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# THE NUMERICAL INDEX OF THE $L_p$ SPACE

ELMOULOUDI ED-DARI AND MOHAMED AMINE KHAMSI

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Abstract. We give a partial answer to the problem of computing the numerical index of $L_p[0,1]$ for $1 < p < \infty$.

## 1. Introduction

Given a Banach space $X$ over $\mathbb{R}$ or $\mathbb{C}$, we write $B_X$ for the closed unit ball and $S_X$ for the unit sphere of $X$. The dual space is denoted by $X^*$, and $B(X)$ is the Banach algebra of all bounded linear operators on $X$. The *numerical range* of an operator $T \in B(X)$ is the subset $V(T)$ of the scalar field defined by

$$V(T) = \{x^*(Tx) : \ x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}.$$

The *numerical radius* is then given by

$$v(T) = \sup\{|\lambda| : \ \lambda \in V(T)\}.$$

Clearly, $v$ is a semi-norm on $B(X)$, and $v(T) \le \|T\|$ for every $T \in B(X)$. It was shown by Glickfeld [8] (and essentially by Bohnenblust and Karlin [3]) that if $X$ is a complex space, then $e^{-1}\|T\| \le v(T)$ for every $T \in B(X)$ where $e = \exp 1$, so that for complex spaces $v$ is always a norm and is equivalent to the operator norm $\|\ \|$. Thus it is natural to consider the so-called *numerical index* of the Banach space $X$, namely the constant $n(X)$ defined by

$$n(X) = \inf\{v(T) : \ T \in S_{B(X)}\}.$$

Obviously, $n(X)$ is the greatest constant $k \ge 0$ such that $k\|T\| \le v(T)$ for every $T \in B(X)$. Note that for any complex Banach space $X$, $e^{-1} \le n(X) \le 1$.

The concept of the numerical index was first suggested by G. Lumer [10] in a lecture at the North British Functional Analysis Seminar in 1968. At that time, it was known that if $X$ is a complex Hilbert space (with $dim X > 1$), then $n(X) = \frac{1}{2}$, and if it is real, then $n(X) = 0$ so that for real spaces, $0 \le n(X) \le 1$.

Later, Duncan, McGregor, Pryce and White [5] determined the range of values of the numerical index. More precisely they proved that

$$\{n(X) : \ X \text{ real Banach space}\} = [0,1],$$
$$\{n(X) : \ X \text{ complex Banach space}\} = [e^{-1},1].$$

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Recently, Finet, Martín and Payá [7] also studied the values of the numerical index from the isomorphic point of view, and they proved that

$$\mathcal{N}(c_0) = \mathcal{N}(l_1) = \mathcal{N}(l_\infty) = \begin{cases} [0,1] & \text{in the real case,} \\ [e^{-1}, 1] & \text{in the complex case,} \end{cases}$$

where $\mathcal{N}(X) := \{n(X,p) : p \in \mathcal{E}(X)\}, \mathcal{E}(X)$ denotes the set of all equivalent norms on the Banach space $X$ and $n(X,p)$ is the numerical index of $X$ equipped with the norm $p$.

The authors in [5] proved that the extreme case $n(X) = 1$ occurs for a large class of interesting spaces including all real or complex L-spaces and M-spaces. Lately, López, Martín and Payá [11] studied some real Banach spaces with numerical index 1. In fact, they proved that an infinite-dimensional real Banach space with numerical index 1 and satisfying the Radon-Nikodym property contains $l_1$. This result is a partial answer to the conjecture [12]: Every infinite-dimensional Banach space with numerical index 1 contains either $l_1$ or $c_0$. Very recently, Martín and Payá [13] studied the numerical index of vector-valued function spaces, and they proved that if $K$ is a compact Hausdorff space and $\mu$ is a positive measure, then the Banach spaces $C(K,X)$ and $L_1(\mu, X)$ have the same numerical index as the Banach space $X$. For general information and background on numerical ranges, we refer to the books by Bonsall and Duncan [1], [2]. Further developments in the Hilbert space may be found in [9].

The computation of $n(L_p)$ for $1 < p < \infty$, $p \neq 2$ is much more complicated, in fact it is an open problem since 1968.

In this paper we give a partial answer to this problem (Theorem 2.1). Actually, we prove that for $1 \leq p < \infty$, the numerical index of the Banach space $L_p([0,1], \mu)$, where $\mu$ is the Lebesgue measure on the unit interval, is equal to the numerical index of the $l_p$ space. It is also known that the numerical index of the Banach space $l_p$ is the limit of the sequence of numerical index of finite-dimensional subspaces $l_p^m$, $m = 1, 2, ...,$ [6]. The computation of the numerical index of the $l_p^m$-space then gives a complete answer to the problem of the numerical index of the $L_p$-space.

## 2. Main results

**Theorem 2.1.** *For $1 < p < \infty$, the numerical index of the Banach space $L_p[0,1]$ is equal to the numerical index of $l_p$, i.e.,*

$$n(L_p[0,1]) = n(l_p).$$

Before, we prove this theorem, we need to introduce some notation.

Let $\mu$ be the Lebesgue measure on $\Omega = [0,1]$. For each integer $n \geq 1$ we denote by $\pi_{2^n}$ the partition of $\Omega$ into $2^n$ dyadic intervals: $[0, \frac{1}{2^n}[, [\frac{1}{2^n}, \frac{2}{2^n}[, \ldots, [\frac{2^n-1}{2^n}, 1]$. By $V_n$ we denote the subspace of $L_p(\Omega, \mu)$ defined by

$$V_n = \left\{ \sum_{k=1}^{2^n} a_k 1_{[\frac{k-1}{2^n}, \frac{k}{2^n}[} : \ a_k \in \mathbb{K} \right\},$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. $P_n$ denotes the projection of $L_p(\Omega, \mu)$ onto $V_n$ defined by

$$P_n(f) = \sum_{k=1}^{2^n} \left[ \frac{1}{2^{-n}} \int_{[\frac{k-1}{2^n}, \frac{k}{2^n}]} f(t)\, d\mu(t) \right] 1_{[\frac{k-1}{2^n}, \frac{k}{2^n}]},$$

and $V$ denotes the union of all subspaces $V_n$ of $L_p(\Omega,\mu)$. We recall that $V$ is a dense subspace of $L_p(\Omega,\mu)$ ([4], p. 140), that is, for each $f \in L_p(\Omega,\mu)$ the sequence $(P_n f)_n$ converges to $f$ in $L_p(\Omega,\mu)$.

**Lemma 2.2.** *For each integer $n \geq 1$ and $T \in B(V_n)$ there exists $\tilde{T} \in B(V_{n+1})$ satisfying the following conditions:*
(i) $\tilde{T}_{|V_n} = T$,
(ii) $\|\tilde{T}\| = \|T\|$,
(iii) $v(\tilde{T}) = v(T)$.

*Proof.* Let $E_k^{2^n}$ denote the interval $[\frac{k-1}{2^n}, \frac{k}{2^n}[$ for $k = 1, 2, ..., 2^n$.

STEP 1. *We shall first prove that the conditions in Lemma* 2.2 *are satisfied for* $n = 1$.

Let $T \in B(V_1)$ be represented by its matrix $\left[\begin{smallmatrix} t_{11} & t_{12} \\ t_{21} & t_{22}\end{smallmatrix}\right]$ on the unit normal basis $\left\{\frac{1_{E_1^{2^1}}}{\mu(E_1^{2^1})^{\frac{1}{p}}}, \frac{1_{E_2^{2^1}}}{\mu(E_2^{2^1})^{\frac{1}{p}}}\right\}$ of $V_1$. Since $\mu(E_1^{2^1})^{\frac{1}{p}} = \mu(E_2^{2^1})^{\frac{1}{p}} = 2^{-\frac{1}{p}}$, the same matrix represents $T$ on the basis $\left\{1_{E_1^{2^1}}, 1_{E_2^{2^1}}\right\}$. Consider the operator $\tilde{T} \in B(V_2)$ represented by its matrix

$$\begin{bmatrix} t_{11} & 0 & t_{12} & 0 \\ 0 & t_{11} & 0 & t_{12} \\ t_{21} & 0 & t_{22} & 0 \\ 0 & t_{21} & 0 & t_{22} \end{bmatrix}$$

on the normal basis $\left\{\frac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{p}}}\right\}_{k=1,\dots,4}$ of $V_2$. Also, since $\mu(E_k^{2^2}) = 2^{\frac{-2^2}{p}}$, $k = 1, ..., 4$, the same matrix represents $\tilde{T}$ on the basis $\{1_{E_k^{2^2}}\}_{k=1,\dots,4}$.

For (i), we have

$$\begin{aligned} \tilde{T}(1_{E_1^{2^1}}) &= \tilde{T}(1_{E_1^{2^2}}) + \tilde{T}(1_{E_2^{2^2}}) \\ &= t_{11}1_{E_1^{2^2}} + t_{21}1_{E_3^{2^2}} + t_{11}1_{E_2^{2^2}} + t_{21}1_{E_4^{2^2}} = T(1_{E_1^{2^1}}). \end{aligned}$$

Also, $\tilde{T}(1_{E_2^{2^1}}) = T(1_{E_2^{2^1}})$. This means that $\tilde{T}_{|V_1} = T$.

For (ii), let $x = \sum_{k=1}^4 x_k \frac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{p}}} \in V_2$. Clearly, $\|x\|^p = \sum_{k=1}^4 |x_k|^p$, and we have

$$\tilde{T}(x) = \begin{bmatrix} t_{11} & 0 & t_{12} & 0 \\ 0 & t_{11} & 0 & t_{12} \\ t_{21} & 0 & t_{22} & 0 \\ 0 & t_{21} & 0 & t_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t_{11}x_1 + t_{12}x_3 \\ t_{11}x_2 + t_{12}x_4 \\ t_{21}x_1 + t_{22}x_3 \\ t_{21}x_2 + t_{22}x_4 \end{bmatrix}.$$

From this we obtain

$$\begin{aligned} \|\tilde{T}(x)\|^p &= |t_{11}x_1 + t_{12}x_3|^p + |t_{11}x_2 + t_{12}x_4|^p + |t_{21}x_1 + t_{22}x_3|^p + |t_{21}x_2 + t_{22}x_4|^p \\ &= \left\| \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \right\|^p + \left\| \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \right\|^p \\ &\leq \|T\|^p(|x_1|^p + |x_2|^p + |x_3|^p + |x_4|^p). \end{aligned}$$

This means that $\|\tilde{T}\| \le \|T\|$ and since $\tilde{T}_{|V_1} = T$ we have $\|T\| \le \|\tilde{T}\|$. Therefore $\|\tilde{T}\| = \|T\|$.

For (iii), we have

$$v(\tilde{T}) = \sup\{|x_x^*(\tilde{T}x)| : \ x \in S_{V_2}\} = |x_x^*(\tilde{T}x)|$$

for some

$$x = \sum_{k=1}^{4} x_k \frac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{p}}}, \quad \|x\|^p = \sum_{k=1}^{4} |x_k|^p = 1, \quad x_x^* = \sum_{k=1}^{4} \varepsilon_k |x_k|^{p-1} \frac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{q}}}$$

where $\varepsilon_k$ is a scalar number such that $\varepsilon_k x_k = |x_k|$. From the previous expression of $\tilde{T}(x)$ we have

$$\begin{aligned} x_x^*(\tilde{T}x) &= \Big(\varepsilon_1|x_1|^{p-1}, ..., \varepsilon_4|x_4|^{p-1}\Big) \begin{bmatrix} t_{11}x_1 + t_{12}x_3 \\ t_{11}x_2 + t_{12}x_4 \\ t_{21}x_1 + t_{22}x_3 \\ t_{21}x_2 + t_{22}x_4 \end{bmatrix} \\ &= \Big(\varepsilon_1|x_1|^{p-1}, \varepsilon_3|x_3|^{p-1}\Big) \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \\ &\quad + \Big(\varepsilon_2|x_2|^{p-1}, \varepsilon_4|x_4|^{p-1}\Big) \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \big|x_x^*(\tilde{T}x)\big| &\le v(T)\big\|\big(\varepsilon_1|x_1|^{p-1}, \varepsilon_3|x_3|^{p-1}\big)\big\|_q \big\|(x_1, x_3)\big\|_p \\ &\quad + v(T)\big\|\big(\varepsilon_2|x_2|^{p-1}, \varepsilon_4|x_4|^{p-1}\big)\big\|_q \big\|\big(x_2, x_4\big)\big\|_p \le v(T). \end{aligned}$$

Since $\tilde{T}_{|V_1} = T$, we also have $v(T) \le v(\tilde{T})$ and consequently $v(\tilde{T}) = v(T)$.

STEP 2. *For every integer $n \ge 1$ and $T \in B(V_n)$, there exists $\tilde{T} \in B(V_{n+1})$ satisfying conditions* (i), (ii) *and* (iii).

Indeed, let $n \ge 1$ and let $T \in B(V_n)$ be represented by its matrix

$$\begin{bmatrix} t_{11} & \cdots & t_{12^n} \\ \vdots & & \vdots \\ t_{2^n1} & \cdots & t_{2^n2^n} \end{bmatrix}$$

on the normal basis $\left\{\frac{1_{E_k^{2^n}}}{\mu(E_k^{2^n})^{\frac{1}{p}}}\right\}_{k=1,\dots,2^n}$ of $V_n$. As in Step 1, we check that the operator $\tilde{T} \in B(V_{n+1})$ defined by its matrix

$$\begin{bmatrix} \begin{pmatrix} t_{11} & 0 \\ 0 & t_{11} \end{pmatrix} & \cdots & \begin{pmatrix} t_{12^n} & 0 \\ 0 & t_{12^n} \end{pmatrix} \\ \vdots & & \vdots \\ \begin{pmatrix} t_{2^n1} & 0 \\ 0 & t_{2^n1} \end{pmatrix} & \cdots & \begin{pmatrix} t_{2^n2^n} & 0 \\ 0 & t_{2^n2^n} \end{pmatrix} \end{bmatrix}$$

as in the normal basis $\left\{\frac{1_{E_k^{2^{(n+1)}}}}{\mu(E_k^{2^{(n+1)}})^{\frac{1}{p}}}\right\}_k$, as in $\left\{1_{E_k^{2^{(n+1)}}}\right\}_k$ of $V_{n+1}$, satisfies the three conditions (i), (ii) and (iii). $\square$

*Proof of Theorem* 2.1. We shall prove that the numerical index of $L_p[0,1]$ is less than the numerical index of $l_p$.

Let $n \geq 1$ and let $U := T_n \in B(V_n)$. Following the previous lemma we can find a sequence of operators $\{T_m\}_{m\geq n}$, $T_m : V_m \to V_m$, satisfying the following conditions:

(i) $T_{m+1|_{V_m}} = T_m$,
(ii) $\|T_{m+1}\| = \|T_m\|$,
(iii) $v(T_{m+1}) = v(T_m)$.

Let $x \in L_p[0,1]$ and let $x_m \in V_m$ for $m \geq n$ such that $x_m$ converges to $x$ in $L_p[0,1]$. The sequence $\{T_m(x_m)\}_{m\geq n}$ converges in $L_p[0,1]$. Indeed, for $m, m^{'} \geq n$ with $m^{'} \geq m$, we have

$$T_{m'}(x_{m'}) - T_m(x_m) = T_{m'}(x_{m'} - x_m) + (T_{m'} - T_m)(x_m).$$

Since $T_{m'}{}_{|_{V_m}} = T_m$ and $\|T_{m'}\| = \|T_n\|$, we obtain

$$\|T_{m'}(x_{m'}) - T_m(x_m)\| = \|T_{m'}(x_{m'} - x_m)\| \leq \|T_n\|\|x_{m'} - x_m\|.$$

Let $T(x) = \lim_m T_m(x_m)$ (note that $T(x)$ is independent of the choice of $\{x_m\}_m$). Clearly, $T$ is a bounded linear operator on $L_p[0,1]$. Moreover,

(i) $T_{|_{V_n}} = T_n$,
(ii) $\|T\| = \|T_n\|$,
(iii) $v(T) = v(T_n)$.

The first conditions (i) and (ii) are clear. For (iii), let $\varepsilon > 0$. There exists $x \in L_p[0,1]$ such that

$$v(T) - \varepsilon \leq |x_x^*(Tx)|.$$

Here $x_x^* = \varepsilon_x |x|^{p-1}$, where $\varepsilon_x$ is the sign function of $x$, that is, $\varepsilon_x(t)x(t) = |x(t)|$ for all $t \in [0,1]$. But $T(x) = \lim_m T_m(x_m)$ for $x_m$ converges to $x$ in $L_p[0,1]$ and $x_m \in S_{V_m}$ for all $m$. Since the norm of $L_p[0,1]$ is Fréchet differentiable [4], the sequence $x_{x_m}^*$ converges to $x_x^*$ in $L_q[0,1]$. Consequently

$$v(T) - \varepsilon \leq |x_x^*(Tx)| = \lim_m |x_{x_m}^*(T_m x_m)| \leq \lim_m v(T_m) = v(T_n),$$

which means that $v(T) \leq v(T_n)$. Since $T_{|_{V_n}} = T_n$, we have also $v(T_n) \leq v(T)$ and therefore $v(T) = v(T_n)$. It follows from this that

$$\{v(U) : \ U \in S_{V_n}\} \subset \{v(T) : \ T \in S_{L_p[0,1]}\},$$

and then

$$n(L_p[0,1]) \leq n(V_n).$$

Since $V_n$ is isometric to $l_p^{2^n}$, we obtain

$$n(L_p[0,1]) \leq n(l_p^{2^n}).$$

According to [6, Theorem 2.5] we get

$$n(L_p[0,1]) \leq \lim_n n(l_p^{2^n}) = n(l_p).$$

Also, it was proved [6, Theorem 3.1] that

$$n(L_p(\mu)) \geq n(l_p)$$

for any Banach space $L_p(\mu), 1 \leq p < \infty$. This completes the proof of Theorem 2.1. □

D. Li and C. Finet asked (in a communication) if the numerical index of the real Banach space $l_p$ is equal to 0 for some $p \neq 2$.

In the following we prove that the numerical index of all Banach spaces $l_p^m, p \neq 2, m = 1, 2, ...$, cannot be equal to 0.

**Theorem 2.3.** *For every real number $p \neq 2$ and $m = 2, 3, \ldots$, the numerical index of the Banach space $l_p^m$ is positive.*

*Proof.* It is enough to give the proof for the case $1 < p < 2$, since $n(X) = n(X^*)$ when $X$ is a reflexive space. Fix a real number $p$, $1 < p < 2$, and let $m$ be an integer with $m \geq 2$. Consider

$$y = \frac{e_i + \varepsilon t e_j}{(1+t^p)^{\frac{1}{p}}} \text{ and } y^* = \frac{e_i^* + \varepsilon t^{p-1} e_j^*}{(1+t^p)^{\frac{p-1}{p}}}, \ \ 0 \leq t \leq 1, \ \ \varepsilon \in \{-1, 1\}.$$

Here $\{e_i\}$ is the canonical basis of $l_p^m$. Clearly, $\|y\|_{l_p^m} = \|y^*\|_{l_q^m} = y^*(y) = 1$. For $T = (t_{lk})_{lk} \in B(l_p^m)$ and by a simple calculus we have

$$y^*(Ty) = \frac{1}{1+t^p}(t_{ii} + \varepsilon t \ t_{ij} + \varepsilon t^{p-1} t_{ji} + t^p t_{jj}).$$

From this we obtain

$$|t_{ii} + \varepsilon t \ t_{ij} + \varepsilon t^{p-1} t_{ji} + t^p t_{jj}| \leq (1+t^p) v(T).$$

Then

$$\max_{\varepsilon \in \{-1,1\}} |t_{ii} + \varepsilon t \ t_{ij} + \varepsilon t^{p-1} t_{ji} + t^p t_{jj}| = |t_{ii} + t^p t_{jj}| + |t \ t_{ij} + t^{p-1} t_{ji}| \leq (1+t^p) v(T).$$

Since

$$|t_{ii} + t^p t_{jj}| \geq |t_{ii}| - t^p |t_{jj}| \ \text{ and } \ |t \ t_{ij} + t^{p-1} t_{ji}| \geq t^{p-1}|t_{ji}| - t|t_{ij}|,$$

we then have

$$|t_{ii}| - t^p|t_{jj}| + t^{p-1}|t_{ji}| - t|t_{ij}| \leq (1+t^p) v(T).$$

Similarly

$$|t_{jj}| - t^p|t_{ii}| + t^{p-1}|t_{ij}| - t|t_{ji}| \leq (1+t^p) v(T).$$

From the last two inequalities we obtain

$$(1-t^p)(|t_{ii}| + |t_{jj}|) + (t^{p-1} - t)(|t_{ij}| + |t_{ji}|) \leq 2(1+t^p) v(T).$$

Since $(1-t^p) \geq t^p - t$, we have

$$(t^{p-1} - t)(|t_{ii}| + |t_{ij}| + |t_{ji}| + |t_{jj}|) \leq 2(1+t^p) v(T).$$

Hence

$$(*) \qquad \frac{M_p}{2}(|t_{ii}| + |t_{ij}| + |t_{ji}| + |t_{jj}|) \leq v(T),$$

where $M_p = \sup_{t \in [0,1]} \dfrac{t^{p-1} - t}{1+t^p}$. Thus if $v(T) = 0$ for $T \in B(l_p^m)$, we have necessarily $\|T\| = 0$ and consequently $n(l_p^m) = v(T) > 0$ for some $T \in S_{l_p^m}$. □

*Remark.* Following $(*)$ in the previous proof, $v$ is a norm on $B(l_p)$ for $p \neq 2$. It is still unknown if it is an equivalent norm to the operator norm.

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Faculté des Sciences Jean Perrin, Université D'Artois, SP 18, 62307-Lens Cedex, France

*E-mail address*: Elmouloudi.Eddari@euler.univ.artois.fr

Department of Mathematical Sciences, University of Texas at El Paso, 500 West University Avenue, El Paso, Texas 79968-0514

*E-mail address*: mohamed@math.utep.edu