

## ON THE INVARIANT TRANSLATION APPROXIMATION PROPERTY FOR DISCRETE GROUPS

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ABSTRACT. Recently J. Roe considered the question of whether for a discrete group the reduced group  $C^*$ -algebra  $C_r^*(\Gamma)$  is the fixed point algebra of  $\{\text{Ad}(\rho_t) \mid t \in \Gamma\}$  acting on the uniform Roe algebra  $UC_r^*(\Gamma)$ .  $\Gamma$  is said to have the invariant translation approximation property in this case. We consider a slight generalization of this property which, for exact  $\Gamma$ , is equivalent to the operator space approximation property of  $C_r^*(\Gamma)$ . We also give a new characterization of exactness and a short proof of the equivalence of exactness of  $\Gamma$  and exactness of  $C_r^*(\Gamma)$  for discrete groups.

### 1. INTRODUCTION

A discrete group  $\Gamma$  has a natural coarse structure which allows us to define the uniform Roe algebra  $UC_r^*(\Gamma)$ . This algebra may be thought of as the closure of scalar  $\Gamma \times \Gamma$ -matrices  $[\alpha_{s,t}]$  of finite width (i.e.  $\{st^{-1} \mid \alpha_{s,t} \neq 0\}$  is finite) with uniformly bounded entries acting on  $\ell^2(\Gamma)$ . The reduced group  $C^*$ -algebra  $C_r^*(\Gamma)$  is naturally contained in  $UC_r^*(\Gamma)$ . Indeed the left translation  $\lambda_r$  on  $\ell^2(\Gamma)$  is given by the matrix where  $\alpha_{s,t} = 1$  if  $st^{-1} = r$  and  $\alpha_{s,t} = 0$  otherwise; so the matrices  $[\alpha_{s,t}]$  of finite width such that  $\alpha_{sr,tr} = \alpha_{s,t}$  for all  $s, t, r \in \Gamma$  form precisely the group ring  $\mathbb{C}[\Gamma]$ . They may also be characterized as those finite width matrices fixed by all automorphisms of the form  $\text{Ad}(\rho_t)$ , where  $t \in \Gamma$  and  $\rho$  is the right regular representation of  $\Gamma$ .

Now consider the set of fixed points  $UC_r^*(\Gamma)^\Gamma$  of this action in the whole of  $UC_r^*(\Gamma)$ , not just in the dense subalgebra of finite width matrices. According to [Ro]  $\Gamma$  is said to have the invariant translation approximation property if  $C_r^*(\Gamma) = UC_r^*(\Gamma)^\Gamma$ . One might wonder whether all discrete groups have this property.

In this note we do not answer this question but consider a slightly stronger invariance property defined as follows. One may define  $UC_r^*(\Gamma, S)$  with coefficients in a (concrete) operator space  $S$ .  $UC_r^*(\Gamma, S)$  is an operator space in general and is a  $C^*$ -algebra if  $S$  is a  $C^*$ -algebra. This very natural idea to allow nontrivial coefficients in the definition of the uniform Roe algebra might be useful elsewhere.  $UC_r^*(\Gamma, -)$  may be regarded as a functor (on  $C^*$ -algebras or on operator spaces). In Theorem 2.3 we show that  $\Gamma$  is exact iff this functor is exact on  $C^*$ -algebras and

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give a quick proof of the result of Kirchberg and Wassermann on the equivalence of exactness of  $\Gamma$  and exactness of  $C_r^*(\Gamma)$  for discrete groups ([KW]) together with another characterization of exactness.

Instead of looking at the fixed points in  $UC_r^*(\Gamma)$  we consider fixed points in  $UC_r^*(\Gamma, S)$ . Using results and techniques from [HK] we show that for exact  $\Gamma$  the following conditions are equivalent:

- (1)  $UC_r^*(\Gamma, S)^\Gamma = C_r^*(\Gamma) \otimes S$  for all closed subspaces  $S$  of the compact operators  $\mathcal{K}$  (on  $\ell^2(\mathbb{N})$ ).
- (2)  $(UC_r^*(\Gamma) \otimes S)^\Gamma = C_r^*(\Gamma) \otimes S$  for all closed subspaces  $S$  of  $\mathcal{K}$ .
- (3) For any operator space  $S$  (not necessarily contained in  $\mathcal{K}$ )  $UC_r^*(\Gamma, S)^\Gamma = (UC_r^*(\Gamma) \otimes S)^\Gamma = C_r^*(\Gamma) \otimes S$ .
- (4)  $C_r^*(\Gamma)$  has the operator approximation property (OAP) or equivalently  $\Gamma$  has the AP of [HK].

Here and throughout  $\otimes$  denotes the minimal tensor product. For group  $C^*$ -algebras of discrete groups the OAP implies exactness, and so condition (4) implies the other conditions for all discrete groups  $\Gamma$ . Moreover, the invariant translation approximation property with coefficients implies the one without coefficients, so if  $\Gamma$  has the AP, then  $C_r^*(\Gamma) = UC_r^*(\Gamma)^\Gamma$ . We also remark that it is not known whether there are exact groups without the AP, but it has been conjectured that  $SL_3(\mathbb{Z})$  is such a group (cf. [HK] or [ER]).

## 2. PRELIMINARIES

A  $C^*$ -algebra  $A$  is *exact* if given any exact sequence  $0 \rightarrow J \rightarrow B \rightarrow C \rightarrow 0$  of  $C^*$ -algebras the sequence  $0 \rightarrow A \otimes J \rightarrow A \otimes B \rightarrow A \otimes C \rightarrow 0$  is again exact, i.e.  $A \otimes -$  is an exact functor. Inspection shows that for arbitrary  $A$  the kernel of the map  $A \otimes B \rightarrow A \otimes C$  is the *Fubini product*  $F(A, J) = \{x \in A \otimes B \mid (\varphi \otimes \text{id})(x) \in J \forall \varphi \in A^*\}$  (cf. [Wa]). Thus exactness of  $A$  is equivalent to requiring  $F(A, J) = A \otimes J$  for all pairs  $J \subseteq B$ , where  $B$  is a  $C^*$ -algebra and  $J$  a closed ideal of  $B$ .

Kraus [Kr] considered the same condition for subspaces, i.e.  $F(A, S) = A \otimes S$  for all pairs  $S \subseteq B$ , where  $B$  is a  $C^*$ -algebra and  $S$  a closed subspace of  $B$ . (Wassermann had considered this earlier for subalgebras.) He showed (cf. [Kr], [ER]) that this condition, the *slice map property for subspaces*, is equivalent to the *strong operator approximation property (strong OAP)* which means the following: for every  $C^*$ -algebra  $B$  there is a net  $(\phi_\alpha)$  of finite rank maps such that  $\phi_\alpha \otimes \text{id}(x) \rightarrow x$  for all  $x \in A \otimes B$ . In particular if  $A$  has the strong OAP, then it is exact. A weaker variant is the *operator approximation property (OAP)* which requires only a net  $(\phi_\alpha)$  of finite rank maps such that  $\phi_\alpha \otimes \text{id}(x) \rightarrow x$  for all  $x \in A \otimes \mathcal{K}$  ( $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ ). It is the operator space version of Grothendieck's AP (cf. [ER]). The OAP is equivalent to the slice map property for closed subspaces of  $\mathcal{K}$ .

Most regularity properties of  $C^*$ -algebras define, when required for  $C_r^*(\Gamma)$ , reasonable regularity properties for the group  $\Gamma$ , e.g. nuclearity of  $C_r^*(\Gamma)$  is equivalent to amenability of  $\Gamma$ . Similarly  $C_r^*(\Gamma)$  is an exact  $C^*$ -algebra iff  $\Gamma$  is amenable at infinity. Another formally stronger property is exactness of  $\Gamma$  defined as follows ([KW]). Let  $\alpha : \Gamma \rightarrow \text{Aut}(A)$  be a  $\Gamma$ -action on a  $C^*$ -algebra  $A$  such that  $\alpha_s(J) = J$  for all  $s \in \Gamma$ ; then  $\dot{\alpha}_s(a + J) = \alpha_s(a) + J$  defines an action  $\dot{\alpha}$  on  $B = A/J$ . Denoting the restriction of  $\alpha$  to  $J$  by  $\alpha$  again, we obtain the exact sequence of  $\Gamma$ -dynamical

systems  $0 \rightarrow (J, \alpha) \rightarrow (A, \alpha) \rightarrow (B, \dot{\alpha}) \rightarrow 0$ .  $\Gamma$  is said to be *exact* if the corresponding sequence of reduced crossed products  $0 \rightarrow J \rtimes_{\alpha, r} \Gamma \rightarrow A \rtimes_{\alpha, r} \Gamma \rightarrow B \rtimes_{\dot{\alpha}, r} \Gamma \rightarrow 0$  is exact. A discrete group  $\Gamma$  is exact iff  $C_r^*(\Gamma)$  is an exact  $C^*$ -algebra ([KW]), and we will give a short proof of this fact in Theorem 2.3.

As for the strong OAP and the OAP, Haagerup and Kraus [HK] show that they are equivalent for  $C_r^*(\Gamma)$  if  $\Gamma$  is a discrete group. They also establish an equivalent condition in terms of the Fourier algebra  $A(\Gamma)$  which they call the approximation property (AP). The Fourier algebra  $A(\Gamma)$  is the set of all coefficients  $s \mapsto \langle \xi, \lambda_s \eta \rangle$  in the regular representation of  $\Gamma$  with the pointwise product and the norm  $\|u\| = \inf\{\|\xi\| \|\eta\| \mid u(s) = \langle \xi, \lambda_s \eta \rangle \text{ for all } s \in \Gamma\}$ . Amenability of  $\Gamma$  is equivalent to the existence of a bounded approximate unit in  $A(\Gamma)$  (even one consisting of positive definite functions). In order to explain the AP recall that a complex function  $u$  on  $\Gamma$  is a *multiplier* of  $A(\Gamma)$  if  $uv \in A(\Gamma)$  for all  $v \in A(\Gamma)$ . In  $A(\Gamma)$  convergence in norm implies pointwise convergence, and therefore, by the closed graph theorem, such a multiplier  $u$  defines a bounded linear map on  $A(\Gamma)$ . Since  $A(\Gamma) = VN(\Gamma)_*$ , by duality, it also defines a bounded linear  $\overline{M}_u \in B(VN(\Gamma))$ , and since  $\overline{M}_u$  leaves the group ring  $\mathbb{C}[\Gamma]$  invariant we finally obtain a bounded  $M_u \in B(C_r^*(\Gamma))$ .  $u$  is said to be a completely bounded multiplier if  $M_u$  is a completely bounded map, i.e.  $M_u \in CB(C_r^*(\Gamma))$ . The space  $M_0A(\Gamma)$  of completely bounded multipliers can then be equipped with the completely bounded norm  $\|u\|_{M_0}$ . As cited in [HK] (cf. [Pi])  $u \in M_0A(\Gamma)$  iff there is a Hilbert space  $K$  and bounded maps  $p, q : \Gamma \rightarrow K$  such that  $u(st^{-1}) = \langle p(s), q(t) \rangle$  for all  $s, t \in \Gamma$ . Moreover, the completely bounded norm is given by

$$\|u\|_{M_0} = \inf\{\|p\|_\infty \|q\|_\infty \mid \exists K \text{ s.t. } p, q : \Gamma \rightarrow K, u(st^{-1}) = \langle p(s), q(t) \rangle \forall s, t \in \Gamma\}.$$

Since all  $u \in M_0A(\Gamma)$  are bounded functions, every summable sequence of scalars  $(\alpha_s) \in \ell^1(\Gamma)$  defines a functional on  $M_0A(\Gamma)$  by  $\omega_{(\alpha_s)}(u) = \sum_s \alpha_s u(s)$ . Let  $Q(\Gamma)$  be the closure of  $\{\omega_{(\alpha_s)} \mid (\alpha_s) \in \ell^1(\Gamma)\}$  in  $M_0A(\Gamma)^*$ . Then  $\Gamma$  is said to have the *approximation property (AP)* if there is a net  $(u_\alpha) \subseteq A(\Gamma)$  converging to 1 in the  $\sigma(M_0A(\Gamma), Q(\Gamma))$ -topology. By ([HK], 2.1)  $C_r^*(\Gamma)$  has the OAP iff  $\Gamma$  has the AP. It is pointed out in [HK] that one may assume  $u_\alpha \in A_c(\Gamma)$ , i.e. all the  $u_\alpha$  to have finite support.

A matrix  $[\alpha_{s,t}]_{s,t \in \Gamma}$  is said to have *width  $F$* , where  $F \subseteq \Gamma$  is a subset if  $\alpha_{s,t} \neq 0$  implies  $st^{-1} \in F$ . If  $[\alpha_{s,t}]_{s,t \in \Gamma}$  has finite width (i.e. width  $F$  for some finite  $F \subseteq \Gamma$ ) and there is  $M \geq 0$  such that  $|\alpha_{s,t}| \leq M$  for all  $s, t \in \Gamma$ , then this matrix defines a bounded operator acting on  $\ell^2(\Gamma)$ . Such matrices form a  $*$ -algebra and the uniform Roe algebra  $UC_r^*(\Gamma)$  is the closure of it in  $B(\ell^2(\Gamma))$ . We have  $UC_r^*(\Gamma) = \ell^\infty(\Gamma) \rtimes_{\tau, r} \Gamma$ , where  $\tau$  is the translation action on  $\ell^\infty(\Gamma)$ .  $UC_r^*(\Gamma)$  is closely related to exactness of  $\Gamma$ . More precisely Guentner, Kaminker ([GK]) and Ozawa ([Oz]) showed the following important result (cf. [Pi] for an excellent exposition).

**Theorem 2.1.** *For a discrete group  $\Gamma$  the following conditions are equivalent:*

- (1)  $C_r^*(\Gamma)$  is exact.
- (2)  $UC_r^*(\Gamma)$  is nuclear.
- (3) *There exists a net of finite width positive definite kernels  $k_\alpha : \Gamma \times \Gamma \rightarrow \mathbb{C}$  such that for all  $\varepsilon > 0$  and every finite subset  $F \subseteq \Gamma$  there is  $\alpha_0$  such that  $|k_\alpha(s, t) - 1| < \varepsilon$  whenever  $st^{-1} \in F$  and  $\alpha \geq \alpha_0$ .*

In (3)  $k_\alpha$  positive definite means  $\sum_{s,t} k_\alpha(s, t) \bar{\alpha}_s \alpha_t \geq 0$  for all  $(\alpha_s)_{s \in \Gamma}$  of finite support. Condition (3) is quite similar to classical amenability which is equivalent

to the existence of a net of positive definite functions on  $\Gamma$  with analogous properties. Another equivalent condition says that  $\Gamma$  is amenable at infinity, i.e. acts amenably on a compact space.

By a GNS-type construction attributed to Kolmogoroff any positive definite kernel  $k : \Gamma \times \Gamma \rightarrow \mathbb{C}$  is of the form  $k(s, t) = \langle v(s), v(t) \rangle$ , where  $v : \Gamma \rightarrow K$  is a map into a suitable Hilbert space  $K$ . Thus  $V(e_s) = e_s \otimes v(s)$  defines a linear map  $V : \ell^2(\Gamma) \rightarrow \ell^2(\Gamma) \otimes K$  such that  $\|V^*V\| = \sup\{\|v(s)\|^2 \mid s \in \Gamma\}$ , and if  $X \in B(\ell^2(\Gamma))$  has the matrix representation  $[\alpha_{s,t}]$ , then  $V^*(X \otimes 1)V$  has the matrix representation  $[k(s, t)\alpha_{s,t}]$ . Thus  $\Theta_k([\alpha_{s,t}]) = [k(s, t)\alpha_{s,t}]$  defines a completely positive map (Schur multiplier)  $\Theta_k \in B(B(\ell^2(\Gamma)))$  which is used in [Oz]. If  $k$  has finite width, then  $\Theta_k(B(\ell^2(\Gamma)))$  consists of finite width matrices. Moreover, if  $(k_\alpha)$  is as in Theorem 2.1(3), then  $\Theta_{k_\alpha}(x) \rightarrow x$  for  $x \in UC_r^*(\Gamma)$  since this is true for finite width matrices and  $\|\Theta_{k_\alpha}\| \rightarrow 1$  as  $\alpha \rightarrow \infty$ .

If  $S \subseteq B(H)$  is a closed subspace (i.e. a concrete operator space) we may define the operator space  $UC_r^*(\Gamma, S)$  as the closure of finite width matrices  $[a_{s,t}]_{s,t \in \Gamma}$ , where  $a_{s,t} \in S$  and  $\|a_{s,t}\|$  is uniformly bounded for all  $s, t \in \Gamma$  acting on  $\ell^2(\Gamma) \otimes H$ . Clearly  $UC_r^*(\Gamma) \otimes S \subseteq UC_r^*(\Gamma, S)$ . Similar to tensor product functors like  $C_r^*(\Gamma) \otimes -$ , we may regard  $UC_r^*(\Gamma, -)$  as a functor on the category of  $C^*$ -algebras and  $*$ -homomorphisms, and will say that  $UC_r^*(\Gamma, -)$  is an exact functor if for every exact sequence  $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$  of  $C^*$ -algebras the induced sequence  $0 \rightarrow UC_r^*(\Gamma, J) \rightarrow UC_r^*(\Gamma, A) \rightarrow UC_r^*(\Gamma, B) \rightarrow 0$  is exact. In Theorem 2.3 we will see that exactness of  $\Gamma$  and of  $UC_r^*(\Gamma, -)$  are equivalent. The proof uses Theorem 2.1 and ideas from [KW] and [Ki]. We start with some preliminaries.

Let  $0 \rightarrow (J, \alpha) \rightarrow (A, \alpha) \xrightarrow{p} (B, \dot{\alpha}) \rightarrow 0$  be an exact sequence of  $\Gamma$ -dynamical systems. Denote by  $p_\alpha$  the surjective  $*$ -homomorphism  $p_\alpha : A \rtimes_{\alpha, r} \Gamma \rightarrow B \rtimes_{\dot{\alpha}, r} \Gamma$  induced by  $p$ .  $A \rtimes_{\alpha, r} \Gamma$  may be regarded naturally as a subalgebra of  $UC_r^*(\Gamma, A)$  since for  $a \in A$  and  $s \in \Gamma$  the element  $a\lambda_s \in A \rtimes_{\alpha, r} \Gamma$  corresponds to the matrix  $\sum_{t \in \Gamma} e_{t, s^{-1}t} \otimes \alpha_{t^{-1}}(a) \in UC_r^*(\Gamma, A)$ . Let  $(e_\lambda)_{\lambda \in \Lambda}$  be an approximate unit in  $J$  consisting of positive contractions. Then  $\hat{e}_\lambda = \sum_{t \in \Gamma} e_{t, t} \otimes \alpha_{t^{-1}}(e_\lambda)$  corresponds to the element  $e_\lambda$  in the regular representation and is an approximate unit for  $J \rtimes_{\alpha, r} \Gamma$ . Since  $\hat{e}_\lambda(A \rtimes_{\alpha, r} \Gamma)\hat{e}_\lambda \subseteq J \rtimes_{\alpha, r} \Gamma$  and  $\hat{e}_\lambda(a\lambda_s)\hat{e}_\lambda \rightarrow a\lambda_s$  for all  $a \in J$  and  $s \in \Gamma$ , it follows that  $x \in A \rtimes_{\alpha, r} \Gamma$  lies in  $J \rtimes_{\alpha, r} \Gamma$  iff  $\hat{e}_\lambda x \hat{e}_\lambda \rightarrow x$  as  $\lambda \rightarrow \infty$ .

Let  $\ell^\infty(\Lambda, A)$  be the bounded functions on  $\Lambda$  with values in  $A$  and  $c_0(\Lambda, A) = \{(a_\lambda) \in \ell^\infty(\Lambda, A) \mid a_\lambda \rightarrow 0 \text{ as } \lambda \rightarrow \infty\}$ . Denoting the quotient  $C^*$ -algebra  $\ell^\infty(\Lambda, A)/c_0(\Lambda, A)$  by  $q^\infty(\Lambda, A)$  we obtain the exact sequence

$$0 \longrightarrow c_0(\Lambda, A) \longrightarrow \ell^\infty(\Lambda, A) \xrightarrow{q} q^\infty(\Lambda, A) \longrightarrow 0$$

and the induced surjective  $*$ -homomorphism

$$\hat{q} : UC_r^*(\Gamma, \ell^\infty(\Lambda, A)) \longrightarrow UC_r^*(\Gamma, q^\infty(\Lambda, A)).$$

Following ideas in [Ki] we define the completely bounded map

$$\delta : A \rtimes_{\alpha, r} \Gamma \longrightarrow UC_r^*(\Gamma, \ell^\infty(\Lambda, A))$$

by  $\delta(x)_\lambda = x - \hat{e}_\lambda x \hat{e}_\lambda$ , where we think of  $UC_r^*(\Gamma, \ell^\infty(\Lambda, A)) \subseteq \ell^\infty(\Lambda, UC_r^*(\Gamma, A))$  as the closure of families  $(x_\lambda)$ , where all  $x_\lambda \in UC_r^*(\Gamma, A)$  have the same finite width. Then it is clear that  $\delta(a\lambda_s) \in UC_r^*(\Gamma, \ell^\infty(\Lambda, A))$  for all  $a \in A$  and  $s \in \Gamma$ , and so  $\delta(A \rtimes_{\alpha, r} \Gamma) \subseteq UC_r^*(\Gamma, \ell^\infty(\Lambda, A))$ .

**Lemma 2.2.** For  $x \in A \rtimes_{\alpha,r} \Gamma$  consider the following conditions:

- (1)  $x \in J \rtimes_{\alpha,r} \Gamma$ .
- (2)  $\delta(x) \in UC_r^*(\Gamma, c_0(\Lambda, A))$ .
- (3)  $x \in \ker(p_\alpha : A \rtimes_{\alpha,r} \Gamma \rightarrow B \rtimes_{\dot{\alpha},r} \Gamma)$ .
- (4)  $x_{s,t} \in J$  for all  $s, t \in \Gamma$ , where  $x_{s,t}$  denote the matrix elements of  $x$ .
- (5)  $\delta(x) \in \ker(\hat{q} : UC_r^*(\Gamma, \ell^\infty(\Lambda, A)) \rightarrow UC_r^*(\Gamma, q^\infty(\Lambda, A)))$ .

Then (1)  $\Leftrightarrow$  (2) and (3)  $\Leftrightarrow$  (4)  $\Leftrightarrow$  (5).

*Proof.* (1)  $\Leftrightarrow$  (2): Follows from the fact that  $\hat{e}_\lambda x \hat{e}_\lambda \xrightarrow{\lambda \rightarrow \infty} x$  iff  $x \in J \rtimes_{\alpha,r} \Gamma$ .

(3)  $\Leftrightarrow$  (4): This is evident if we note that  $p_\alpha : A \rtimes_{\alpha,r} \Gamma \rightarrow B \rtimes_{\dot{\alpha},r} \Gamma$  is simply the restriction of  $\hat{p} : UC_r^*(\Gamma, A) \rightarrow UC_r^*(\Gamma, B)$  and the kernel of  $\hat{p}$  consists precisely of those matrices with entries in  $J$ .

(4)  $\Leftrightarrow$  (5): We have

$$\ker \hat{q} = \{X \in UC_r^*(\Gamma, \ell^\infty(\Lambda, A)) \mid X_{s,t} \in c_0(\Lambda, A) \text{ for all } s, t \in \Gamma\}.$$

Regarding  $\delta(x)$  as in its definition as a family  $(\delta(x)_\lambda)_{\lambda \in \Lambda}$ , we have

$$\delta(x)_{\lambda;s,t} = x_{s,t} - \alpha_{s^{-1}}(e_\lambda)x_{s,t}\alpha_{t^{-1}}(e_\lambda).$$

$(\alpha_{s^{-1}}(e_\lambda))_{\lambda \in \Lambda}$  and  $(\alpha_{t^{-1}}(e_\lambda))_{\lambda \in \Lambda}$  are also approximate units in  $J$ , whenever  $s, t \in \Gamma$  are arbitrary and fixed so that

$$x_{s,t} \in J \iff \delta(x)_{\lambda;s,t} \rightarrow 0 \text{ as } \lambda \rightarrow \infty$$

or  $x_{s,t} \in J$  iff  $(\lambda \mapsto \delta(x)_{\lambda;s,t}) \in c_0(\Lambda, A)$ . This proves (4)  $\Leftrightarrow$  (5). □

We are now ready to prove our characterizations of exactness. Condition (3) of the following theorem will be used later.

**Theorem 2.3.** For a discrete group the following conditions are equivalent:

- (1)  $\Gamma$  is exact.
- (2)  $C_r^*(\Gamma)$  is exact.
- (3) For all Hilbert spaces  $H$  and closed subspaces  $S \subseteq B(H)$ ,

$$UC_r^*(\Gamma, S) = \{x \in UC_r^*(\Gamma, B(H)) \mid x_{s,t} \in S \text{ for all } s, t \in \Gamma\}.$$

- (4)  $UC_r^*(\Gamma, -)$  is an exact functor.

*Proof.* (1)  $\Rightarrow$  (2): This is clear since  $0 \rightarrow J \rtimes_{\alpha,r} \Gamma \rightarrow A \rtimes_{\alpha,r} \Gamma \rightarrow B \rtimes_{\dot{\alpha},r} \Gamma \rightarrow 0$  is just  $0 \rightarrow J \otimes C_r^*(\Gamma) \rightarrow A \otimes C_r^*(\Gamma) \rightarrow B \otimes C_r^*(\Gamma) \rightarrow 0$  when  $\alpha$  is the trivial action  $\alpha_s = \text{id}$  for all  $s$ .

(2)  $\Rightarrow$  (3): Clearly  $\subseteq$  in (3) always holds. For the reverse inclusion note that if  $C_r^*(\Gamma)$  is exact, then there exists a net  $(k_\alpha)$  as in Theorem 2.1(3), and the Schur multiplier  $\Theta_{k_\alpha}$  defines a completely positive map on  $B(\ell^2(\Gamma) \otimes H)$  such that  $\Theta_{k_\alpha}(x) \rightarrow x$  is norm for all  $x \in UC_r^*(\Gamma, B(H))$ . Moreover, all the images  $\Theta_{k_\alpha}(\{x \in UC_r^*(\Gamma, B(H)) \mid x_{s,t} \in S \text{ for all } s, t \in \Gamma\})$  consist of finite width matrices only, which must have entries in  $S$  whose norms are uniformly bounded. Therefore they form subsets of  $UC_r^*(\Gamma, S)$ . Since  $UC_r^*(\Gamma, S)$  is norm closed and  $\Theta_{k_\alpha}(x) \rightarrow x$  for  $x \in UC_r^*(\Gamma, B(H))$ , the claim follows.

(3)  $\Rightarrow$  (4): Let  $0 \rightarrow J \rightarrow A \xrightarrow{p} B \rightarrow 0$  be any exact sequence of  $C^*$ -algebras (without a  $\Gamma$ -action).  $p$  induces a surjective  $*$ -homomorphism  $\hat{p} : UC_r^*(\Gamma, A) \rightarrow UC_r^*(\Gamma, B)$ , where  $\ker \hat{p} = \{x \in UC_r^*(\Gamma, A) \mid x_{s,t} \in J \text{ for all } s, t \in \Gamma\}$ . Thinking of

$A$  as being faithfully represented on a Hilbert space  $H$ , i.e.  $A \subseteq B(H)$ , we have  $UC_r^*(\Gamma, J) \subseteq \ker \hat{p} \subseteq \{x \in UC_r^*(\Gamma, B(H)) \mid x_{s,t} \in J \text{ for all } s, t \in \Gamma\} = UC_r^*(\Gamma, J)$  by assumption, hence  $\ker \hat{p} = UC_r^*(\Gamma, J)$ .

(4)  $\Rightarrow$  (1): Let  $0 \rightarrow (J, \alpha) \rightarrow (A, \alpha) \rightarrow (B, \dot{\alpha}) \rightarrow 0$  be an exact sequence of  $\Gamma$ -dynamical systems and  $p_\alpha : A \rtimes_{\alpha,r} \Gamma \rightarrow B \rtimes_{\dot{\alpha},r} \Gamma$  the induced surjective \*-homomorphism. Let  $0 \rightarrow c_0(\Lambda, A) \rightarrow \ell^\infty(\Lambda, A) \xrightarrow{\hat{q}} q^\infty(\Lambda, A) \rightarrow 0$  be the exact sequence before Lemma 2.2. Then by assumption the sequence

$$0 \longrightarrow UC_r^*(\Gamma, c_0(\Lambda, A)) \longrightarrow UC_r^*(\Gamma, \ell^\infty(\Lambda, A)) \xrightarrow{\hat{q}} UC_r^*(\Gamma, q^\infty(\Lambda, A)) \longrightarrow 0$$

is exact. Suppose that  $x \in \ker p_\alpha$ . Then by Lemma 2.2(3) $\Rightarrow$ (5) we know that  $\delta(x) \in \ker \hat{q} = UC_r^*(\Gamma, c_0(\Lambda, A))$  and by Lemma 2.2(2) $\Rightarrow$ (1) it follows that  $x \in J \rtimes_{\alpha,r} \Gamma$  and hence  $\Gamma$  is an exact group.  $\square$

Now we get back to the OAP, respectively AP for  $\Gamma$ . Let us first note that a completely bounded multiplier  $u \in M_0A(\Gamma)$  defines a completely bounded map  $\hat{M}_u \in CB(UC_r^*(\Gamma, S))$  for any operator space  $S \subseteq B(H)$  by  $\hat{M}_u([x_{s,t}]) = [u(st^{-1})x_{s,t}]$ . (Here  $CB$  stands for completely bounded linear map.) Indeed, as in the definition of Schur multipliers, let  $K$  be a Hilbert space and  $p, q : \Gamma \rightarrow K$  be bounded with  $u(st^{-1}) = \langle p(s), q(t) \rangle$  for all  $s, t \in \Gamma$ . Define bounded linear maps  $V, W : \ell^2(\Gamma) \otimes H \rightarrow \ell^2(\Gamma) \otimes H \otimes K$  by

$$V(e_s \otimes \xi) = e_s \otimes \xi \otimes p(s) \text{ and } W(e_s \otimes \xi) = e_s \otimes \xi \otimes q(s),$$

where  $(e_s) \subseteq \ell^2(\Gamma)$  is the canonical orthonormal basis in  $\ell^2(\Gamma)$  and  $\xi \in H$ . Then  $V^*(x \otimes 1_K)W = \hat{M}_u(x)$  for all  $x \in UC_r^*(\Gamma, S)$ . This map is clearly completely bounded with  $\text{cb-norm} \leq \|V\| \|W\|$ . It follows that  $\|\hat{M}_u\| \leq \|u\|_{M_0}$  and  $\sup_S \|\hat{M}_u\| = \|u\|_{M_0}$ , where the supremum is taken over all operator spaces.

For convergence properties of  $\hat{M}_{u_\alpha}$  (as  $\alpha \rightarrow \infty$ ), where  $(u_\alpha)$  is a net in  $A_c(\Gamma)$ , we can now use the same kind of arguments as in [HK] for tensor products which we sketch for convenience. By a Hahn-Banach type argument the point-norm closure and the point-weak closure of the subspace  $\{\hat{M}_u \mid u \in A_c(\Gamma)\} \subseteq B(UC_r^*(\Gamma, S))$  are identical. Thus the identity map is in the point-norm closure of this subspace iff there is a net  $(u_\alpha) \subseteq A_c(\Gamma)$  such that  $\varphi(\hat{M}_{u_\alpha}(x)) \rightarrow \varphi(x)$  for all  $\varphi \in UC_r^*(\Gamma, S)^*$  and  $x \in UC_r^*(\Gamma, S)$ . Now let us view  $u \mapsto \varphi(\hat{M}_u(x))$  as a bounded linear functional  $\omega_{\varphi,x} \in M_0A(\Gamma)^*$ . Then  $u_\alpha \rightarrow 1$  in the  $\sigma(M_0A(\Gamma), Q(\Gamma))$ -topology implies  $\varphi(\hat{M}_{u_\alpha}(x)) \rightarrow \varphi(x)$  for all  $\varphi \in UC_r^*(\Gamma, S)^*$  and  $x \in UC_r^*(\Gamma, S)$ , provided we can show the following.

**Lemma 2.4.**  $\omega_{\varphi,x}$  is in  $Q(\Gamma)$  for all  $\varphi \in UC_r^*(\Gamma, S)^*$  and  $x \in UC_r^*(\Gamma, S)$ .

*Proof.* Note first that  $\|\hat{M}_u\| \leq \|u\|_{M_0}$  implies  $\|\omega_{\varphi,x}\| \leq \|\varphi\| \|x\|$ . Since  $Q(\Gamma)$  is complete we may thus assume that  $x$  has finite width. But then  $\omega_{\varphi,x}(u)$  actually only depends on the value of the function  $u : \Gamma \rightarrow \mathbb{C}$  on a finite subset of  $\Gamma$  which clearly means that  $u \mapsto \omega_{\varphi,x}(u)$  is in  $Q(\Gamma)$  in this case.  $\square$

Thus the existence of  $(u_\alpha)$  in  $A_c(\Gamma)$  with  $u_\alpha \rightarrow 1$  in the  $\sigma(M_0A(\Gamma), Q(\Gamma))$ -topology guarantees that we may choose  $(u_\alpha)$  so that  $\hat{M}_{u_\alpha}(x) \rightarrow x$  for all  $x \in UC_r^*(\Gamma, S)$ .

3. THE INVARIANT TRANSLATION APPROXIMATION PROPERTY AND THE AP

Note that if  $S \subseteq B(H)$  is a concrete operator space, then  $\text{Ad}(\lambda_t \otimes \text{id})$  and  $\text{Ad}(\rho_t \otimes \text{id})$  both leave  $UC_r^*(\Gamma, S) \subseteq B(\ell^2(\Gamma) \otimes H)$  invariant. We say that a discrete group  $\Gamma$  has the *operator invariant translation approximation property* if for any closed subspace  $S$  of the compact operators  $\mathcal{K}$  (on  $\ell^2(\mathbb{N})$ ) the equality

$$UC_r^*(\Gamma, S)^\Gamma = C_r^*(\Gamma) \otimes S$$

is obtained, where  $UC_r^*(\Gamma, S)^\Gamma$  denotes the set of elements in  $UC_r^*(\Gamma, S)$  which are fixed under  $\text{Ad}(\rho_t \otimes \text{id})$  for all  $t \in \Gamma$ . If  $\Gamma$  is exact we may replace this condition by  $(UC_r^*(\Gamma) \otimes S)^\Gamma = C_r^*(\Gamma) \otimes S$ :

**Lemma 3.1.** *Suppose that  $\Gamma$  is exact and  $S \subseteq B(H)$  is an arbitrary closed subspace. Then  $UC_r^*(\Gamma, S)^\Gamma = (UC_r^*(\Gamma) \otimes S)^\Gamma$ .*

*Proof.* Let  $(k_\alpha)$  be as in Theorem 2.1(3). We show first that  $\Theta_{k_\alpha}(UC_r^*(\Gamma, S)^\Gamma) \subseteq UC_r^*(\Gamma) \odot S$ , where  $\odot$  denotes the algebraic tensor product. Given  $x \in UC_r^*(\Gamma, S)^\Gamma$  we have  $x_{s,t} = x_{sr,tr} \in S$  for all  $s, t, r \in \Gamma$ . So  $x_{s,t}$  only depends on the value of  $st^{-1}$ . In particular  $x_{s,t} = x_{st^{-1},e}$  for all  $s, t \in \Gamma$ . Let  $F \subseteq \Gamma$  be finite and  $x^F$  the element obtained from  $x$  by replacing  $x_{s,t}$  by 0 whenever  $st^{-1} \notin F$ . Then  $x^F = \sum_{r \in F} \lambda_r \otimes x_{r,e}$  is in  $\mathbb{C}[\Gamma] \odot S$ , where  $\mathbb{C}[\Gamma]$  is the group ring. Moreover,  $\Theta_{k_\alpha}(x) = \Theta_{k_\alpha}(x^F)$  provided  $k_\alpha$  has width  $F$ . But  $\Theta_{k_\alpha}(x^F) = \sum_{r \in F} \Theta_{k_\alpha}(\lambda_r) \otimes x_{r,e} \in UC_r^*(\Gamma) \odot S$ , thus  $\Theta_{k_\alpha}(UC_r^*(\Gamma, S)^\Gamma) \subseteq UC_r^*(\Gamma) \odot S$  for all  $\alpha$ . Since  $\Theta_{k_\alpha}(x) \rightarrow x$  for  $x \in UC_r^*(\Gamma, S)$ , it follows that  $UC_r^*(\Gamma, S)^\Gamma = UC_r^*(\Gamma) \otimes S \cap UC_r^*(\Gamma, S)^\Gamma = (UC_r^*(\Gamma) \otimes S)^\Gamma$ .  $\square$

Finally we can prove our result.

**Theorem 3.2.** *For a discrete exact group  $\Gamma$  the following conditions are equivalent:*

- (1)  $\Gamma$  has the operator invariant translation approximation property.
- (2)  $UC_r^*(\Gamma, S)^\Gamma = (UC_r^*(\Gamma) \otimes S)^\Gamma = C_r^*(\Gamma) \otimes S$  for any closed subspace  $S \subseteq \mathcal{K}$ .
- (3)  $UC_r^*(\Gamma, S)^\Gamma = (UC_r^*(\Gamma) \otimes S)^\Gamma = C_r^*(\Gamma) \otimes S$  for any Hilbert space  $H$  and any closed subspace  $S \subseteq B(H)$ .
- (4)  $\Gamma$  has the AP.

*Proof.* (1)  $\Leftrightarrow$  (2): follows from Lemma 3.1.

(1)  $\Rightarrow$  (4): given  $\Gamma$  exact with the operator invariant translation approximation property, it suffices to show that  $C_r^*(\Gamma)$  has the slice map property for closed subspaces  $S \subseteq \mathcal{K}$ . Thus given such an  $S$  we have to show that  $F(C_r^*(\Gamma), S) = \{x \in C_r^*(\Gamma) \otimes \mathcal{K} \mid (\varphi \otimes \text{id})(x) \in S \text{ for all } \varphi \in C_r^*(\Gamma)^*\}$  equals  $C_r^*(\Gamma) \otimes S$ . Now  $C_r^*(\Gamma) \otimes \mathcal{K} \subseteq UC_r^*(\Gamma, \mathcal{K})$ , so we may view  $x \in F(C_r^*(\Gamma), S)$  as a  $\Gamma \times \Gamma$ -matrix with entries in  $\mathcal{K}$ . Since  $((e_s \cdot e_t) \otimes \text{id})(x) = x_{s,t}$ , all its entries must be in  $S$  and if all entries  $x_{s,t}$  are in  $S$ , then one checks that  $x \in F(C_r^*(\Gamma), S)$ . Since  $\Gamma$  is exact Theorem 2.3(3) now implies that  $F(C_r^*(\Gamma), S) = C_r^*(\Gamma) \otimes \mathcal{K} \cap UC_r^*(\Gamma, S)$ , and this equals  $UC_r^*(\Gamma, \mathcal{K})^\Gamma \cap UC_r^*(\Gamma, S) = UC_r^*(\Gamma, S)^\Gamma = C_r^*(\Gamma) \otimes S$  if  $\Gamma$  has the operator invariant translation approximation property.

(4)  $\Rightarrow$  (3): let  $(u_\alpha) \subseteq A_c(\Gamma)$  (all of finite support) such that  $\hat{M}_{u_\alpha}(x) \rightarrow x$  for all  $x \in UC_r^*(\Gamma, B(H))$ .  $\hat{M}_{u_\alpha}$  commutes with  $\text{Ad}\rho_t$  for all  $t \in \Gamma$ , so for all closed subspaces  $S \subseteq B(H)$  the completely bounded operator  $\hat{M}_{u_\alpha} \in CB(UC_r^*(\Gamma, S))$  maps invariant elements into invariant elements. Moreover,  $\hat{M}_{u_\alpha}(UC_r^*(\Gamma, S)^\Gamma) \subseteq \mathbb{C}[\Gamma] \odot S$  by a similar argument as in the proof of Lemma 3.1. Since  $\hat{M}_{u_\alpha}(x) \rightarrow x$

whenever  $x \in UC_r^*(\Gamma, S)$ , it follows that  $UC_r^*(\Gamma, S)^\Gamma \subseteq C_r^*(\Gamma) \otimes S$ , i.e.  $UC_r^*(\Gamma, S)^\Gamma = C_r^*(\Gamma) \otimes S$ .

(3)  $\Rightarrow$  (1): obvious.  $\square$

*Remark.* Is it true that if  $\Gamma$  is a discrete group, then  $(UC_r^*(\Gamma) \otimes S)^\Gamma = UC_r^*(\Gamma)^\Gamma \otimes S$  for all closed subspaces  $S \subseteq \mathcal{K}$ ? If so, then Roe's invariant translation approximation property is equivalent to the OAP at least if  $\Gamma$  is exact. We leave this question open.

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