

ON HOMEOMORPHISMS AND QUASI-ISOMETRIES OF THE REAL LINE

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ABSTRACT. We show that the group of piecewise-linear homeomorphisms of \mathbb{R} having bounded slopes surjects onto the group $QI(\mathbb{R})$ of all quasi-isometries of \mathbb{R} . We prove that the following groups can be imbedded in $QI(\mathbb{R})$: the group of compactly supported piecewise-linear homeomorphisms of \mathbb{R} , the Richard Thompson group F , and the free group of continuous rank.

1. INTRODUCTION

We begin by recalling the notion of quasi-isometry. Let $f : X \rightarrow X'$ be a map (which is not assumed to be continuous) between metric spaces. We say that f is a C -quasi-isometric embedding if there exists a $C > 1$ such that

$$(*) \quad C^{-1}d(x, y) - C \leq d'(f(x), f(y)) \leq Cd(x, y) + C$$

for all $x, y \in X$. Here d, d' denote the metrics on X, X' , respectively. If, further, every $x' \in X'$ is within distance C from the image of f , we say that f is a C -quasi-isometry. If f is a quasi-isometry (for some C), then there exists a quasi-isometry $f' : X' \rightarrow X$ (for a possibly different constant C') such that $f' \circ f$ (resp. $f \circ f'$) is quasi-isometry equivalent to the identity map of X (resp. X'). (Two maps $f, g : X \rightarrow X$ are said to be quasi-isometrically equivalent if there exists a constant M such that $d(f(x), g(x)) \leq M$ for all $x \in X$.) Let $[f]$ denote the equivalence class of a quasi-isometry $f : X \rightarrow X$. The set $QI(X)$ of all equivalence classes of quasi-isometries of X is a group under composition: $[f].[g] = [f \circ g]$ for $[f], [g] \in QI(X)$. If X' is quasi-isometry equivalent to X , then $QI(X')$ is isomorphic to $QI(X)$. We refer the reader to [1] for basic facts concerning quasi-isometry. For example $t \mapsto [t]$ is a quasi-isometry from \mathbb{R} to \mathbb{Z} .

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any homeomorphism of \mathbb{R} . Denote by $B(f)$ the set of break points of f , i.e., points where f fails to have derivative and by $\Lambda(f)$ the set of slopes of f , i.e., $\Lambda(f) = \{f'(t) \mid t \in \mathbb{R} \setminus B(f)\}$. Note that $B(f) \subset \mathbb{R}$ is discrete if f is piecewise differentiable.

Definition 1.1. We say that a subset Λ of \mathbb{R}^* , the set of non-zero real numbers, is *bounded* if there exists an $M > 1$ such that $M^{-1} < |\lambda| < M$ for all $\lambda \in \Lambda$. We say that a homeomorphism f of \mathbb{R} which is piecewise differentiable has *bounded slopes* if $\Lambda(f)$ is bounded.

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We denote by $PL_\delta(\mathbb{R})$ the set of all those piecewise-linear homeomorphisms f of \mathbb{R} such that $\Lambda(f)$ is bounded. It is clear that $PL_\delta(\mathbb{R})$ is a subgroup of the group $PL(\mathbb{R})$ of all piecewise-linear homeomorphisms of \mathbb{R} .

It is easy to see that each $f \in PL_\delta(\mathbb{R})$ is a quasi-isometry. (See Lemma 2.1 below.) One has a natural homomorphism $\varphi : PL_\delta(\mathbb{R}) \rightarrow QI(\mathbb{R})$, where $\varphi(f) = [f]$ for all $f \in PL_\delta(\mathbb{R})$.

Theorem 1.2. *The natural homomorphism $\varphi : PL_\delta(\mathbb{R}) \rightarrow QI(\mathbb{R})$, defined as $f \mapsto [f]$, is surjective.*

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism, recall that $\text{Supp}(f)$, the support of f , is the closure of the set $\{x \in \mathbb{R} \mid f(x) \neq x\}$ of all points moved by f . Denote by $PL_\kappa(\mathbb{R})$ the group of all piecewise-linear homeomorphisms of \mathbb{R} which have compact support. It is obvious that $PL_\kappa(\mathbb{R}) \subset \ker(\varphi)$.

Let Γ be a group of homeomorphisms of \mathbb{S}^1 . Any $f \in \Gamma$ can be lifted to obtain a homeomorphism \tilde{f} of \mathbb{R} over the covering projection $p : \mathbb{R} \rightarrow \mathbb{S}^1$, $t \mapsto \exp(2\pi\sqrt{-1}t)$. The set $\tilde{\Gamma}$ of all homeomorphisms of \mathbb{R} which are lifts of elements of Γ is a subgroup of the group $\text{Homeo}(\mathbb{R})$ of all homeomorphisms of \mathbb{R} . Indeed $\tilde{\Gamma}$ is a central extension of Γ by the infinite cyclic group generated by translation by 1: $x \mapsto x + 1$. Denote by $\text{Diff}(\mathbb{S}^1)$ the group of all C^∞ diffeomorphisms of the circle. When Γ is one of the groups $PL(\mathbb{S}^1)$, $\text{Diff}(\mathbb{S}^1)$, any element of $\tilde{f} \in \tilde{\Gamma}$ has bounded slope and is quasi-isometrically equivalent to the identity map of \mathbb{R} (since $\tilde{f}(x+n) = \tilde{f}(x) + n$ for $n \in \mathbb{Z}$).

Recall that Richard Thompson discovered the group

$$F = \langle x_0, x_1, \dots \mid x_i x_j x_i^{-1} = x_{j+1}, i < j \rangle$$

and used it in some constructions in logic related to word problems. The group F is finitely presentable with two generators x_0, x_1 and two relations. This group and a closely related larger group G have since then appeared in several contexts including homotopy theory [6], homological group theory [3], Teichmüller theory [9], etc. The group F is isomorphic to the subgroup of piecewise-linear homeomorphisms of \mathbb{R} which are the identity outside the unit interval I such that $B(f)$ is contained in dyadic rationals and $\Lambda(f)$ is contained in the subgroup of \mathbb{R}^* generated by 2. Although F satisfies no (non-trivial) group law, it contains no non-abelian free group. The group G is the group of piecewise-linear homeomorphisms f of the circle $\mathbb{S}^1 = I/\{0, 1\}$ with $B(f)$ contained in dyadic rationals and $\Lambda(\tilde{f})$ contained in the multiplicative subgroup of \mathbb{R}^* generated by 2 for some lift \tilde{f} of f . It is the first known example of a finitely presented infinite simple group. We recommend the beautiful survey article [4] for further information about Richard Thompson's groups.

We shall prove the following theorem:

Theorem 1.3. *The following groups can be imbedded in $QI(\mathbb{R})$:*

- (i) *the groups $\widetilde{\text{Diff}}(\mathbb{S}^1)$ and $\widetilde{PL}(\mathbb{S}^1)$,*
- (ii) *the group $PL_\kappa(\mathbb{R})$,*
- (iii) *the Thompson's group F , and*
- (iv) *the free group of rank c , the continuum.*

Our proofs are completely elementary. We explain the main idea of the proof of Theorem 1.3. Take for example the group $PL_\kappa(\mathbb{R})$. The first step is to realise

this as a subgroup Γ_1 of $PL_\kappa(\mathbb{R})$ having support in $(0, 1)$. This is easily achieved by imbedding \mathbb{R} in the interval $(0, 1)$. The group Γ_1 can be thought of as a group of piecewise-linear homeomorphisms of the circle. Lifting this back to \mathbb{R} via the covering projection, we now obtain a group $\tilde{\Gamma}_1$ which no longer has compact support. However each element of this group is quasi-isometric to id . So we conjugate this group by a piecewise-linear homeomorphism whose slope grows exponentially. The result is that the features of each element of $\tilde{\Gamma}_1$ get magnified, resulting in a quasi-isometry not representing 1. The same trick works for $\widetilde{Diff}(\mathbb{S}^1)$ as well. Parts (iii) and (iv) follow from known embeddings of the relevant groups.

2. PROOF OF THEOREM 1.2

We first establish the following basic observation.

Lemma 2.1. *Let f be a piecewise differentiable homeomorphism of \mathbb{R} with $\Lambda(f) \subset \mathbb{R}^*$ bounded. Then f is a quasi-isometry.*

Proof. Replacing f by $-f$ if necessary, one may assume without loss of generality that f is monotone increasing.

Suppose that $\Lambda(f) \subset (1/M, M)$. If f is differentiable everywhere, then it is an M -quasi-isometry.

Suppose that $B(f) \neq \emptyset$. Let $a \in \mathbb{R}$. For any $b > a$, let $a_1 < \dots < a_k$ be the points of (a, b) where f is non-differentiable. Then, applying the mean value theorem, $f(b) - f(a) = \sum_{0 \leq i \leq k} (f(a_{i+1}) - f(a_i)) = \sum_{0 \leq i \leq k} f'(c_i)(a_{i+1} - a_i)$ for some $c_i \in (a_i, a_{i+1})$. Since $\Lambda(f) \subset (1/M, M)$, it follows that $M^{-1}(b - a) < f(b) - f(a) < M(b - a)$. Since $a, b \in \mathbb{R}$ are arbitrary, we conclude that f is a quasi-isometry. \square

One has a well-defined map $\varphi : PL_\delta(\mathbb{R}) \rightarrow QI(\mathbb{R})$ which is a homomorphism. We now prove that φ is surjective.

Lemma 2.2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C -quasi-isometry that preserves the ends of \mathbb{R} . Let $x \in \mathbb{R}$. Then (i) there exists y such that $y - x \leq 4C^2$ is positive integer and $f(y) > f(x)$; (ii) there exists v such that $x - v \leq 4C^2$ is a positive integer and $f(x) > f(v)$.*

Proof. If $f(x + 1) > f(x)$, then $y = x + 1$ meets our requirements.

Assume that $f(x) > f(y)$ for all y such that $x + 1 \leq y < 4C^2 + x$. Let $z \geq x + 2$ be the smallest real number such that $z - x$ is a positive integer and $f(z) > f(x) \geq f(z - 1)$. Such a z exists since $f(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. By our assumption $z - x \geq 4C^2 + 1$. Set $u = z - 1$. Then the inequality (*) implies $f(u) < f(x) + C - C^{-1}(u - x) \leq f(x) - 3C$ and $f(z) - f(u) < C(z - u) + C = 2C$. Hence $f(z) < f(u) + 2C < f(x) - C$, i.e., $f(z) - f(x) < -C$. This contradicts our hypothesis that $f(z) > f(x)$, completing the proof of part (i). The proof of part (ii) is similar. \square

Proof of Theorem 1.2. Since the subgroup $QI^+(\mathbb{R}) \subset QI(\mathbb{R})$ that preserves the ends $\{+\infty, -\infty\}$ of \mathbb{R} is of index 2 and since $PL_\delta(\mathbb{R})$ contains elements which are orientation reversing, it suffices to show that $QI^+(\mathbb{R})$ is contained in the image of φ , where $QI^+(\mathbb{R}) \subset QI(\mathbb{R})$ is the index 2 subgroup whose elements preserve the ends of \mathbb{R} .

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C -quasi isometry, with $C > 1$, which preserves the ends of \mathbb{R} . We assume, as we may, that C is a positive integer.

Set $x_0 = 0$. We define $x_k \in \mathbb{Z}$ for any integer k as follows: Let $k \geq 1$. Having defined x_{k-1} inductively, choose $x_k > x_{k-1}$ to be the smallest integer such that $f(x_k) > f(x_{k-1})$. For any negative integer k , we define x_k analogously (by downward induction) as the greatest integer such that $x_k < x_{k+1}$ and $f(x_k) < f(x_{k+1})$.

Set $y_k := x_{C^3k}$, and let $B := \{y_k \mid k \in \mathbb{Z}\} \subset \mathbb{Z}$. By Lemma 2.2, we see that B is a discrete subset of \mathbb{R} which is $4C^5$ -dense in \mathbb{R} . Note that for any $k \in \mathbb{Z}$, $y_k - y_{k-1} \geq C^3$.

Since $f(y_k) > f(y_{k-1})$ for all $k \in \mathbb{Z}$, there exists a unique piecewise-linear homeomorphism $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(y_k) = f(y_k)$ and is linear on the interval $[y_{k-1}, y_k]$ for every $k \in \mathbb{Z}$. We claim that g has bounded slopes. Since g is linear on each of the intervals $[y_{k-1}, y_k]$, we need only bound $\frac{g(y_k) - g(y_{k-1})}{y_k - y_{k-1}}$. Indeed,

$$\frac{g(y_k) - g(y_{k-1})}{y_k - y_{k-1}} = \frac{f(y_k) - f(y_{k-1})}{y_k - y_{k-1}} < C + \frac{C}{y_k - y_{k-1}} \leq C + C^{-2}$$

as $y_k - y_{k-1} \geq C^3$. Similarly,

$$\frac{g(y_k) - g(y_{k-1})}{y_k - y_{k-1}} > C^{-1} - C^{-2}.$$

It follows that $\Lambda(g) \subset [C^{-1} - C^{-2}, C + C^{-2}]$ and $g \in PL_\delta(\mathbb{R})$.

Since f and g agree on the quasi-dense set B , we see that $[f] = [g]$. This completes the proof. \square

Remark 2.3. (i) By setting $g(y_k)$ equal to a rational number sufficiently close to $f(y_k)$ in the above proof, we see that since $y_k \in \mathbb{Z}$, the element $g \in PL_\delta(\mathbb{R})$ has rational slopes. Consequently it follows that φ restricted to the subgroup $PL_{\mathbb{Q}}^*(\mathbb{R})$ of $PL_\delta(\mathbb{R})$ consisting of those $g \in PL_\delta(\mathbb{R})$ having slopes in \mathbb{Q}^* and $B(g)$ contained in \mathbb{Q} is surjective.

(ii) The kernel of φ contains the group of all piecewise-linear homeomorphisms which have slope 1 outside a compact interval. This latter group equals the derived group $PLF'(\mathbb{R})$, where $PLF(\mathbb{R})$ denotes the subgroup of $PL_\delta(\mathbb{R})$ consisting of homeomorphisms f for which $B(f)$ is finite. Also $PL_\kappa(\mathbb{R}) = PLF''(\mathbb{R})$. See [2].

3. PROOF OF THEOREM 1.3

Let $h_1 : \mathbb{R} \rightarrow (0, 1)$ be the homeomorphism defined by $h_1(-x) = 1 - h_1(x)$ for every $x \in \mathbb{R}$, $h_1(n) = 1 - 1/(n + 2)$ for each integer $n \geq 0$, and is linear on each interval $[n, n + 1]$ for $n \in \mathbb{Z}$. If f is any compactly supported (piecewise-linear) homeomorphism of \mathbb{R} , then $h_1 \circ f \circ h_1^{-1}$ is a compactly supported (piecewise-linear) homeomorphism of $(0, 1)$. Since $\mathbb{S}^1 = I/\{0, 1\}$, we also get an embedding $\bar{\eta} : PL_\kappa(\mathbb{R}) \rightarrow PL(\mathbb{S}^1)$, where $\bar{\eta}(f)$ is defined to be the extension of $h_1 \circ f \circ h_1^{-1}$ to \mathbb{S}^1 . We define $\eta : PL_\kappa(\mathbb{R}) \rightarrow PL_\delta(\mathbb{R})$ as the imbedding $f \mapsto \eta(f)$, where $\eta(f)(n) = n$ for $n \in \mathbb{Z}$ and $\eta(f)(x) = n + h_1 f h_1^{-1}(x - n)$ for $n < x < n + 1$.

Let $h_0 : \mathbb{R} \rightarrow \mathbb{R}$ be the piecewise-linear homeomorphism defined as follows: $h_0(-x) = -h_0(x) \forall x \in \mathbb{R}$, $h_0(x) = x$ for $0 \leq x \leq 1$ and maps the interval $[n, n + 1]$ onto $[2^{n-1}, 2^n]$ linearly for each positive integer n .

Suppose $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is an orientation-preserving piecewise-linear homeomorphism or a diffeomorphism. Let $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ be any lift of f so that $p \circ \tilde{f} = f \circ p$, where $p : \mathbb{R} \rightarrow \mathbb{S}^1$ is the covering projection $t \mapsto \exp(2\pi\sqrt{-1}t)$. Then $[f] = 1$ in

$QI(\mathbb{R})$. (Indeed one has $\tilde{f}(x+n) = n + \tilde{f}(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ and so $|\tilde{f} - id| \leq |\tilde{f}(0)| + 1$.)

Let Γ be one of the groups $PL(\mathbb{S}^1)$ or $Diff(\mathbb{S}^1)$ and let $\tilde{\Gamma}$ be the group of homeomorphisms of \mathbb{R} which are lifts of elements of Γ with respect to the covering projection p . For $\tilde{f} \in \tilde{\Gamma}$ set $f_0 := h_0 \tilde{f} h_0^{-1}$. Clearly, $\tilde{f} \mapsto f_0$ is a monomorphism of groups $\tilde{\Gamma} \rightarrow Homeo(\mathbb{R})$. We claim that for any $\tilde{f} \in \tilde{\Gamma}$, f_0 is a quasi-isometry. To see this, we assume without loss of generality that \tilde{f} is orientation preserving. It is clear that f_0 is differentiable outside a discrete subset of \mathbb{R} . We claim that f_0 has bounded slopes. Since f_0 has continuous derivatives on each interval on which \tilde{f} has derivatives, it suffices to show that the set $\{f_0'(t)\}$ as t varies in $\mathbb{R} \setminus B$ is bounded, where B is any discrete set which contains $B(f_0)$. We set $B := B(h_0) \cup h_0 B(\tilde{f}) \cup h_0 \tilde{f}^{-1} B(h_0)$.

Let $0 < m < M$ be such that $m < \tilde{f}'(x) < M$ for $x \in \mathbb{R}$. Let $t \in \mathbb{R} \setminus B$ and set $s = h_0^{-1}(t)$, $u = \tilde{f}(s)$ so that h_0^{-1}, \tilde{f}, h_0 are differentiable at t, s, u , respectively. Consequently f_0 is differentiable at t .

Since $|u - s| = |\tilde{f}(s) - s| < q$, where $q := \lceil |\tilde{f}(0)| \rceil + 2$, we see that $2^{-q} < h_0'(u)/h_0'(s) < 2^q$. Using the chain rule, it follows that $f_0'(t) = h_0'(u)\tilde{f}'(s)(h_0^{-1})'(t) = \tilde{f}'(s)h_0'(u)/h_0'(s)$ lies in the interval $(2^{-q}m, 2^qM)$. It follows from Lemma 2.1 that f_0 is a quasi-isometry.

It is clear that the map $\psi : \tilde{\Gamma} \rightarrow QI(\mathbb{R})$ defined as $\tilde{f} \mapsto [f_0]$ is a homomorphism.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. We use the above notations throughout the proof.

(i) We prove that $\psi : \tilde{\Gamma} \rightarrow QI(\mathbb{R})$ is a monomorphism where $\Gamma = PL(\mathbb{S}^1)$ or $Diff(\mathbb{S}^1)$. Suppose that $\tilde{f} \in \tilde{\Gamma}$, $\tilde{f} \neq id$. We shall show that $|f_0 - id|$ is unbounded. Choose x in the interval $[0, 1)$ such that $\tilde{f}(x) \neq x$. Set $k = \lceil \tilde{f}(x) \rceil$ so that $\tilde{f}(x) = k + y$, $0 \leq y < 1$. Replacing \tilde{f} by its inverse if necessary, we assume without loss of generality that $x < \tilde{f}(x)$. This implies that $k \geq 0$ with equality only if $y > x$. For any positive integer n , we have $f_0(2^n + 2^n x) = h_0 \tilde{f} h_0^{-1}(2^n + 2^n x) = h_0 \tilde{f}(n + 1 + x) = h_0(n + 1 + \tilde{f}(x)) = h_0(n + 1 + k + y) = 2^{n+k} + 2^{n+k}y$.

If $k = 0$, then $y > x$ and so $f_0(2^n + 2^n x) - (2^n + 2^n x) = 2^n(y - x)$. Thus $|f_0 - id|$ is unbounded.

If $k > 0$, then $f_0(2^n + 2^n x) - (2^n + 2^n x) = 2^{n+k} + 2^{n+k}y - 2^n - 2^n x \geq 2^{n+1} - 2^n - 2^n x = 2^n(1 - x)$. As $0 \leq x < 1$, again it follows that $|f_0 - id|$ is unbounded.

(ii) As observed earlier, $\eta : PL_\kappa(\mathbb{R}) \rightarrow PL_\delta(\mathbb{R})$ is a monomorphism. It is evident that the image of η is contained in $\widetilde{PL}(\mathbb{S}^1)$. Since $\psi : \widetilde{PL}(\mathbb{S}^1) \rightarrow QI(\mathbb{R})$ is a monomorphism by (i), assertion (ii) follows.

(iii) Now statement (iii) follows from (ii) above and the fact that Thompson's group F is isomorphic to the subgroup of $PL_\kappa(\mathbb{R})$ of all piecewise-linear homeomorphisms which have support in $[0, 1]$ having break points contained in the set of dyadic rationals in $[0, 1]$ and slopes contained in the multiplicative subgroup of \mathbb{R}^* generated by 2.

(iv) To prove (iv), recall that Grabowski [8] has shown that the free group of rank c the continuum embeds in the group of compactly supported C^k diffeomorphisms ($1 \leq k \leq \infty$) of any positive-dimensional manifold. In particular, this is true of

$Diff(\mathbb{S}^1)$. It follows easily that $\widetilde{Diff}(\mathbb{S}^1)$ also contains a free group of rank the continuum. By part (i), this completes the proof. \square

Lemma 3.1. *The group $QI^+(\mathbb{R})$ is torsion-free.*

Proof. Let $f \in PL_\delta(\mathbb{R})$ be such that $[f] \neq 1 \in QI^+(\mathbb{R})$. Thus $f - id$ is unbounded. Choose a sequence (a_n) of real numbers such that $a_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and $|f(a_n) - a_n| \rightarrow +\infty$. Let $k > 1$ be any integer. Suppose that $f(a_n) > a_n$. Since f is order preserving, for each n we have $a_n < f(a_n) < \cdots < f^k(a_n)$. In particular $f^k(a_n) - a_n > f(a_n) - a_n$. Similarly, $a_n - f^k(a_n) > a_n - f(a_n)$ in case $a_n > f(a_n)$. Therefore $|f^k(a_n) - a_n| > |f(a_n) - a_n| \forall n$ and hence $f^k - id$ is unbounded. Hence $[f^k] \neq 1$ in $QI^+(\mathbb{R})$ for $k > 1$. \square

Remark 3.2. Thompson's group G does not imbed in $QI(\mathbb{R})$ since it has an element of order 3, whereas it follows from Lemma 3.1 that all torsion elements in $QI(\mathbb{R})$ are of order 2.

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