

## A HYPERFINITE INEQUALITY FOR FREE ENTROPY DIMENSION

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*For H-town*

ABSTRACT. If  $X, Y$ , and  $Z$  are finite sets of selfadjoint elements in a tracial von Neumann algebra and  $X$  generates a hyperfinite von Neumann algebra, then  $\delta_0(X \cup Y \cup Z) \leq \delta_0(X \cup Y) + \delta_0(X \cup Z) - \delta_0(X)$ .

In [11] Voiculescu describes the role of entropy in free probability. He discusses several problems in the area, one of which is the free entropy dimension problem. Free entropy dimension ([8], [9]) associates to an  $n$ -tuple of selfadjoint operators,  $X = \{x_1, \dots, x_n\}$ , in a tracial von Neumann algebra  $M$  a number  $\delta_0(X)$  called the (modified) free entropy dimension of  $X$ .  $\delta_0(X)$  is an asymptotic Minkowski or packing dimension of sets of  $n$ -tuples of matrices which model the behavior of  $X$ . The free entropy dimension problem simply asks whether  $\delta_0(X) = \delta_0(Y)$  for any other  $m$ -tuple of selfadjoint elements  $Y$  satisfying  $Y'' = X''$ . It is known from [10] that  $\delta_0$  is an algebraic invariant, i.e.,  $\delta_0(X) = \delta_0(Y)$  if  $X$  and  $Y$  generate the same algebra.

The origin of this remark started with two extremely special and highly tractable cases of this problem, the first being: If  $X, Y$  and  $Z$  are finite sets of selfadjoint elements in  $M$  such that  $X'' = Z''$  is hyperfinite, then is it true that

$$\delta_0(X \cup Y) = \delta_0(Y \cup Z)?$$

The second problem concerns invariance of  $\delta_0$  over the center: If  $Y$  is an arbitrary set of selfadjoint elements in  $M$  and  $y$  is any element in the center of  $Y''$ , then is it true that

$$\delta_0(Y \cup \{y\}) = \delta_0(Y)?$$

Both questions have affirmative answers and follow from a kind of hyperfinite inequality for  $\delta_0$ : If  $X, Y, Z$ , are sets of selfadjoint elements in  $M$  and  $X$  generates a hyperfinite von Neumann algebra, then

$$\delta_0(X \cup Y \cup Z) \leq \delta_0(X \cup Y) + \delta_0(X \cup Z) - \delta_0(X).$$

Related inequalities of this nature can be found in Gaboriau's work on the cost of equivalence relations [3].

The proof of the microstates inequality above is an application of the work in [5] paired with the packing formulation of  $\delta_0$  in [6].

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In addition to answering the two specialized invariance questions, the hyperfinite inequality for  $\delta_0$  has other applications. One involves a certain property of  $\Delta(\cdot)$  introduced by Connes and Shlyakhtenko in [2]. They show that if  $F$  is a finite set of selfadjoints in  $M$  with  $F''$  having diffuse center, then  $\Delta(F) = 1$ . They also show that  $\delta_0 \leq \Delta$  so this implies  $\delta_0(F) \leq 1$ . We use the hyperfinite inequality to give a microstates proof of this property. In another direction we show that if  $\{s_1, \dots, s_n\}$  is a free semicircular family with  $n > 1$  and  $x = x^* \in \{s_1, \dots, s_n\}'' = L(F_n)$ , then for some  $1 \leq j \leq n$ ,  $\delta_0(x, s_j) > 1$ . It follows from [4] that the von Neumann algebra generated by  $x$  and  $s_j$  is prime and has no Cartan subalgebras.

This paper has four sections. The first is a short list of assumptions. Motivated by the recent work of Belinschi and Bercovici ([1]), we find in the second section a slightly simpler formulation of  $\delta_0$  where the operator cutoff constants are removed. The third section presents the hyperfinite inequality. The fourth and last section presents several corollaries and include the proofs to the two invariance questions mentioned above.

## 1. PRELIMINARIES

Throughout, suppose  $M$  is a von Neumann algebra with a normal, tracial, faithful state  $\varphi$ . For any  $n \in \mathbb{N}$ ,  $|\cdot|_2$  denotes the norm on  $(M_k^{sa}(\mathbb{C}))^n$  given by  $|(x_1, \dots, x_n)|_2 = (\sum_{j=1}^n \text{tr}_k(x_j^2))^{1/2}$  where  $\text{tr}_k$  is the tracial state on the  $k \times k$  complex matrices, and  $|\cdot|_\infty$  denotes the operator norm.  $U_k$  denotes the  $k \times k$  unitary matrices. For any  $k, n \in \mathbb{N}$ ,  $u \in U_k$ , and  $x = (x_1, \dots, x_n) \in (M_k^{sa}(\mathbb{C}))^n$ , define  $uxu^* = (ux_1u^*, \dots, ux_nu^*)$ . We will maintain the notation introduced in [6], [8], and [9]. If  $F = \{a_1, \dots, a_n\}$  is a finite set of selfadjoint elements in  $M$ , we abbreviate  $\Gamma_R(a_1, \dots, a_n; m, k, \gamma)$  by  $\Gamma_R(F; m, k, \gamma)$  and in a similar way we write the associated microstate sets and quantities introduced in [6], [8], and [9]:  $\delta_0(F)$ ,  $\mathbb{P}_\epsilon(F)$ ,  $\mathbb{K}_\epsilon(F)$ . Also, if  $G = \{b_1, \dots, b_p\}$  is another finite set of selfadjoint elements in  $M$ , then we denote by  $\Gamma_R(F \cup G; m, k, \gamma)$  the set  $\Gamma_R(a_1, \dots, a_n, b_1, \dots, b_p; m, k, \gamma)$  and write all the associated microstate quantities  $\delta_0(F \cup G)$ ,  $\mathbb{P}_\epsilon(F \cup G)$ , and  $\mathbb{K}_\epsilon(F \cup G)$  with respect to  $\Gamma_R(F \cup G; m, k, \gamma)$ . Finally,  $\Gamma(F; m, k, \gamma)$  will denote the set of all  $k \times k$  microstates (no restrictions on the operator norms) with degree of approximation  $(m, \gamma)$ .

## 2. CUTOFF CONSTANTS

Recall that in [1] Belinschi and Bercovici have lifted the operator norm cutoff constants in the definition of  $\chi$ . In other words, if  $\chi(X)$  is the normal definition as conceived of by Voiculescu, and  $\chi_\infty(X)$  is the quantity obtained by replacing the microstate spaces  $\Gamma_R(X; m, k, \gamma)$  with  $\Gamma(X; m, k, \gamma)$ , then Belinschi and Bercovici showed that one always has

$$\chi(X) = \chi_\infty(X).$$

We want to show the same thing for the packing formulation of  $\delta_0$ . Voiculescu defined  $\delta_0(X)$  by

$$\delta_0(X) = n + \limsup_{\epsilon \rightarrow 0} \frac{\chi(x_1 + \epsilon s_1, \dots, x_n + \epsilon s_n : s_1, \dots, s_n)}{|\log \epsilon|}$$

where  $s_1, \dots, s_n$  is a semicircular family free with respect to  $X$ . One can actually also define  $\delta_0(X)$  in terms of metric space packings. One associates to each  $\epsilon > 0$

an asymptotic  $\epsilon$  packing number  $\mathbb{P}_\epsilon(X)$  and an  $\epsilon$  covering number  $\mathbb{K}_\epsilon(X)$ . These definitions also make use of cutoff constants. We recall the definitions. For any metric space  $\Omega$  and  $\epsilon > 0$  denote by  $P_\epsilon(\Omega)$  the maximum number in a collection of mutually disjoint open  $\epsilon$  balls of  $\Omega$  and by  $K_\epsilon(\Omega)$  the minimum number of open  $\epsilon$  balls required to cover  $\Omega$ . In what follows all spaces are endowed with the  $|\cdot|_2$  metric. Define successively:

$$\begin{aligned} \mathbb{P}_{\epsilon,R}(X; m, \gamma) &= \limsup_{k \rightarrow \infty} k^{-2} \cdot \log(P_\epsilon(\Gamma_R(X; m, k, \gamma))), \\ \mathbb{P}_{\epsilon,R}(X) &= \inf\{\mathbb{P}_{\epsilon,R}(X; m, \gamma) : m \in \mathbb{N}, \gamma > 0\}, \\ \mathbb{P}_\epsilon(X) &= \sup_{R > 0} \mathbb{P}_{\epsilon,R}(X). \end{aligned}$$

Similarly, we define  $\mathbb{K}_\epsilon(X)$  by replacing all the  $P_\epsilon$  above with  $K_\epsilon$ . It was shown in [6] that

$$\delta_0(X) = \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{P}_\epsilon(X)}{|\log \epsilon|} = \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_\epsilon(X)}{|\log \epsilon|}.$$

Now define successively:

$$\begin{aligned} \mathbb{P}_{\epsilon,\infty}(X; m, \gamma) &= \limsup_{k \rightarrow \infty} k^{-2} \cdot \log P_\epsilon(\Gamma(X; m, k, \gamma)), \\ \mathbb{P}_{\epsilon,\infty}(X) &= \inf\{\mathbb{P}_{\epsilon,\infty}(X; m, \gamma) : m \in \mathbb{N}, \gamma > 0\}. \end{aligned}$$

Similarly, we define  $\mathbb{K}_{\epsilon,\infty}$ . We want to show that the packing formulation for  $\delta_0$  holds when  $\mathbb{P}_\epsilon(X)$  is replaced with  $\mathbb{P}_{\epsilon,\infty}(X)$  and  $\mathbb{K}_\epsilon(X)$  is replaced with  $\mathbb{K}_{\epsilon,\infty}(X)$ . Assume for the remainder of this section that  $X = \{x_1, \dots, x_n\}$  is a finite set of selfadjoint elements in  $M$  and that  $R \geq 1$  is a constant greater than or equal to the maximum of the operator norms of the elements of  $X$ . We need one easy lemma which is undoubtedly known, but which we will prove for completeness:

**Lemma 2.1.** *For  $m_0 \in \mathbb{N}$ , and  $1 > \epsilon > \gamma_0 > 0$  there exists an  $m \in \mathbb{N}$  and  $\gamma > 0$  such that if  $\xi = (\xi_1, \dots, \xi_n) \in \Gamma(X; m, k, \gamma)$ , then  $|\xi - F_R(\xi)|_2 < \epsilon$  and  $F_R(\xi) \in \Gamma_R(X; m_0, k, \gamma_0)$  where  $F_r : \mathbb{R} \rightarrow [-r, r]$  is the monotone function equal to the identity on  $(-r, r)$  and  $F_r(\xi) = (F_r(\xi_1), \dots, F_r(\xi_n))$ .*

*Proof.* Denote by  $C$  the maximum over all numbers of the form  $(R + 1)^{2m_0}$  where  $1 \leq p \leq m_0$  and  $1 \leq i_1, \dots, i_p \leq n$  (this constant  $C$  is used to satisfy the second condition). Choose  $\gamma < \frac{\gamma_0}{100Cm_0n}$  and  $m \in \mathbb{N}$  so large that  $m > 2m_0$  and  $\frac{R^m + \gamma}{(R + \gamma)^m} < \frac{\gamma^2}{100n^2(R^4 + \gamma)}$ . Suppose  $\xi = (\xi_1, \dots, \xi_n) \in \Gamma(X; m, k, \gamma)$  and denote by  $\lambda_{i_1}, \dots, \lambda_{i_k}$  the eigenvalues of  $\xi_i$  with multiplicity. Set  $B_i = \{j \in \mathbb{N} : 1 \leq j \leq k, |\lambda_{ij}| > R + \gamma\}$ . We have

$$\begin{aligned} \#B_i \cdot (R + \gamma)^m &\leq \sum_{j \in B_i} |\lambda_{ij}|^m &\leq \sum_{j=1}^k |\lambda_{ij}|^m \\ &&\leq |\text{Tr}(\xi_i^m)| \\ &&< k(R^m + \gamma). \end{aligned}$$

Consequently,  $\frac{\#B_i}{k} \leq \frac{R^m + \gamma}{(R + \gamma)^m}$ . By the Cauchy-Schwarz inequality

$$\begin{aligned} |\xi_i - F_{R+\gamma}(\xi_i)|_2^2 &\leq \frac{1}{k} \cdot \sum_{j \in B_i} |\lambda_{ij}|^2 \\ &\leq |\xi_i^2|_2 \cdot \left(\frac{\#B_i}{k}\right)^{\frac{1}{2}} \\ &< (R^4 + \gamma)^{\frac{1}{2}} \cdot \sqrt{\frac{\gamma^2}{100n^2(R^4 + \gamma)}} \\ &< \frac{\gamma}{10n} \\ &< \frac{\epsilon}{10n}. \end{aligned}$$

$|F_{R+\gamma}(\xi_i) - F_R(\xi_i)|_2 \leq \gamma$  whence it follows that  $|\xi - F_R(\xi)|_2 < 2n\gamma < \epsilon$ . To see that the second claim is satisfied observe that for any  $1 \leq p \leq m_0$  and  $1 \leq i_1, \dots, i_p \leq n$ , Cauchy-Schwarz again yields

$$|tr_k(\xi_{i_1} \cdots \xi_{i_p}) - tr_k(F_R(\xi_{i_1}) \cdots F_R(\xi_{i_p}))| \leq C \cdot p \cdot \max_{1 \leq i \leq p} |\xi_i - F_R(\xi_i)|_2 < \gamma_0/2. \quad \square$$

**Lemma 2.2.**

$$\begin{aligned} \delta_0(X) = \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{P}_{\epsilon, \infty}(X)}{|\log \epsilon|} &= \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_{\epsilon, \infty}(X)}{|\log \epsilon|} \\ &= \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{P}_{\epsilon}(X)}{|\log \epsilon|} \\ &= \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_{\epsilon}(X)}{|\log \epsilon|}. \end{aligned}$$

*Proof.* Suppose  $m_0 \in \mathbb{N}$  and  $1 > \epsilon > \gamma_0 > 0$ . There exist by Lemma 2.1 an  $m \in \mathbb{N}$  and  $\gamma > 0$  such that if  $\xi \in \Gamma(X; m, k, \gamma)$ , then  $|\xi - F_R(\xi)|_2 < \epsilon$  and  $F_R(\xi) \in \Gamma_R(X; m_0, k, \gamma_0)$ . It follows from this that  $K_{2\epsilon}(\Gamma(X; m, k, \gamma)) \leq K_{\epsilon}(\Gamma_R(X; m_0, k, \gamma_0))$  whence

$$\mathbb{K}_{2\epsilon, \infty}(X) \leq \mathbb{K}_{2\epsilon}(X; m, \gamma) \leq \mathbb{K}_{\epsilon, R}(X, m_0, \gamma_0).$$

So  $\mathbb{K}_{2\epsilon, \infty}(X) \leq \mathbb{K}_{\epsilon, R}(X) \leq \mathbb{K}_{\epsilon}(X)$ . Now clearly,

$$\delta_0(X) = \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_{\epsilon}(X)}{|\log \epsilon|} \leq \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_{2\epsilon, \infty}(X)}{|\log 2\epsilon|} \leq \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_{\epsilon}(X)}{|\log 2\epsilon|} = \delta_0(X).$$

$\mathbb{P}_{\epsilon, \infty}(X) \geq \mathbb{K}_{2\epsilon, \infty}(X) \geq \mathbb{P}_{4\epsilon, \infty}(X)$  so this completes the proof. □

3. THE HYPERFINITE INEQUALITY FOR  $\delta_0$

Throughout, assume  $X, Y, Z$  and  $F$  are finite sets of selfadjoint elements in  $M$ . Assume further that  $X$  generates a hyperfinite von Neumann algebra, an assumption we will restate for emphasis in some of the corollaries.

**Definition 3.1.** Suppose for each  $m \in \mathbb{N}$  and  $\gamma > 0$ ,  $\langle \xi_k \rangle_{k=1}^{\infty}$  is a sequence such that for large enough  $k$ ,  $\xi_k \in \Gamma(X; m, k, \gamma)$ . The set of microstates  $\Xi(F; m, k, \gamma)$  for  $F$  relative to the  $\xi_k$  is

$$\Xi(F; m, k, \gamma) = \{\eta : (\xi_k, \eta) \in \Gamma(X \cup F; m, k, \gamma)\}.$$

Define successively for  $\epsilon > 0$ ,

$$\begin{aligned} \mathbb{K}_\epsilon(\Xi(F; m, \gamma)) &= \limsup_{k \rightarrow \infty} k^{-2} \cdot \log K_\epsilon(\Xi(F; m, k, \gamma)), \\ \mathbb{K}_\epsilon(\Xi(F)) &= \inf\{\mathbb{K}_\epsilon(F; m, \gamma) : m \in \mathbb{N}, \gamma > 0\}, \end{aligned}$$

where the packing quantities are taken with respect to  $|\cdot|_2$ . In a similar fashion, we define  $\mathbb{P}_\epsilon(\Xi(F))$  by replacing the  $K_\epsilon$  above with  $P_\epsilon$ .

**Lemma 3.2.** *For each  $m \in \mathbb{N}$  and  $\gamma > 0$  there exists a sequence  $\langle \xi_k \rangle_{k=1}^\infty$  satisfying  $\xi_k \in \Gamma(X; m, k, \gamma)$  for sufficiently large  $k$  such that if  $\mathbb{K}_\epsilon(\Xi(F))$  is taken relative to these sequences  $\langle \xi_k \rangle_{k=1}^\infty$ ,*

$$\delta_0(X \cup F) = \delta_0(X) + \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_\epsilon(\Xi(F))}{|\log \epsilon|}.$$

*Proof.* By [5] we can find for each  $m \in \mathbb{N}$  and  $\gamma > 0$  a sequence  $\langle \xi_k \rangle_{k=1}^\infty$  such that if  $\tau > 0$ , then for sufficiently large  $k$ ,  $\xi_k \in \Gamma_R(X; m, k, \gamma)$  and  $\dim \xi'_k \geq k^2(1 - \delta_0(X) - \tau)$ . Consider the associated  $\Xi(F; m, k, \gamma)$  relative to these sequences.

First we show that the left-hand side is greater than or equal to the right-hand side. Suppose  $t > 0$  is given. By [5] and [6] there exists an  $\epsilon_0 > 0$  such that for all  $\epsilon_0 > \epsilon > 0$  and any  $m \in \mathbb{N}, \gamma > 0$ ,  $\liminf_{k \rightarrow \infty} k^{-2} \cdot \log(P_\epsilon(\Gamma(X; m, k, \gamma))) > (\delta_0(X) - t)|\log 2\epsilon|$ . Now suppose  $m \in \mathbb{N}$  and  $\gamma > 0$  are fixed. Consider the  $\langle \xi_k \rangle_{k=1}^\infty$  corresponding to the  $m$  and  $\gamma$ . Because the von Neumann algebra generated by  $X$  is hyperfinite by [5] we can find a set of unitaries  $\langle v_{\lambda k} \rangle_{\lambda \in \Lambda_k}$  such that  $\langle v_{\lambda k} \xi_k v_{\lambda k}^* \rangle_{\lambda \in \Lambda_k}$  is an  $\epsilon$ -separated set with respect to the  $|\cdot|_2$  norm and  $\liminf_{k \rightarrow \infty} k^{-2} \cdot \log \#\Lambda_k > (\delta_0(X) - t)|\log 2\epsilon|$ . For each  $k$  pick an  $\epsilon$  separated subset  $\langle \zeta_{jk} \rangle_{k \in \Lambda_k}$  of minimal cardinality for  $\Xi(F; m, k, \gamma)$  (the set of microstates for  $F$  relative to  $\xi_k$ ). Now it is manifest that

$$\langle (v_{\lambda k} \xi_k v_{\lambda k}^*, v_{\lambda k} \zeta_{jk} v_{\lambda k}^*) \rangle_{(\lambda, j) \in \Lambda_k \times J_k}$$

is a subset of  $\Gamma(X \cup F; m, k, \gamma)$  and, moreover, it is easily checked that this set is  $\epsilon$ -separated with respect to the  $|\cdot|_2$  norm. Hence, for  $m$  large enough and  $\gamma$  small enough

$$\begin{aligned} \mathbb{P}_{\epsilon, \infty}(X \cup F; m, \gamma) &\geq \limsup_{k \rightarrow \infty} k^{-2} \cdot \log(\#\Lambda_k \cdot \#J_k) \\ &\geq \liminf_{k \rightarrow \infty} k^{-2} \cdot \log \#\Lambda_k + \limsup_{k \rightarrow \infty} k^{-2} \cdot \log P_\epsilon(\Xi(F; m, k, \gamma)) \\ &\geq (\delta_0(X) - t)|\log 2\epsilon| + \limsup_{k \rightarrow \infty} k^{-2} \cdot \log P_\epsilon(\Xi(F; m, k, \gamma)) \end{aligned}$$

so that for  $\epsilon_0 > \epsilon > 0$ ,

$$\mathbb{P}_{\epsilon, \infty}(X \cup F) \geq (\delta_0(X) - t)|\log 2\epsilon| + \mathbb{P}_\epsilon(\Xi(F)).$$

Using the packing formulation of  $\delta_0$  in [6], Lemma 2.2, and the fact that for any metric space  $\Omega$ ,  $P_\epsilon(\Omega) \geq K_{2\epsilon}(\Omega) \geq P_{4\epsilon}(\Omega)$ ,

$$\begin{aligned} \delta_0(X \cup F) &= \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{P}_{\epsilon, \infty}(X \cup F)}{|\log \epsilon|} \geq \limsup_{\epsilon \rightarrow 0} \frac{(\delta_0(X) - t)|\log 2\epsilon| + \mathbb{P}_\epsilon(\Xi(F))}{|\log \epsilon|} \\ &= \delta_0(X) - t + \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{P}_\epsilon(\Xi(F))}{|\log \epsilon|} \\ &= \delta_0(X) - t + \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_\epsilon(\Xi(F))}{|\log \epsilon|}. \end{aligned}$$

$t > 0$  being arbitrary we have

$$\delta_0(X \cup F) \geq \delta_0(X) + \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{P}_\epsilon(\Xi(F))}{|\log \epsilon|}.$$

For the reverse inequality by [6] there are  $C, \epsilon_0 > 0$  such that for  $\epsilon_0 > \epsilon > 0$  and for any  $k \in \mathbb{N}$  and tractable subgroup  $H$  of  $U_k$  (in the sense of [5]) there exists an  $\epsilon$ -net for  $U_k/H$  with respect to the quotient metric induced by  $|\cdot|_\infty$  with cardinality no greater than  $\left(\frac{C}{\epsilon}\right)^{\dim(U_k/H)}$ . Suppose  $R > 1$  is some number strictly greater than the maximum of the operator norms over elements in  $X \cup F$ . Suppose  $m \in \mathbb{N}$  and  $\tau, \gamma > 0$ . Observe that there exists  $\epsilon > r > 0$  so small that if  $(\xi, \eta) \in \Gamma_R(X \cup F; m, k, \gamma/2)$ ,  $|(\xi, \eta) - (x, a)|_2 < r$ , and the operator norms of any of the entries of  $x$  or  $a$  is less than  $R$ , then  $(x, a) \in \Gamma_R(X \cup F; m, k, \gamma)$ . There also exist  $m_1 \in \mathbb{N}$  and  $\gamma_1 > 0$  such that if  $\xi, x \in \Gamma_R(X; m_1, k, \gamma_1)$ , then there exists a  $u \in U_k$  satisfying  $|u\xi u^* - x|_2 < r$ . Set  $m_2 = m + m_1$  and  $\gamma_2 = \min\{\gamma/2, \gamma_1\}$ .

For each  $k$  find an  $\epsilon$ -net  $\langle \eta_{jk} \rangle_{j \in J_k}$  for  $\Xi(F; m, k, \gamma)$  with respect to  $|\cdot|_2$  of minimum cardinality. Define  $H_k$  to be the unitary group of  $\xi'_k$ . By the way the  $\xi_k$  were selected we can find for each  $k$  large enough a set of unitaries  $\langle u_{gk} \rangle_{g \in G_k}$  such that their images in  $U_k/H_k$  is an  $\epsilon$ -net with respect to the quotient metric induced by  $|\cdot|_\infty$  and such that

$$\#\Lambda_k \leq \left(\frac{C}{\epsilon}\right)^{(\delta_0(X) + \tau)k^2}.$$

Consider

$$\langle (u_{gk} \xi_k u_{gk}^*, u_{gk} \eta_{jk} u_{gk}^*) \rangle_{(g,j) \in G_k \times J_k}.$$

We claim that this set is a  $5\epsilon R(\#X + \#F)$ -net for  $\Gamma_R(X \cup F; m_2, k, \gamma_2)$ .

To see this suppose  $(\xi, \eta) \in \Gamma_R(X \cup F; m_2, k, \gamma_2)$ . By the selection of  $m_1$  and  $\gamma_1$  there exists a  $u \in U_k$  such that  $|u^* \xi_k u - \xi|_2 < r$ . Taking into account the stipulation on  $r$  this implies that  $(u^* \xi_k u, \eta) \in \Gamma_R(X \cup F; m, k, \gamma) \iff (\xi_k, u\eta u^*) \in \Gamma_R(X \cup F; m, k, \gamma)$ , whence  $u\eta u^* \in \Xi(F; m, k, \gamma)$ . There exists an  $g \in G_k$  and an  $h \in H_k$  such that  $|u - u_{gk} h|_\infty < \epsilon$ . Consequently,

$$|u_{gk} \xi_k u_{gk}^* - \xi|_2 = |u_{gk} h \xi_k h^* u_{gk} - \xi|_2 \leq 2\epsilon R \#X + |u \xi_k u^* - \xi|_2 \leq 3\epsilon R \#X.$$

Now  $u\eta u^* \in \Xi(F; m, k, \gamma)$ , so there exists a  $j \in J_k$  such that  $|\eta_{jk} - u\eta u^*|_2 < \epsilon$ . Because  $\Xi(F; m, k, \gamma)$  is invariant under the action of  $H_k$  it follows that  $h\eta_{jk} h^* \in \Xi(F; m, k, \gamma)$  and so there exists an  $\ell \in J_k$  such that  $|\eta_{\ell k} - h\eta_{jk} h^*|_2 < \epsilon$ . So again we have

$$|u_{gk} \eta_{\ell k} u_{gk}^* - \eta|_2 < |u_{gk} h \eta_{jk} h^* u_{gk}^* - \eta|_2 + \epsilon < |u \eta_{jk} u^* - \eta|_2 + 3\epsilon R \#F < 4\epsilon R \#F,$$

and we have the desired claim.

It follows that

$$\begin{aligned} \mathbb{K}_{5\epsilon R(\#X + \#F), R}(X \cup F; m_2, \gamma_2) &\leq \limsup_{k \rightarrow \infty} k^{-2} \cdot \log(\#G_k \cdot \#J_k) \\ &\leq \log C + (\delta_0(X) + \tau) \cdot |\log \epsilon| + \limsup_{k \rightarrow \infty} k^{-2} \cdot \log K_\epsilon(\Xi(F; m, k, \gamma)). \end{aligned}$$

Given any  $m \in \mathbb{N}$  and  $\gamma > 0$  we produced  $m_2 \in \mathbb{N}$  and  $\gamma_2 > 0$  so that the above inequality holds for  $0 < \epsilon < \epsilon_0$ . Thus

$$\mathbb{K}_{5\epsilon R(\#X + \#F), R}(X \cup F) \leq \log C + (\delta_0(X) + \tau) \cdot |\log \epsilon| + \mathbb{K}_\epsilon(\Xi(F)).$$

Taking  $\limsup_{\epsilon \rightarrow 0}$  on both sides and again using the packing formulation of  $\delta_0$  in [6] we have

$$\begin{aligned} \delta_0(X \cup F) &= \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_{5\epsilon R(\#X + \#F), R}(X \cup F)}{|\log 5\epsilon R(\#X + \#F)|} \\ &\leq \limsup_{\epsilon \rightarrow 0} \frac{\log C + (\delta_0(X) + \tau) \cdot |\log \epsilon| + \mathbb{K}_\epsilon(\Xi(F))}{|\log 5\epsilon R(\#X + \#F)|} \\ &= \delta_0(X) + \tau + \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_\epsilon(\Xi(F))}{|\log \epsilon|}. \end{aligned}$$

$\tau > 0$  being arbitrary, we have the desired inequality. □

**Hyperfinite inequality for  $\delta_0$ .** *If  $X''$  is hyperfinite, then*

$$\delta_0(X \cup Y \cup Z) \leq \delta_0(X \cup Y) + \delta_0(X \cup Z) - \delta_0(X).$$

*Proof.*  $X$  has finite dimensional approximants, so for each  $m \in \mathbb{N}$  and  $\gamma > 0$  we can find sequences  $\langle \xi \rangle_{k=1}^\infty$  satisfying the conditions of Lemma 2.2 and consider all relative microstates with respect to these fixed sequences. For each  $k$ ,  $\Xi(Y \cup Z; m, k, \gamma) \subset \Xi(Y; m, k, \gamma) \times \Xi(Z; m, k, \gamma)$ , so that

$$K_{2\epsilon}(\Xi(Y \cup Z; m, k, \gamma)) \leq K_\epsilon(\Xi(Y; m, k, \gamma)) \cdot K_\epsilon(\Xi(Z; m, k, \gamma)).$$

It follows that

$$\limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_\epsilon(\Xi(Y \cup Z))}{|\log \epsilon|} \leq \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_\epsilon(\Xi(Y))}{|\log \epsilon|} + \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_\epsilon(\Xi(Z))}{|\log \epsilon|}.$$

By the preceding lemma and the inequality above

$$\begin{aligned} \delta_0(X \cup Y \cup Z) &= \delta_0(X) + \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_\epsilon(\Xi(Y \cup Z))}{|\log \epsilon|} \\ &\leq \delta_0(X) + \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_\epsilon(\Xi(Y))}{|\log \epsilon|} + \limsup_{\epsilon \rightarrow 0} \frac{\mathbb{K}_\epsilon(\Xi(Z))}{|\log \epsilon|} \\ &= \delta_0(X \cup Y) + \delta_0(X \cup Z) - \delta_0(X). \end{aligned} \quad \square$$

*Remark 3.3.* The hyperfinite assumption on  $X''$  is necessary. To see this consider the group inclusion  $\mathbb{F}_3 \subset \mathbb{F}_2 \subset \mathbb{F}_3$  where  $\mathbb{F}_n$  is the free group on  $n$  generators. On the von Neumann algebra level this translates to  $L(\mathbb{F}_3) \simeq M_1 \subset L(\mathbb{F}_2) \simeq M_2 \subset L(\mathbb{F}_3) \simeq M_3$ . Take  $X, Y$ , and  $Z$  to be the canonical sets of freely independent semicirculars associated to  $M_1, M_2$ , and  $M_3$ , respectively. Then we can see that  $\delta_0(X \cup Y) + \delta_0(X \cup Z) - \delta_0(X) = 2 + 3 - 3 = 2 < 3 = \delta_0(X \cup Y \cup Z)$ .

#### 4. SEVEN COROLLARIES

In this section  $X, Y$ , and  $Z$  are again finite sets of selfadjoint elements in  $M$ . Here are some corollaries of the hyperfinite inequality for  $\delta_0$ .

**Corollary 4.1.** *Suppose  $X''$  is hyperfinite. Assume one of the following holds:*

- $Z \subset X''$ .
- $X''$  is diffuse,  $\delta_0(X \cup Z) \leq 1$ , and  $Z \subset (X \cup Y)''$ .

*Then  $\delta_0(X \cup Y) = \delta_0(X \cup Y \cup Z)$ .*

*Proof.* In either of the two cases  $Z$  is contained in the von Neumann algebra generated by  $X$  and  $Y$ , so by [10]  $\delta_0(X \cup Y) \leq \delta_0(X \cup Y \cup Z)$ . For the reverse inequality observe that either situation implies  $\delta_0(X \cup Z) = \delta_0(X)$ . This follows in the first case from invariance of  $\delta_0$  for hyperfinite von Neumann algebras ([5]). In the second case we have by assumption and hyperfinite monotonicity that  $1 \geq \delta_0(X \cup Z) \geq \delta_0(X) \geq 1$ . In either case  $\delta_0(X \cup Z) = \delta_0(X)$ , so by the hyperfinite inequality,

$$\delta_0(X \cup Y \cup Z) \leq \delta_0(X \cup Y) + \delta_0(X \cup Z) - \delta_0(X) = \delta_0(X \cup Y).$$

Thus,  $\delta_0(X \cup Y) = \delta_0(X \cup Y \cup Z)$ . □

We can now settle the first of the two questions broached in the introduction.

**Corollary 4.2.** *If  $X'' = Z''$  is hyperfinite, then  $\delta_0(X \cup Y) = \delta_0(Y \cup Z)$ .*

*Proof.* This follows from Corollary 4.1. □

**Corollary 4.3** (1-Inequality for  $\delta_0$ ). *If  $X''$  is diffuse and  $\delta_0(X) \leq 1$ , then*

$$\delta_0(X \cup Y \cup Z) \leq \delta_0(X \cup Y) + \delta_0(X \cup Z) - 1.$$

*Proof.* The inequality is trivially satisfied if  $X$  does not have finite dimensional approximants, so without loss of generality assume  $X$  has finite dimensional approximants. Find a sequence  $\langle x_k \rangle_{k=1}^\infty$  of selfadjoint elements in the algebra generated by  $X$  such that the  $x_k$  converge strongly to a diffuse element  $x \in X''$  as  $k \rightarrow \infty$ . By algebraic invariance of  $\delta_0$  and the hyperfinite inequality for  $\delta_0$ ,

$$\begin{aligned} \delta_0(X \cup Y \cup Z) &= \delta_0(X \cup Y \cup X \cup Z \cup \{x_k\}) \\ &\leq \delta_0(X \cup Y \cup \{x_k\}) + \delta_0(X \cup Z \cup \{x_k\}) - \delta_0(x_k) \\ &= \delta_0(X \cup Y) + \delta_0(X \cup Z) - \delta_0(x_k). \end{aligned}$$

By [8] and [9],  $\liminf_{k \rightarrow \infty} \delta_0(x_k) \geq \delta_0(x) = 1$ . The above now yields the desired inequality. □

**Corollary 4.4.** *Suppose  $X'' = Z''$  is a diffuse von Neumann algebra. If for any finite set of generators  $F$  for  $X''$   $\delta_0(F) \leq 1$ , then  $\delta_0(X \cup Y) = \delta_0(Y \cup Z)$ . In particular, this will happen when  $X'' = Z''$  has a Cartan subalgebra, or when it can be written as a tensor product of two diffuse von Neumann algebras.*

*Proof.* By symmetry it will suffice to show that  $\delta_0(X \cup Y \cup Z) = \delta_0(X \cup Y)$ . Clearly,  $\delta_0(X \cup Y) \leq \delta_0(X \cup Y \cup Z)$ . By the 1-inequality for  $\delta_0$ ,

$$\begin{aligned} \delta_0(X \cup Y \cup Z) &\leq \delta_0(X \cup Y) + \delta_0(X \cup Z) - 1 \\ &= \delta_0(X \cup Y). \end{aligned}$$

The second statement follows from [4] and [9]. □

Now to answer the second question discussed in the introduction:

**Corollary 4.5.** *If  $y = y^*$  lies in the center of the von Neumann algebra generated by  $Y$ , then*

$$\delta_0(Y \cup \{y\}) = \delta_0(Y).$$

*Proof.* Again by [10]  $\delta_0(Y) \leq \delta_0(Y \cup \{y\})$ . For the reverse inequality set  $\alpha = \sup\{\delta_0(x) : x = x^* \in Y''\}$  (actually the supremum is achieved but we won't need that). Suppose  $\epsilon > 0$ . Find  $x = x^* \in Y''$  such that  $\alpha - \epsilon < \delta_0(x)$ . Take a sequence  $\langle x_k \rangle_{k=1}^\infty$  such that for each  $k$ ,  $x_k = x_k^*$  lies in the \*-algebra generated by  $Y$  and such that  $x_k \rightarrow x$  strongly. Now for every  $k$  there exists an  $a_k = a_k^*$  such that the von Neumann algebra generated by  $a_k$  is equal to the von Neumann algebra generated by  $x_k$  and  $y$  and thus  $\delta_0(a_k) = \delta_0(x_k, y)$ . Using the fact that  $\delta_0$  is an algebraic invariant we have by the hyperfinite inequality for  $\delta_0$ ,

$$\begin{aligned} \delta_0(Y \cup \{y\}) = \delta_0(\{x_k\} \cup Y \cup \{y\}) &\leq \delta_0(\{x_k\} \cup Y) + \delta_0(x_k, y) - \delta_0(x_k) \\ &= \delta_0(Y) + \delta_0(a_k) - \delta_0(x_k) \\ &\leq \delta_0(Y) + \alpha - \delta_0(x_k). \end{aligned}$$

Forcing  $k \rightarrow \infty$  and using the fact that  $\liminf_{k \rightarrow \infty} \delta_0(x_k) \geq \delta_0(x)$  (by [8]) we have that  $\delta_0(Y \cup \{y\}) \leq \delta_0(Y) + \alpha - (\alpha - \epsilon) = \delta_0(Y) + \epsilon$ .  $\epsilon > 0$  being arbitrary,  $\delta_0(Y \cup \{y\}) \leq \delta_0(Y)$ . Thus,  $\delta_0(Y \cup \{y\}) = \delta_0(Y)$ .  $\square$

**Corollary 4.6.** *Suppose  $x = x^* \in M$ ,  $\delta_0(\{x\} \cup Y) = \alpha$ ,  $\delta_0(Z) = \beta$ ,  $\{x\} \cup Y \subset Z''$ , and  $Z = \{z_1, \dots, z_n\}$ . Then*

$$\beta - \alpha + n \cdot \delta_0(x) \leq \sum_{j=1}^n \delta_0(x, z_j).$$

*Thus if  $n < \beta - \alpha + n \cdot \delta_0(x)$ , then for some  $j$ ,  $1 < \delta_0(x, y_j)$ . In particular, if  $Z$  consists of  $2 \leq \beta \in \mathbb{N}$  freely independent semicircular elements,  $Z = \{s_1, \dots, s_\beta\}$  and  $x$  is any self-adjoint element in  $Z''$  with no atoms, then for some  $1 \leq j \leq \beta$ ,  $1 < \delta_0(x, s_j)$ .*

*Proof.*  $\{x\} \cup Y \subset Z''$  so by [9] and the hyperfinite inequality,

$$\begin{aligned} \beta = \delta_0(Z) &\leq \delta_0(\{x\} \cup Y \cup \{z_1, \dots, z_n\}) \\ &\leq \delta_0(\{x\} \cup Y \cup \{z_1, \dots, z_{n-1}\}) + \delta_0(x, z_n) - \delta_0(x). \end{aligned}$$

Repeating this  $n$  times we arrive at

$$\beta \leq \delta_0(\{x\} \cup Y) + \sum_{j=1}^n \delta_0(x, z_j) - n \cdot \delta_0(x) = \alpha - n \cdot \delta_0(x) + \sum_{j=1}^n \delta_0(x, z_j),$$

whence  $\beta - \alpha + n \cdot \delta_0(x) \leq \sum_{j=1}^n \delta_0(x, z_j)$ . Everything else is obvious.  $\square$

*Remark 4.7.* Recall from [4] that for a finite set of selfadjoint elements  $F$  in  $M$ , if  $\delta_0(F) > 1$ , then the von Neumann algebra generated by  $F$  cannot be generated by a sequence of Haar unitaries  $\langle u_j \rangle_{j=1}^s$  satisfying the condition  $u_{j+1}u_ju_{j+1}^* \in \{u_1, \dots, u_j\}''$ . In particular,  $F''$  is prime and has no Cartan subalgebras. Thus, in the context of Corollary 4.4 for some  $j$ ,  $\{x, s_j\}''$  is prime and has no Cartan subalgebra.

We conclude with a microstates proof of a property of the Connes-Shlyakhtenko dimension  $\Delta$ :

**Corollary 4.8.** *If  $X = \{x, x_1, \dots, x_n\}$ ,  $xx_i = x_ix$  for all  $1 \leq i \leq n$ , and the spectrum of  $x$  is diffuse, then  $\delta_0(X) \leq 1$ . If, in addition,  $X$  has finite dimensional approximants, then  $\delta_0(X) = 1$ . Consequently, if  $F$  is a finite set of selfadjoint elements in  $M$  which has finite dimensional approximants and such that the von Neumann algebra generated by  $F$  has diffuse center, then  $\delta_0(F) = 1$ .*

*Proof.* By the hyperfinite inequality for  $\delta_0$ , the diffuseness of  $x$ , and [9],

$$\begin{aligned}\delta_0(X) &\leq \delta_0(x, \dots, x_{n-1}) + \delta_0(x, x_n) - \delta_0(x) \\ &\leq \delta_0(x, \dots, x_{n-1}) + 1 - 1 = \delta_0(x, x_1, \dots, x_{n-1}).\end{aligned}$$

Continuing inductively we have  $\delta_0(X) \leq \delta_0(x) = 1$  as promised. If  $X$  has finite dimensional approximants, then by [5]  $\delta_0(X) \geq \delta_0(x) = 1$  and consequently,  $\delta_0(X) = 1$ . The claim concerning  $F$  is immediate.  $\square$

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