kappa$, there exist $h: \lambda^{<\kappa} \to \nu$ with the property that $\bigcap_{\xi < \kappa \leq \lambda} (Q \cap S^x_\xi) \subseteq I_{\kappa,\mu}^+$ for every nonempty $x \in P_\kappa(\lambda)$. Then for each $F: P_\kappa(\lambda) \times P_\kappa(\mu) \to \nu$, one can find $A \in I_{\kappa,\lambda}^+$ and $h: A \to \nu$ such that $(a \cap \mu: a \in A) \subseteq Q$, and (b) $F(a,b \cap \mu) \neq h(a)$ for every $(a,b) \in [A]^2$.

**Proof.** Select $C \subseteq I_{\kappa,\lambda}^+$ so that $|C| = u(\kappa,\lambda)$, and $D \subseteq P_\kappa(\lambda \setminus \mu)$ so that $|D| = u(\kappa,\lambda)$ and $|D \cap P(\mu)| < \kappa$ for all $a \in P_\kappa(\lambda)$. Pick bijections $c: u(\kappa,\lambda) \to C$ and $d: u(\kappa,\lambda) \to D$. Now fix $F: P_\kappa(\lambda) \times P_\kappa(\mu) \to \nu$. For $a \in P_\kappa(\lambda)$ and $i \in \nu$, set $E^a_i = \{e \in Q: F(a,e) \neq i\}$. Select $h: P_\kappa(\lambda) \to \nu$ so that $\bigcap_{a \in x} E^a_i \subseteq I_{\kappa,\mu}^+$ for every $x \in P_\kappa(\lambda\setminus\{\emptyset\})$. Proceeding by induction, we define $a_\gamma \in P_\kappa(\lambda)$ for $\gamma < u(\kappa,\lambda)$ so that:

(i) $c(\gamma) \cup d(\gamma) \subseteq a_\gamma$.
(ii) $a_\gamma \notin a_\beta$ whenever $\beta < \gamma$.
(iii) $a_\gamma \cap \mu \in E^a_{\beta(a)}$ whenever $\beta < \gamma$ and $d(\beta) \subseteq a_\gamma$.
(iv) $a_0 \cap \mu = Q$.

The definition of $a_0$ is easy. Now assume $\gamma > 0$ and we have already constructed $a_\beta$ for $\beta < \gamma$. Pick $b_\gamma \in P_\kappa(\lambda)$ so that $c(\gamma) \cap d(\gamma) \subseteq b_\gamma$ and $b_\gamma \not\subseteq a_\beta$ for all $\beta < \gamma$. Put $y = \{\beta < \gamma: d(\beta) \subseteq b_\gamma\}$ and select $e_\gamma \in \bigcap_{\beta \in y} E^a_{\beta(a)}$ so that $b_\gamma \subseteq e_\gamma$. Finally, let $a_\gamma = e_\gamma \cup (b_\gamma \setminus \mu)$.

Set $A = \{a_\gamma: \gamma < u(\kappa,\lambda)\}$. Clearly, $A \in I_{\kappa,\lambda}^+$ and $\{a \cap \mu: a \in A\} \subseteq Q$. Given $(a,b) \in [A]^2$, let $a = a_\beta$ and $b = a_\gamma$, where $\beta, \gamma \in u(\kappa,\lambda)$. We have $\beta < \gamma$ and $d_\beta \subseteq a_\beta$, so $a_\gamma \cap \mu \in E^a_{\beta(a)}$. Hence, $F(a,b \cap \mu) \neq h(a)$.
The following is a simple consequence of Lemma [10]

**Proposition 11.**
1. Suppose $\kappa < \lambda$ and $u(\kappa, \lambda) \in T_{\kappa, \lambda}$. Let $\nu$ be a cardinal such that $2 \leq \nu < \kappa$ and there exists a $\nu$-saturated ideal on $\kappa$. Then $\{P_\kappa(\lambda)\} \overset{\subset}{\longrightarrow} [I_{\kappa, \lambda}^+]^2_\nu$ holds.
2. Suppose $\mu < \lambda$ and $u(\kappa, \lambda) \in T_{\kappa, \lambda}$. Let $\nu$ be a cardinal such that $2 \leq \nu < \kappa$ and there exists a $\nu$-saturated ideal on $P_\mu(\mu)$. Then $\{P_\kappa(\lambda)\} \overset{\subset}{\longrightarrow} [I_{\kappa, \lambda}^+]^2_\nu$ holds.

**Corollary 12.** Suppose $\mu < \lambda$ and $\kappa$ is $\mu$-compact. Then $\{P_\kappa(\lambda)\} \overset{\subset}{\longrightarrow} [I_{\kappa, \lambda}^+]^2_\mu$ holds.

Given a cardinal $\nu \geq \kappa$, $\kappa$ is said to be mildly $\nu$-ineffable if given $t_a : a \to 2$ for $a \in P_\kappa(\nu)$, there is $g : \nu \to 2$ such that for every $a \in P_\kappa(\nu)$, $\{b \in P_\kappa(\nu) : a \subseteq b\}$ and $t_b \upharpoonright a = g \upharpoonright a \in I_{\kappa, \nu}^{-}$.

It is simple to see that if $\kappa$ is $\mu$-compact, then $\kappa$ is mildly $\nu$-ineffable. Carr [2] proved that if $\kappa$ is mildly $\nu$-ineffable, then $\kappa$ is weakly compact.

**Proposition 13.** Suppose $\mu < \lambda$ and $\kappa$ is mildly $\lambda^{<\kappa}$-ineffable. Then $\{P_\kappa(\lambda)\} \overset{\subset}{\longrightarrow} [I_{\kappa, \lambda}^+]^2_\mu$.

**Proof.** This follows from Lemma [10] and the fact [8] that if $\kappa$ is mildly $\lambda^{<\kappa}$-ineffable and $\{A_i : i < 2\}$ is a partition of $P_\kappa(\lambda)$ for each $\xi < \lambda^{<\kappa}$, then there is $h : \lambda^{<\kappa} \to 2$ such that $\bigcap_{\xi \in \chi} A_{h(\xi)} = I_{\kappa, \lambda}^{-}$ for every nonempty $\chi \in P_\kappa(\lambda^{<\kappa})$. □

If $2^{2^\mu} \leq \lambda^{<\kappa}$ and $\{P_\kappa(\lambda)\} \overset{\subset}{\longrightarrow} [I_{\kappa, \lambda}^+]^2_\mu$ holds, then by Corollary [10] $\kappa$ must be measurable. On the other hand, a result of Levinski can be used to establish the consistency of $\{P_\kappa(\kappa^+)\} \overset{\subset}{\longrightarrow} [I_{\kappa, \kappa^+}^+]^2_\mu$ and “$\kappa$ is not measurable”:

**Proposition 14.** It is consistent relative to the existence of a measurable cardinal that there is a nonmeasurable regular infinite cardinal $\chi$ such that $\{P_\chi(\chi^+)\} \overset{\subset}{\longrightarrow} [I_{\chi, \chi^+}^+]^2_\mu$.

**Proof.** Starting from a model where GCH holds and $\chi$ is a measurable uncountable cardinal, Levinski [7] constructs a generic extension that satisfies: (i) $\chi$ is completely ineffable, (ii) $\chi$ is the least infinite cardinal $\rho$ such that $2^\rho > \rho^+$, and (iii) given a partition $\{S_i : i < 2\}$ of $\chi$ for each $\xi < \chi^+$, there is $h : \chi^+ \to 2$ such that $|\bigcap_{\xi \in x} S_{h(\xi)}| = \kappa$ for every nonempty $x \in P_\chi(\chi^+)$. By Lemma [10] $\{P_\chi(\chi^+)\} \overset{\subset}{\longrightarrow} [I_{\chi, \chi^+}^+]^2_\mu$ holds in this generic extension. □

Concerning the consistency of $\text{Part}(\kappa, \kappa^+)$ at a nonmeasurable cardinal, see [10].

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