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ON THE LOCAL STRUCTURE OF RANK-ONE CONVEX HULLS

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ABSTRACT. In this note we prove that if K is a compact set of $m \times n$ matrices containing an isolated point X with no rank-one connection into the convex hull of $K \setminus \{X\}$, then the rank-one convex hull separates as

$$K^{rc} = \big(K \setminus \{X\}\big)^{rc} \cup \{X\}.$$

This is an extension of a result of P. Pedregal, which holds for 2×2 matrices.

1. INTRODUCTION

In recent years there have been a number of results concerning the construction of counterexamples to various problems using convex integration, for example [7], [9], [1] and [2]. In such results a key point is to construct a laminate (see definition below) or sequence of laminates satisfying certain conditions coming from the specific problem and supported in a finite set of matrices. In order to make the construction of such examples more systematic, one would like to have easy criteria for deciding whether a given set of matrices can support nontrivial (i.e. non-Dirac) laminates. This amounts to finding necessary conditions for *nontrivial inclusion-minimal* configurations, i.e. sets $K \subset \mathbb{R}^{m\times n}$ with the property that $K^{rc} \neq K$ but $\tilde{K}^{rc} = \tilde{K}$ for any proper subset $\tilde{K} \subset K$. Such configurations in $\mathbb{R}^{2\times 2}$ have been completely classified in [10], but in higher dimensions such a classification is beyond hope, partly due to the fact that there exist arbitrarily large finite nontrivial inclusion-minimal configurations (see [6] and [5]).

A natural necessary condition for nontrivial inclusion-minimality that one would expect is that for isolated points $X \in K$ there should be a point Y in the convex hull of $K \setminus \{X\}$ such that $\operatorname{rank}(Y - X) = 1$. This is suggested by the fact that it holds for all known examples. Moreover, this was stated (for finite sets) in Corollary 5.6 in [8]. However, as we shall demonstrate below, the proof given there only works for the case $\mathbb{R}^{2\times 2}$.

For simplicity, from now on we will take the origin as the isolated point, and consider the rank-one convex hull of the set $\{0\} \cup K$, where K is compact, convex and does not contain 0. Our result is the following theorem.

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Theorem 1. *Let $K \subset \mathbb{R}^{m\times n}$ be a convex, compact set not containing the origin, and suppose that*

$$(\{0\} \cup K)^{rc} \neq \{0\} \cup K. \tag{1}$$

Then K contains a matrix of rank 1.

Observe that (1) holds, in particular if $K \cup \{0\}$ is assumed to be nontrivial inclusion-minimal (cf. Theorem 2). It should be pointed out that the statement of the theorem does not imply the existence of a rank-one line connecting 0 with K which is **contained** in the hull $(\{0\} \cup K)^{rc}$. To the best of our knowledge this remains an open problem (see also p. 87 in [3]).

We will denote by $\mathbf{m}$ the vector of all 2×2 minors, so that

$$\mathbf{m} : \mathbb{R}^{m\times n} \to \mathbb{R}^{\binom{m}{2}\binom{n}{2}}.$$

First we give a warning example to highlight the difference between $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^{m\times n}$ and show why Pedregal's proof fails in the general case. In Pedregal's proof the idea is to prove that locally near the origin the polyconvex hull is trivial, and hence the rank-one convex hull needs to separate. Recall that the *polyconvex hull* of a compact set of matrices $K \subset \mathbb{R}^{m\times n}$ is defined as

$$K^{pc} = \Big\{ P \in \mathbb{R}^{m\times n} : P = \sum_{i=1}^{k} \lambda_i X_i \text{ for some } k \in \mathbb{N}, \lambda_i \geqq 0, \sum_{i=1}^{k} \lambda_i = 1, X_i \in K$$
$$\text{such that } m(P) = \sum_{i=1}^{k} \lambda_i m(X_i) \text{ for all minors } m \Big\}.$$

In other words the aim is to show that $\big(\{0\}\cup K\big)^{\mathrm{pc}} \cap B_\epsilon(0) = \{0\}$ for sufficiently small ϵ. Standard locality results (cf. Theorem 2) would then imply that $\big(\{0\} \cup K\big)^{rc} = \{0\} \cup K^{rc}$. In the case $m = n = 2$ this happens exactly when $\det(X)$ does not change sign as $X \in K$ varies. However, in higher dimensions the situation is quite different:

Example 1. Let us consider *separate convexity* in $\mathbb{R}^3$, corresponding to rank-one convexity in the subspace $\begin{pmatrix} x & 0 & z \\ 0 & y & z \end{pmatrix}$, so that $\mathbf{m}\big((x, y, z)\big) = (xy, xz, yz)$. Let $K = \{X_1, X_2, X_3, X_4\}^{co}$, where

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, X_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, X_3 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, X_4 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

From Figure 1 we see that K does not intersect the coordinate axes: in fact the convex hull lies in the set given by the inequalities

$$x + y \geq 1,\ x + z \geq 1,\ y + z \geq 1.$$

Hence K contains no matrix of rank 1.

On the other hand the whole line segment $[0, X_4]$ is contained in the polyconvex hull $\big(\{0\} \cup K\big)^{pc}$. To see this let

$$P(t) = (1 + t - 2\sqrt{t})X_0 + \frac{2}{3}(\sqrt{t} - t)(X_1 + X_2 + X_3) + tX_4,$$

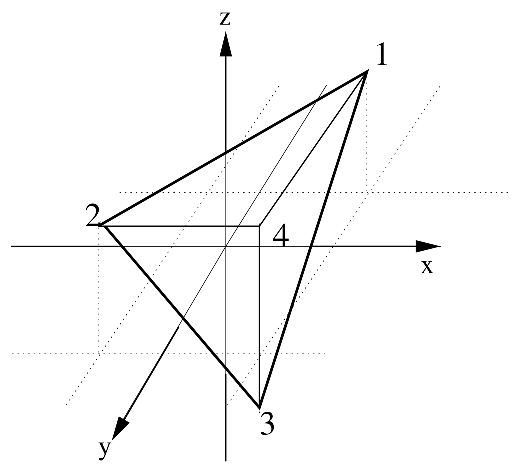

FIGURE 1. An example with nontrivial local polyconvex hull.

where we write $X_0 = 0$. Since $X_1 + X_2 + X_3 = \frac{3}{2}X_4$, we have $P(t) = \sqrt{t}X_4$. Moreover, since $\mathbf{m}(X_1) + \mathbf{m}(X_2) + \mathbf{m}(X_3) = 0$, we also have

$$\begin{aligned}\mathbf{m}(P(t)) &= t\mathbf{m}(X_4)\\ &= (1 + t - 2\sqrt{t})\mathbf{m}(X_0) + \frac{2}{3}(\sqrt{t} - t)\big(\mathbf{m}(X_1) + \mathbf{m}(X_2) + \mathbf{m}(X_3)\big) + t\mathbf{m}(X_4).\end{aligned}$$

But this means that $P(t)$ is in the polyconvex hull of $\{X_0, \dots, X_4\}$, as claimed.

2. LOCAL PROPERTIES OF RANK-ONE CONVEX HULLS

In this section we recall some of the definitions and results related to rank-one convexity. For a compact set $K \subset \mathbb{R}^{m\times n}$ the *rank-one convex hull* is defined as

$$K^{rc} = \{X \in \mathbb{R}^{m\times n} : f(X) \leqq \sup_K f \quad \forall f : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ rank-one convex}\}.$$

The dual objects to rank-one convex functions are *laminates*, introduced in [8]. Laminates are probability measures ν supported in $\mathbb{R}^{m\times n}$ which satisfy

$$\int f d\mu \geqq f(\bar{\nu}) \quad \text{ for all rank-one convex } f : \mathbb{R}^{m\times n} \to \mathbb{R},$$

and the connection with rank-one convex hulls is that

$$K^{rc} = \{\bar{\nu} : \nu \text{ is a laminate with supp } \nu \subset K\}.$$

A crucial property of rank-one convex hulls is locality.

Theorem 2 (B. Kirchheim [3]; see also [6] and [8])**.** *Let K and B be compact sets in $\mathbb{R}^{m\times n}$. Then*

$$K^{rc} \cap B = \big((B \cap K) \cup (\partial B \cap K^{rc})\big)^{rc} \cap B.$$

In particular, if $C_1, \dots, C_k$ are disjoint compact sets such that $K^{rc} \subset \bigcup_{i=1}^k C_i$, then

$$K^{rc} = \bigcup_{i=1}^{k} (K \cap C_i)^{rc}.$$

A particularly useful instance of the above result is the following observation which has been used in e.g. [8] and [4]. We give the proof for the convenience of the reader.

Lemma 1. *Let K be a compact set in $\mathbb{R}^{m\times n}$ which does not contain 0, and suppose that for some $\alpha \in \mathbb{R}$ we have*

$$\alpha \cdot \mathbf{m}(X) \geqq 1 \text{ for all } X \in K.$$

Then

$$(\{0\} \cup K)^{rc} = \{0\} \cup K^{rc}.$$

Proof. Since K is compact, there exists $R > 0$ such that $|X| < R$ for all $X \in K$. Also, since $\mathbf{m}$ is quadratic, there exists a constant $c > 0$ such that $\alpha \cdot \mathbf{m}(X) \leqq c|X|^2$. Consider the function

$$f(X) = |X| - R\alpha \cdot \mathbf{m}(X).$$

It is immediate that $f \leqq 0$ on $\{0\} \cup K$, but $f(X) > 0$ for $0 < |X| < 1/(cR)$. On the other hand, since f is polyconvex, it is rank-one convex, hence

$$B_{1/(cR)}(0) \cap (\{0\} \cup K)^{rc} = \{0\}$$

from the definition. But then from Theorem 2 we deduce that $(\{0\} \cup K)^{rc} = \{0\} \cup K^{rc}$. □

We conclude this section with a stability result. Here and in the sequel, for a compact set K we denote by $N_\varepsilon(K)$ the open ε-neighbourhood of K.

Lemma 2. *Let K be a compact set in $\mathbb{R}^{m\times n}$ disjoint from 0, and let $K_\varepsilon = \overline{N_\varepsilon(K)}$. Suppose that*

$$(\{0\} \cup K_\varepsilon)^{rc} \neq \{0\} \cup K_\varepsilon^{rc}$$

for all $\varepsilon > 0$. Then also

$$(\{0\} \cup K)^{rc} \neq \{0\} \cup K^{rc}.$$

Proof. Since K is compact and disjoint from 0, there exists a $\delta > 0$ such that

$$B_\delta(0) \cap N_\delta(K) = \emptyset.$$

We assume for contradiction that $(\{0\} \cup K)^{rc} = \{0\} \cup K$, so that in particular $(\{0\} \cup K)^{rc} \cap \partial B_\delta(0) = \emptyset$.

By assumption on the nontriviality of the rank-one convex hulls, and using Theorem 2 which in particular implies that $\big(\{0\} \cup K_\varepsilon\big)^{rc}$ is a connected set, we see that for each $\varepsilon \in (0, \delta)$ there exists $Y_\varepsilon \in (\{0\} \cup K_\varepsilon)^{rc}$ with $|Y_\varepsilon| = \delta$. Correspondingly there exists a laminate ν_ε with support $\operatorname{supp} \nu_\varepsilon \subset \{0\} \cup K_\varepsilon$ and barycenter $\overline{\nu}_\varepsilon = Y_\varepsilon$. Since the measures $\{\nu_\varepsilon\}$ are uniformly supported (in $\{0\} \cup K_\delta$), a suitable subsequence ν_{ε_j} converges, i.e. there exists a laminate ν with support in K_δ such that $\nu_{\varepsilon_j} \overset{*}{\rightharpoonup} \nu$ in the space of Radon measures. Moreover, $\operatorname{supp} \nu \subset \{0\} \cup K$ and $|\overline{\nu}| = \delta$. But this implies that

$$\overline{\nu} \in (\{0\} \cup K)^{rc} \cap \partial B_\delta(0),$$

resulting in a contradiction. □

3. The local polyconvex hull

In this section we will consider compact sets $K \subset \mathbb{R}^{m\times n}$ contained in a hyperplane which are disjoint from the origin, i.e. sets $K \subset H$, where

$$H = \{X \in \mathbb{R}^{m\times n} : \operatorname{tr}(X_0^T X) = 1\}$$

for some nonzero $X_0 \in \mathbb{R}^{m\times n}$. Here X_0^T denotes the transpose of the matrix X_0. In the next section we will see that localisation allows us to reduce to this case. For a compact set $K \subset H$ let $\mathcal{P}(K)$ denote the set of probability measures supported in K, and let

$$K^* = \left\{\overline{\mu} : \mu \in \mathcal{P}(K) \text{ such that } \int \mathbf{m}(X) d\mu(X) = 0\right\}. \tag{2}$$

Recall that $\mathbf{m}$ is defined as the vector of all 2×2 minors, so that K^* can be thought of as the "linearisation" of $\big(\{0\}\cup K\big)^{pc}$ at the origin.

Proposition 1. *Suppose $K \subset H$ is a compact, convex set such that*

$$\big(K \cup \{0\}\big)^{rc} \neq K \cup \{0\}. \tag{3}$$

Then $K^* \subset K$ is also a compact, convex set such that

$$\big(K^* \cup \{0\}\big)^{rc} \neq K^* \cup \{0\}. \tag{4}$$

Proof. It is immediate from the definition (2) that K^* is a compact, convex set contained in K, so it remains to prove (4).

First of all, let us consider the set $\mathbf{m}(K) = \{\mathbf{m}(X) : X \in K\}$. We claim that $0 \in \mathbf{m}(K)^{co}$. Indeed, if not, then there exists a nonzero $\alpha \in \mathbb{R}^{\binom{m}{2}\binom{n}{2}}$ such that $\alpha \cdot \mathbf{m}(X) \geqq 1$ for all $X \in K$, but then Lemma 1 contradicts the assumption (3). Let $L \subset \mathbb{R}^{\binom{m}{2}\binom{n}{2}}$ be the affine space generated by $\mathbf{m}(K)$. Since $0 \in L$ by above, L is a vector subspace.

For $k \in \mathbb{N}$ let $\triangle_k$ denote the k-dimensional standard simplex, i.e.

$$\triangle_k = \{\lambda \in \mathbb{R}^k : \lambda_i \geqq 0 \text{ for all } i \text{ and } \sum_i \lambda_i \leqq 1\},$$

and consider the following set:

$$K^{\mathbf{m}} = \Bigg\{P : P = \sum_{i=1}^k \lambda_i X_i \text{ for some } k \in \mathbb{N}, \lambda \in \triangle_k, X_i \in K \\ \text{such that } \mathbf{m}(P) = \sum_{i=1}^k \lambda_i \mathbf{m}(X_i)\Bigg\}.$$

It is clear that the polyconvex hull of $\{0\}\cup K$ is contained in $K^{\mathbf{m}}$ (with equality if $\min(m,n) = 2$), in particular $K^{\mathbf{m}}$ is nontrivial near 0 by assumption. As $K^{\mathbf{m}}$ is nothing but the projection onto the first coordinate of

$$\big\{(X, \mathbf{m}(X)) : X \in K\big\}^{co} \subset \mathbb{R}^{m\times n} \times L,$$

by Carathéodory's theorem we see that

$$K^{\mathbf{m}} = \bigcup_{X_1,\ldots,X_k \in K} \{X_1, \ldots, X_k\}^{\mathbf{m}}, \tag{5}$$

where $k = mn + \dim L + 1$ and the union is over $k-$tuples $\{X_1, \dots, X_k\}$ whose affine span is equal to the affine span of $\big\{(X, \mathbf{m}(X)) : X \in K\big\}$. In particular the corresponding $k-$tuples $\{\mathbf{m}(X_1), \dots, \mathbf{m}(X_k)\}$ span all of L.

Next we estimate $\{X_1, \dots, X_k\}^{\mathbf{m}}$ locally near 0 using the implicit function theorem. Let

$$F(\lambda) = \sum_{i=1}^{k} \lambda_i \mathbf{m}(X_i) - \mathbf{m}\bigg(\sum_{i=1}^{k} \lambda_i X_i\bigg),$$

and let π be the natural projection $\pi : \lambda \mapsto \sum_{i=1}^{k} \lambda_i X_i$. Then

$$\{X_1, \dots, X_k\}^{\mathbf{m}} = \pi\big(F^{-1}(0) \cap \triangle_k\big).$$

We claim that $F : \mathbb{R}^k \to L$ is a local submersion. To see that $F(\lambda) \in L$, recall that $\mathbf{m}$ is quadratic and let $\mathbf{b}$ be the symmetric bilinear map determined by $\mathbf{m}(X) = \frac{1}{2}\mathbf{b}(X, X)$ for all $X \in \mathbb{R}^{m\times n}$. For any $X, Y \in K$ we have

$$\mathbf{m}(\frac{1}{2}X + \frac{1}{2}Y) = \frac{1}{4}\mathbf{m}(X) + \frac{1}{4}\mathbf{b}(X, Y) + \frac{1}{4}\mathbf{m}(Y).$$

Since K is convex, $\frac{1}{2}X + \frac{1}{2}Y \in K$, and hence $\mathbf{b}(X, Y) \in L$. But then, expanding the expression for F shows that

$$F(\lambda) = \sum_{i=1}^{k} \lambda_i \mathbf{m}(X_i) - \frac{1}{2} \sum_{i,j=1}^{k} \lambda_i \lambda_j \mathbf{b}(X_i, X_j) \in L.$$

Now $DF(0) : \mathbb{R}^k \to L$ is given by

$$\frac{\partial F}{\partial \lambda_i}(0) = \mathbf{m}(X_i),$$

and since $\{\mathbf{m}(X_1), \dots, \mathbf{m}(X_k)\}$ spans all of L, we deduce that $DF(0) : \mathbb{R}^k \to L$ is surjective. Hence $F^{-1}(0) \subset \mathbb{R}^k$ is locally near $0 \in \mathbb{R}^k$ a smooth manifold, with tangent space at 0 given by $\ker DF(0)$. Moreover,

$$D^2F(0)[\lambda, \lambda] = -2\mathbf{m}\bigg(\sum_{i=1}^{k} \lambda_i X_i\bigg),$$

hence $\big|D^2F(0)[\lambda, \lambda]\big| \leqq c|\lambda|^2$, where c is a constant depending on K but independent of the choice of $k-$tuple $\{X_1, \dots, X_k\} \subset K$ (since K is compact). In particular, there exists $\varepsilon_0 > 0$ such that for all $x \in F^{-1}(0)$ with $|x| < \varepsilon_0$ we have

$$\operatorname{dist}\big(x, \ker DF(0)\big) \leqq c_K|x|^2$$

for some constant c_K. This implies that $F^{-1}(0) \cap \triangle_k$ is nontrivial near 0, i.e. that

$$\big(F^{-1}(0) \cap \triangle_k\big) \cap B_\varepsilon(0) \neq \{0\} \quad \text{ for all } \varepsilon < \varepsilon_0$$

only if there exists a nonzero vector $\lambda \in \ker DF(0)$ such that $\lambda_i \geq 0$ for all i (in other words, λ is a vector at 0 pointing inwards into $\triangle_k$). Moreover, with

$$V = \{\lambda \in \ker DF(0) : \lambda_i \geqq 0 \text{ for all } i\},$$

for all $x \in F^{-1}(0) \cap \triangle_k$ with $|x| < \varepsilon_0$ we have

$$\operatorname{dist}\big(x, V\big) \leqq c_K|x|^2.$$

Let

$$W = \bigcup_{X_1,\dots,X_k \in K} \pi(V),$$

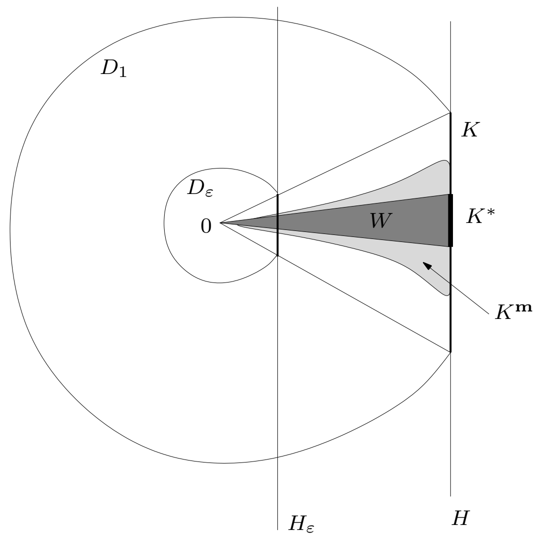

FIGURE 2. Localisation of the rank-one convex hull in D_ε.

where the union is over the same k-tuples as in (5) and π varies with the k-tuple. Note that W is a closed convex cone, in fact W is the cone generated by K^*:

$$W = \{tX \in \mathbb{R}^{m\times n} : X \in K^*, t \geqq 0\}. \tag{6}$$

From the considerations above we conclude that there exists $\varepsilon_1 > 0$ such that for all $X \in K^{\mathbf{m}}$ with $|X| < \varepsilon_1$ we have

$$\operatorname{dist}\big(X, W\big) \leqq c_K |X|^2, \tag{7}$$

where $c_K > 0$ is a constant depending only on K.

Because K is compact, there exists $R > 0$ such that $K \subset B_R(0)$. We define for any $\varepsilon > 0$ the following compact and convex neighbourhoods of the origin:

$$D_\varepsilon = \{X \in B_{\varepsilon R}(0) : \operatorname{tr}(X_0^T X) \leqq \varepsilon\}.$$

The point of the definition is that $\big(\{0\} \cup K\big)^{co} \cap \partial D_\varepsilon = \big(\{0\} \cup K\big)^{co} \cap H_\varepsilon$, where

$$H_\varepsilon = \{X \in \mathbb{R}^{m\times n} : \operatorname{tr}(X_0^T X) = \varepsilon\};$$

see Figure 2. Applying Theorem 2 in D_ε for $\varepsilon \in (0, \varepsilon_1)$ gives

$$(K \cup \{0\})^{rc} \cap D_\varepsilon \subset \Big(\{0\} \cup \big(K^{\mathbf{m}} \cap \partial D_\varepsilon\big)\Big)^{rc}. \tag{8}$$

But (7) implies that

$$K^{\mathbf{m}} \cap \partial D_\varepsilon \subset \overline{N_{c_K \varepsilon^2}(W)} \cap H_\varepsilon,$$

and so using (8) and the assumption (3) on K we deduce that

$$\{0\} \cup \big(\overline{N_{c_K \varepsilon^2}(W)} \cap H_\varepsilon\big)$$

has a nontrivial rank-one convex hull locally near 0. It can be seen directly from the definition of the rank-one convex hull that (local) nontriviality of the hull is scale-invariant. Hence

$$\{0\} \cup \frac{1}{\varepsilon}\big(\overline{N_{c_K \varepsilon^2}(W)} \cap H_\varepsilon\big)$$

also has a nontrivial hull locally near 0, but using (6)

$$\frac{1}{\varepsilon}\left(\overline{N_{c_K\varepsilon^2}(W)}\cap H_\varepsilon\right)=\overline{N_{c_K\varepsilon}(W)}\cap H_1\subset\overline{N_{c_K\varepsilon}(K^*)}.$$

We conclude that assuming (3) implies that

$$\left(\overline{N_\varepsilon(K^*)}\cup\{0\}\right)^{rc}\neq\overline{N_\varepsilon(K^*)}\cup\{0\} \tag{9}$$

for any $\varepsilon>0$. But then Lemma 2 implies (4), thus concluding the proof of the proposition. □

4. Proof of Theorem 1

Example 1 in the Introduction shows that K^* may be nonempty even if K contains no rank-one matrices. Nevertheless the "iterated version" of K^* can only stay nonempty if K contains rank-one matrices. This is the essence of the proof of Theorem 1.

Proof of Theorem 1. Arguing by contradiction, we assume that $\left(\{0\}\cup K\right)^{rc}\neq\{0\}\cup K$ and that K contains no rank-one matrices. First of all we show how to reduce to the case where K lies in a hyperplane, as in Section 3. Let $Y\in\mathbb{R}^{m\times n}$ be a nonzero matrix so that $\mathrm{tr}(Y^TX)\geqq 2$ for all $X\in K$, and let $R>0$ so that $K\subset B_R(0)$. Define

$$D=\{X\in B_R(0):\,\mathrm{tr}(Y^tX)\leqq 1\}.$$

Note that $\partial D\cap\left(\{0\}\cup K\right)^{co}=H\cap\left(\{0\}\cup K\right)^{co}$, where $H=\{X\in\mathbb{R}^{m\times n}:\,\mathrm{tr}(Y^TX)=1\}$. Applying Theorem 2 in D then yields

$$\left(\{0\}\cup K\right)^{rc}\cap D\subset\left(\{0\}\cup\left(H\cap\left(\{0\}\cup K\right)^{co}\right)\right)^{rc}$$

so that if we define

$$K'=H\cap\left(\{0\}\cup K\right)^{co},$$

then K' is a compact, convex set contained in H, containing no matrices of rank 1, and such that $\{0\}\cup K'$ has a nontrivial rank-one convex hull locally near 0.

Define the nested sequence of convex and compact sets K_i as

$$K_1=K'\supset K_2=K_1^*\supset K_3=K_2^*\supset\cdots\supset K_{i+1}=K_i^*,$$

and let

$$K_\infty=\bigcap_{i=1}^{\infty}K_i.$$

Step 1. First of all we claim that K_∞ is nonempty. Indeed, by applying Proposition 1 iteratively, we see that for each i

$$(\{0\}\cup K_i)^{rc}\neq\{0\}\cup K_i,$$

in particular, K_i is nonempty for each i. By compactness it follows that K_∞ is nonempty.

Step 2. Secondly, we claim that $K_\infty^*=K_\infty$. Let $Y\in K_\infty$. We need to show that $Y\in K_\infty^*$. Now $Y\in K_i$ for all i, so by Carathéodory's theorem there exist $X_1^i,\dots,X_k^i\in K_i$ and $\lambda_1^i,\dots,\lambda_k^i\in[0,1]$ such that $\sum_{j=1}^k\lambda_j^i=1$,

$$\sum_{j=1}^k\lambda_j^iX_j^i=Y\ \text{and}\ \sum_{j=1}^k\lambda_j^i\mathbf{m}(X_j^i)=0. \tag{10}$$

By compactness, modulo selecting a suitable subsequence, we may assume that

$$X_j^i \to Y_j \in K' \text{ and } \lambda_j^i \to \nu_j \in [0,1] \text{ as } j \to \infty,$$

and $\sum_{j=1}^k \nu_j = 1$. As the equations (10) also hold in the limit, it remains to prove that $Y_j \in K_\infty$. Suppose for contradiction that $Y_1 \notin K_\infty$. As K_∞ is compact and convex, there exists a hyperplane $\tilde{H}$ separating K_∞ from Y_1. But $X_1^i \to Y_1$, so in particular because K_i is convex, $K_i \cap \tilde{H} \neq \emptyset$. But $K_i \cap \tilde{H}$ is also compact, and so $K_\infty \cap \tilde{H} = \bigcap_i (K_i \cap \tilde{H})$ cannot be empty. This is the required contradiction.

Step 3. To conclude, let $X \in K_\infty$ be a convex extreme point of K_∞ (such a point exists by Step 1). By the second step $X \in K_\infty^*$, but then from the definition (2) we see that necessarily $\mathbf{m}(X) = 0$ and so rank $X = 1$. In fact

$$K_\infty = \{X \in K' : \text{rank } X = 1\}^{co}.$$

This concludes the proof of the theorem. □

5. Separate convexity

In this section we give another proof of Theorem 1 for the case of separate convexity in $\mathbb{R}^n$, which can be canonically identified with rank-one convexity on diagonal $n \times n$ matrices. It uses a standard separation argument using the so-called $\det^{++}$ functions; see for example Section 6 in [4]. Unfortunately, as these functions have no useful rank-one convex extension to $\mathbb{R}^{n\times n}$, the method does not work in the general case. Nevertheless, because of the simplicity and the geometric nature of the proof, it is included in this paper.

Proposition 2. *Let $K \subset \mathbb{R}^n$ be a convex and compact set disjoint from 0 and not intersecting the coordinate axes. Then*

$$(\{0\} \cup K)^{sc} = \{0\} \cup K.$$

Proof. We will use separately convex functions of the following form: for $\epsilon \in \{0,+,-\}^n$ let

$$f(x) = \prod_{i=1}^n [x_i]_{\epsilon_i}.$$

Here $x = (x_1, \dots, x_n)$ in coordinates, and $[x_i]_\pm$ denotes the positive and negative part of x_i, and $[x_i]_0 = 1$.

Let $L_i = \{x \in \mathbb{R}^n : x_i = 0\}$. Since K does not intersect the coordinate axes, its projection onto L_i is a convex set disjoint from the origin (in L_i). Hence there exist vectors $a^i \in \mathbb{R}^n$ for $i = 1, \dots, n$ with $a_i^i = 0$ for all i and such that for all $i = 1, \dots, n$,

$$a^i \cdot x \geqq 1 \text{ for all } x \in K.$$

Define the vectors

$$y^i = \frac{a^i}{2|a^i|^2}, \tag{11}$$

so that in particular $a^i \cdot y^i = \frac{1}{2}$. Let $\epsilon_i^j = -\text{sgn } a_i^j$ for $i \neq j$ with the convention that sgn $0 = 0$, and consider the functions

$$f_\pm^j(x) = [x_j]_\pm \prod_{i \neq j} [x_i - y_i^j]_{\epsilon_i^j}. \tag{12}$$

We claim that $f^j_\pm(x) = 0$ for $x \in K$. Indeed, let us fix $x \in K$ and $j \in \{1, \ldots, n\}$. By the definition of y^j we have $a^j \cdot (x - y^j) \geqq 1/2$, so that there exists $i \neq j$ such that $a^j_i(x_i - y^j_i) > 0$. In particular $\operatorname{sgn} a^j_i = \operatorname{sgn}(x_i - y^j_i)$, and so the product in (12) vanishes. Moreover $f^j_\pm(0) = 0$. But then, since $f^j_\pm$ is separately convex, for each j the set

$$\begin{aligned} C_j =& \{f^j_+ > 0\} \cup \{f^j_- > 0\} \\ =& \{z \in \mathbb{R}^n : z_j \neq 0 \text{ and } a^j_i(z_i - y^j_i) < 0 \text{ for all } i \text{ such that } a^j_i \neq 0\} \end{aligned}$$

is disjoint from $(\{0\} \cup K)^{sc}$. But then we claim that

$$B_r(0) \cap \big(\{0\} \cup K\big)^{sc} = \{0\}$$

for

$$r < r_0 = \min\left\{ \frac{|a^j_i|}{2|a^j|^2} : a^j_i \neq 0 \right\}.$$

Indeed, if $0 < |z| < r_0$, then

$$a^j_i z_i < |a^j_i| r_0 \leqq \frac{|a^j_i|^2}{2|a^j|^2} = a^j_i y^j_i \quad \text{for all } i$$

using (11), hence if j is such that $z_j \neq 0$, then $z \in C_j$, thus proving our claim. Theorem 2 then implies that $(\{0\} \cup K)^{sc} = \{0\} \cup K$. □

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