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EXTENDING INTO ISOMETRIES OF $\mathcal{K}(X, Y)$

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Abstract. In this paper we generalize a result of Hopenwasser and Plastiras (1997) that gives a geometric condition under which into isometries from $\mathcal{K}(\ell^2)$ to $\mathcal{L}(\ell^2)$ have a unique extension to an isometry in $\mathcal{L}(\mathcal{L}(\ell^2))$. We show that when X and Y are separable reflexive Banach spaces having the metric approximation property with X strictly convex and Y smooth and such that $\mathcal{K}(X, Y)$ is a Hahn-Banach smooth subspace of $\mathcal{L}(X, Y)$, any nice into isometry $\Psi_0 : \mathcal{K}(X, Y) \to \mathcal{L}(X, Y)$ has a unique extension to an isometry in $\mathcal{L}(\mathcal{L}(X, Y))$.

1. Introduction

In this paper we study the unique norm-preserving extension of operators in $\mathcal{L}(X, X^{**})$ to $\mathcal{L}(X^{**})$ (we always consider X as canonically embedded in its bidual). We are in particular interested in the question of uniquely extending isometries from $\mathcal{K}(X, Y) \to \mathcal{L}(X, Y)$ to $\mathcal{L}(X, Y)$ without knowing a specific description of the into isometry. We formulate and prove an abstract analogue of a result of Hopenwasser and Plastiras [3] that gives a unique extension under some additional hypothesis, in the case of separable Hilbert spaces. Our result is valid for a class of operators that are isometries and nice operators.

We recall from [8] that X is said to be Hahn-Banach smooth if, under the canonical embeddings, $x^* \in X^{***}$ is the unique norm preserving extension of $x^* \in X^*$. It is known that for any such space, X^* has the Radon-Nikodým property. It is well known that $\mathcal{K}(\ell^2)$ is a Hahn-Banach smooth space. We are interested in the situation when $\mathcal{L}(X, Y)$ is the canonical bidual of $\mathcal{K}(X, Y)$ and $\mathcal{K}(X, Y)$ is Hahn-Banach smooth. See the discussions on page 333 of [2] and the references given therein for several examples of spaces X, Y for which $\mathcal{K}(X, Y)$ is a Hahn-Banach smooth subspace of its bidual $\mathcal{L}(X, Y)$. In particular, for a reflexive space X, $1 < p < \infty$, $\mathcal{K}(\ell^p, X)$ is Hahn-Banach smooth. See also [6]. More generally, when $\mathcal{K}(X, Y)$ is an M-ideal in its bidual $\mathcal{L}(X, Y)$, it is a Hahn-Banach smooth space. See Chapter VI of [2] for several examples of this phenomenon from among classical function spaces which are strictly convex or smooth.

For a Banach space X let X_1 denote the closed unit ball, let S_X denote the unit sphere and let $\partial_e X_1$ denote the set of extreme points. We call a linear map $T : X \to X^{**}$ a nice operator if $x^* \circ T \in S_{X^*}$ for all $x^* \in \partial_e X_1^*$. Note that when

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X is Hahn-Banach smooth, $x^* \in \partial_e X_1^*$ continues to be an extreme point of X_1^{***}. Since $\partial_e \mathcal{K}(\ell^2)_1^* = \{x \otimes y : x, y \in S_{\ell^2}\}$, we see that in our notation the Lemma from [3] reads as any nice isometry $\Psi_0 : \mathcal{K}(\ell^2) \to \mathcal{L}(\ell^2)$ has a unique isometric extension to $\mathcal{L}(\ell^2)$.

In the first part of the paper we consider unique extensions of certain nice operators on Hahn-Banach smooth spaces and use these to deduce the result quoted in the abstract. We also prove a version of the Hopenwasser and Plastiras theorem in the non-reflexive case. Here the unique extension need not be of the form given by the Lemma from [3]. We refer to Chapter VIII of [1] for results on tensor product spaces. We use the subscript π to denote the projective tensor product. The assumptions of reflexivity and metric approximation property assumed here ensure that $\mathcal{K}(X,Y)^* = X \otimes_\pi Y^*$ and hence $\mathcal{K}(X,Y)^{**} = \mathcal{L}(X,Y)$. Thus $\mathcal{L}(X,Y)$ is the canonical bidual of $\mathcal{K}(X,Y)$. It is possible to prove some of the results considered here under assumptions that are weaker than Hahn-Banach smoothness [4], however the author is unaware of good applications of such a generalization to the context of spaces of operators.

2. Main results

The following result and its corollary are one form of abstract analogues of the Lemma from [3].

Theorem 1. *Let X be Hahn-Banach smooth. Suppose $T : X \to X$ is a linear map such that $x^* \circ T \in S_{X^*}$. Then $T^{**} \in \mathcal{L}(X^{**})$ is the unique norm-preserving extension of T.*

Proof. An easy application of the Krein-Milman theorem shows that $\|T\| = 1$. Let $S : X^{**} \to X^{**}$ be such that $\|S\| \leq 1$ and $S = T$ on X. We shall show that $S^* = T^{***}$. Since X_1^* is weak* dense in X_1^{***}, clearly it is enough that the operators agree on X^*. Let $x^* \in \partial_e X_1^*$. For $x \in X$, by our assumption $S^*(x^*)(x) = x^*(S(x)) = x^*(T^{**}(x)) = T^{***}(x^*)(x)$. Since $T^{***}(x^*)$ has a unique norm-preserving extension, we get that $S^*(x^*) = T^{***}(x^*)$ so that $S^* = T^{***}$ on $\partial_e X_1^*$. Since X^* has the Radon-Nikodým property, its unit ball is the norm closed convex hull of its extreme points (see Theorem VII.4.5 of [1]). Thus $S^* = T^{***}$ on X^* so that $S = T^{**}$. □

We denote the fourth dual of X by $X^{(IV)}$. By $\pi_{X^{**}}$ we denote the canonical projection from $X^{(IV)}$ onto X^{**} . We note from Proposition III.2.1 of [2] that when X is an M-ideal in its bidual, this is the only contractive projection from $X^{(IV)}$ to X^{**}.

Corollary 2. *Let X be Hahn-Banach smooth. Let $T : X \to X^{**}$ be a nice operator. Then $T' = \pi_{X^{**}} \circ T^{**}$ is the unique norm-preserving extension of T in $\mathcal{L}(X^{**})$.*

Proof. Let $S \in \mathcal{L}(X^{**})$ be an extension of T. As before we shall show that $S^* = T'^*$ on X^*. Now for $x^* \in \partial_e X_1^*$ and $x \in X$, $S^*(x^*)(x) = x^*(S(x)) = x^*(T'(x)) = T'^*(x^*)(x) = T^*(x^*)(x)$. Thus again by the uniqueness of norm-preserving extensions we obtain the conclusion. □

We are now ready to formulate the Hopenwasser and Plastiras result for separable reflexive Banach spaces satisfying the metric approximation property for which $\mathcal{K}(X,Y)$ is a Hahn-Banach smooth subspace of $\mathcal{L}(X,Y)$. We recall from [7], [9] ([5] for the complex case) that $\partial_e \mathcal{K}(X,Y)_1^* = \{x \otimes y^* : x \in \partial_e X_1, y^* \in \partial_e Y_1^*\}$.

Theorem 3. *Suppose X and Y are separable reflexive Banach spaces with the metric approximation property such that $\mathcal{K}(X, Y)$ is a Hahn-Banach smooth subspace of $\mathcal{L}(X, Y)$. Let $\Psi_0 : \mathcal{K}(X, Y) \to \mathcal{L}(X, Y)$ be an into isometry such that $\|\Psi_0^*(x \otimes y^*)\| = \|x\|\|y^*\|$ for $x \in X$, $y^* \in Y^*$. Then Ψ_0 has a unique extension to an isometry in $\mathcal{L}(\mathcal{L}(X, Y))$.*

Proof. As already remarked, the hypothesis implies that $\mathcal{L}(X, Y)$ is the canonical bidual of $\mathcal{K}(X, Y)$. Thus uniqueness of the extension follows from the above corollary. Let $T \in \mathcal{L}(X, Y)$; to define the extension $\Psi(T)$ we once again use uniqueness of extensions. Since $\mathcal{L}(X, Y) = (X \otimes_\pi Y^*)^*$, we let $\Psi(T)(x \otimes y^*) = \Psi_0^*(x \otimes y^*)(T)$. This by hypothesis is a linear contraction and is an extension of Ψ_0. To show that it is an isometry we proceed as in the proof of the Lemma in [3]. Let $\{T_n\}_{n \geq 1} \subset \mathcal{K}(Y)$ be a sequence of contractions of finite rank such that $T_n \to I$ in the strong operator topology (s. o. t.). Since $T_n T \to T$ in the s. o. t, we have for $x \in X$ and $y^* \in Y^*$, $y^*(\Psi(T)(x)) = lim\ y^*(\Psi_0(T_n T)(x))$. As Ψ_0 is an isometry, we conclude that $\|\Psi(T)\| \geq \|T\|$. ☐

Corollary 4. *Suppose X and Y are separable reflexive Banach spaces with the metric approximation property such that $\mathcal{K}(X, Y)$ is a Hahn-Banach smooth subspace of $\mathcal{L}(X, Y)$. Suppose X is strictly convex and Y is smooth. Let $\Psi_0 : \mathcal{K}(X, Y) \to \mathcal{L}(X, Y)$ be an into, nice isometry . Then Ψ_0 has a unique extension to an isometry in $\mathcal{L}(\mathcal{L}(X, Y))$.*

Proof. We note that if X is strictly convex and Y is smooth, then $x \otimes y^*$ for $x \in S_X$ and $y^* \in S_{Y^*}$ are precisely the extreme points of $\mathcal{K}(X, Y)_1^*$. Now the nice assumption on Ψ_0 implies that the hypothesis of the above theorem is satisfied. Hence the conclusion follows. ☐

The following is a formulation of the above theorem for general Hahn-Banach smooth spaces.

Theorem 5. *Let X be a separable Hahn-Banach smooth space. Let $\Phi : X \to X^{**}$ be an isometry such that $\|\Phi^*(x^*)\| = 1$ for all $x^* \in \partial_e X_1^*$. Then $\Psi = \pi_{X^{**}} \circ \Phi^{**} : X^{**} \to X^{**}$ is the isometry that uniquely extends Φ.*

Proof. As noted before we only need to show that Ψ is an isometry. Let $0 \neq \Lambda \in X^{**}$. Since X^* has the Radon-Nikodým property, the unit ball is the norm closed convex hull of its extreme points. Thus Λ is determined by its values at the extreme points of the unit ball. Also since X is separable, so is (Theorem VII. 2.6 of [1]) X^* and hence X_1 is weak*-sequentially dense in X_1^{**}. Let $\{x_n\}_{n \geq 1} \subset X$ and $x_n \to \Lambda$ in the weak*-topology and such that $\|x_n\| \to \|\Lambda\|$. We now have $\Phi(x_n) = \Phi^{**}(x_n) \to \Phi^{**}(\Lambda)$. Now for any $x^* \in \partial_e X_1^*$, $\Psi(\Lambda)(x^*) = lim\ \Phi(x_n)(x^*)$. Since Φ is an isometry as in the proof of the above theorem, we conclude that $\|\Psi(\Lambda)\| \geq \|\Lambda\|$. Hence Ψ is an isometry. ☐

Now suppose that $\mathcal{K}(X, Y)$ is Hahn-Banach smooth in its bidual. Since this is a hereditary property, we have that X^* and Y are Hahn-Banach smooth, and thus by Lemma 1 of [8] we have that X is reflexive. Now if one assumes that X or Y^* has the metric approximation property, then we see that $\mathcal{L}(X, Y^{**})$ is the bidual of $\mathcal{K}(X, Y)$. Now the following corollary is easy to deduce from the above theorem. Unlike Theorem 3 or Corollary 4 here the extension is not explicitly defined.

Corollary 6. *Suppose $\mathcal{K}(X,Y)$ is a Hahn-Banach smooth space. Assume further that X or Y^* has the metric approximation property and both are separable. Then any nice into isometry $\Phi : \mathcal{K}(X,Y) \to \mathcal{L}(X,Y^{**})$ has a unique extension to an isometry in $\mathcal{L}(\mathcal{L}(X,Y^{**}))$.*

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