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ON KORENBLUM'S MAXIMUM PRINCIPLE

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Abstract. Let $A^2(\mathbb{D})$ be the Bergman space over the open unit disk $\mathbb{D}$ in the complex plane. Korenblum's maximum principle states that there is an absolute constant $c \in (0,1)$, such that whenever $|f(z)| \leq |g(z)|$ $(f, g \in A^2(\mathbb{D}))$ in the annulus $c < |z| < 1$, then $\|f\|_{A^2} \leq \|g\|_{A^2}$. In this paper we prove that Korenblum's maximum principle holds with $c = 0.25018$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The Bergman space $A^2(\mathbb{D})$ consists of analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{A^2} = \left[\frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^2 dA(z)\right]^{\frac{1}{2}} < +\infty,$$

where

$$dA(z) = dx\, dy = r dr\, d\theta, \quad z = x + iy = re^{i\theta},$$

is the Lebesgue area measure. Korenblum [4] conjectured that there is an absolute constant c, $0 < c < 1$, such that whenever $|f(z)| \leq |g(z)|$ in the annulus $c < |z| < 1$ $(f, g \in A^2(\mathbb{D}))$, then $\|f\|_{A^2} \leq \|g\|_{A^2}$.

In 1999 Hayman [1] proved Korenblum's conjecture, and Hinkkanen [2] proved that Korenblum's conjecture is true for the Bergman space $A^p(\mathbb{D})$ $(p \geq 1)$. But the sharp value of c even when $p = 2$ (we use γ to denote this sharp value and call it Korenblum's constant) is still unknown. However, Hayman [1] gave a lower bound on γ: $\gamma \geq 0.04$. Hinkkanen [2] improved Hayman's result that $\gamma \geq 0.15724\cdots$. Recently Schuster [5] has shown that $\gamma \geq 0.21$ in terms of Möbius pseudodistance for the annulus. On the other hand, an upper bound on γ can be found from Martin's example (see [4] or [6]): $\gamma < 0.70450\cdots$. Wang [6] gave an upper bound on γ: $\gamma < 0.69472$, and has recently [7] shown that $\gamma < 0.67795$. In this paper we prove that Korenblum's maximum principle holds with $c = 0.25018$, which is the following theorem.

Theorem 1. *Let $f(z)$ and $g(z)$ be analytic functions in the unit disk $\mathbb{D}$. Suppose that $|f(z)| \leq |g(z)|$ whenever $c < |z| < 1$, where $c = 0.25018$. Then $\|f\|_{A^2} \leq \|g\|_{A^2}$.*

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Let $c \in (0,1)$ be a constant to be determined later. We write $D(c) = \{z \in \mathbb{C} : |z| < c\}$ and $A(c,1) = \{z \in \mathbb{C} : c < |z| < 1\}$. Suppose that $f(z)$ and $g(z)$ are analytic functions in the unit disk $\mathbb{D}$ and that $|f(z)| \leq |g(z)|$ whenever $c < |z| < 1$. To prove Theorem 1, we need only to show that

$$\int_{D(c)} (|f(z)|^2 - |g(z)|^2) dA(z) \leq \int_{A(c,1)} (|g(z)|^2 - |f(z)|^2) dA(z). \tag{1}$$

We may assume that $\|g\|_{A^2}$ is finite, for otherwise the result holds trivially. Hence $\|f\|_{A^2}$ is also finite, since $|f(z)| \leq |g(z)|$ whenever $c < |z| < 1$. We may also assume that g is not identically equal to 0. Define $\omega(z) = f(z)/g(z)$ for $c < |z| < 1$, so that $|\omega| \leq 1$. We may assume that $|\omega(z)| < 1$ in $c < |z| < 1$, since otherwise the result is trivial. Define $\omega_0 = \omega_0(\rho) = \max\{|\omega(z)| : |z| = \rho\}$.

To obtain a better constant c, we follow Hinkkanen [2] and Schuster [5] but use a different estimate. Note that for any $a \in (-1,1)$,

$$|f|^2 - |g|^2 \leq \frac{|f - ag|^2}{1 - a^2}, \tag{2}$$

which is a direct consequence of the following identity:

$$|f|^2 - |g|^2 = \frac{|f - ag|^2 - |af - g|^2}{1 - a^2}.$$

Using (2) and the fact that, for a subharmonic function h in $\mathbb{D}$ and $0 < r_1 < r_2 < 1$, $\int_0^{2\pi} h(r_1 e^{i\theta}) d\theta \leq \int_0^{2\pi} h(r_2 e^{i\theta}) d\theta$, we obtain

$$\begin{aligned}
& \int_0^{2\pi} (|f(re^{i\theta})|^2 - |g(re^{i\theta})|^2) d\theta \\
\leq\ & \int_0^{2\pi} \frac{|f(re^{i\theta}) - ag(re^{i\theta})|^2}{1 - a^2}\, d\theta \\
\leq\ & \int_0^{2\pi} \frac{|f(\rho e^{i\theta}) - ag(\rho e^{i\theta})|^2}{1 - a^2}\, d\theta \\
=\ & \int_0^{2\pi} \frac{|\omega(\rho e^{i\theta}) - a|^2}{(1 - a^2)(1 - |\omega(\rho e^{i\theta})|^2)} (|g(\rho e^{i\theta})|^2 - |f(\rho e^{i\theta})|^2) d\theta \\
\leq\ & \gamma(\rho) \int_0^{2\pi} (|g(\rho e^{i\theta})|^2 - |f(\rho e^{i\theta})|^2) d\theta,
\end{aligned}$$

where $a \in (0,1)$ is a quantity depending on $\rho \in (c,1)$ but not on r and θ, and

$$\gamma(\rho) = \max\left\{ \frac{|\omega(z) - a|^2}{(1 - a^2)(1 - |\omega(z)|^2)} : |z| = \rho \right\}.$$

Let us keep ρ fixed and multiply both sides of this inequality by r, and then integrate with respect to r from 0 to c. We obtain

$$\frac{1}{\gamma(\rho)} \int_{D(c)} (|f(z)|^2 - |g(z)|^2) dA(z) \leq \frac{c^2}{2} \int_0^{2\pi} (|g(\rho e^{i\theta})|^2 - |f(\rho e^{i\theta})|^2) d\theta. \tag{3}$$

Multiplying both sides of (3) by ρ and integrating with respect to ρ from c to 1 yields

$$\int_c^1 \frac{\rho}{\gamma(\rho)}\, d\rho \int_{D(c)} (|f(z)|^2 - |g(z)|^2) dA(z) \leq \frac{c^2}{2} \int_{A(c,1)} (|g(z)|^2 - |f(z)|^2) dA(z). \tag{4}$$

Thus if we can choose c so that

$$\int_c^1 \frac{\rho}{\gamma(\rho)}\, d\rho \geq \frac{c^2}{2}, \tag{5}$$

then Theorem 1 is proved.

So we only need to find a suitable bound for $\gamma(\rho)$.

2. The Möbius pseudodistance

We recall that the pseudohyperbolic distance d between two points $\alpha, \beta \in \mathbb{D}$ is defined by

$$d(\alpha, \beta) = \left| \frac{\alpha - \beta}{1 - \overline{\alpha}\beta} \right|.$$

The Möbius pseudodistance for a domain $D \subset \mathbb{D}$ is defined by

$$c_D^*(a, z) = \sup\{d(\omega(a), \omega(z)) : \omega \in \mathcal{H}(D, \mathbb{D})\},$$

where $a, z \in D$ and $\mathcal{H}(D, \mathbb{D})$ denotes the set of analytic functions from D to $\mathbb{D}$ (see [3] for an excellent survey). Note that both of the two pseudodistances are Möbius invariant.

In [5] Schuster showed that

$$c^*_{A(c,1)}(\rho, \rho e^{i\theta}) = \rho \prod_{n=1}^{\infty} f_n(\rho, c) g_n(\rho, c, \theta),$$

where

$$f_n(\rho, c) = \frac{(1 + \rho^2 c^{2n-1})(1 + \rho^{-2} c^{2n-1})}{(1 + c^{2n-1})^2}$$

and

$$g_n(\rho, c, \theta) = \frac{\sqrt{1 - 2c^{2n-2}\cos\theta + c^{4n-4}}\sqrt{1 - 2c^{2n}\cos\theta + c^{4n}}}{\sqrt{1 - 2\rho^2 c^{2n-2}\cos\theta + \rho^4 c^{4n-4}}\sqrt{1 - 2\rho^{-2} c^{2n}\cos\theta + \rho^{-4} c^{4n}}}.$$

In the following we use $F \sim G$ to indicate that the quotient F/G of two functions F and G is positive. We want to give an estimate for $c^*_{A(c,1)}(\rho, \rho e^{i\theta})$ as in [5]. First, we proceed to show that when $n \geq 3$ and $c \leq 1/\sqrt{2}$,

$$g_n(\rho, c, \theta) \leq g_n(\rho, c, \pi). \tag{6}$$

Since $g_n(\rho, c, \theta)$ is an even function of θ, we may assume that $0 \leq \theta \leq \pi$. So it suffices for us to prove that $\frac{\partial g_n^2(\rho, c, \theta)}{\partial \theta} > 0$ for $0 < \theta < \pi$. A direct calculation shows that for $0 < \theta < \pi$,

$$\begin{aligned}\frac{\partial g_n^2(\rho, c, \theta)}{\partial \theta} &\sim -4c^{4n-2}(1 + c^{4n-2})\cos^2\theta - 4c^{6n-4}(1 + \rho^2)(1 + \rho^{-2}c^2)\cos\theta \\ &+(1 + c^{4n-2})(1 - 2c^{4n-2} + c^{8n-4} - \rho^2 c^{4n-2} - \rho^2 c^{4n-4} - \rho^{-2} c^{4n-2} - \rho^{-2} c^{4n}).\end{aligned}$$

Write

$$\begin{aligned} x_0 &= \frac{-c^{4n-3}(1 + \rho^2)(1 + \rho^{-2}c^2)}{2c^{2n-1}(1 + c^{4n-2})} \\ &\quad + \frac{\sqrt{(1 - \rho^2 c^{4n-4})(1 - \rho^2 c^{4n-2})(1 - \rho^{-2} c^{4n-2})(1 - \rho^{-2} c^{4n})}}{2c^{2n-1}(1 + c^{4n-2})}.\end{aligned}$$

Then we only need to prove $x_0 > 1$, which is equivalent to

$$\begin{aligned}\beta(\rho,c) &= (1-\rho^2c^{4n-4})(1-\rho^2c^{4n-2})(1-\rho^{-2}c^{4n-2})(1-\rho^{-2}c^{4n})\\ &\quad -c^{4n-2}(2(1+c^{4n-2})+c^{2n-2}(1+\rho^2)(1+\rho^{-2}c^2))^2 > 0.\end{aligned}$$

It is easy to check that

$$\frac{\partial\beta(\rho,c)}{\partial\rho} \sim c-\rho^2,$$

and for any $n \geq 3, c \leq 1/\sqrt{2}$,

$$\begin{aligned}\beta(1,c) &= \beta(c,c) = (1-c^{4n-4})(1-c^{4n-2})^2(1-c^{4n})\\ &\quad -4c^{4n-2}(1+c^{4n-2}+c^{2n-2}(1+c^2))^2\\ &\quad \sim 1+c^{8n-4}-6c^{4n-2}-c^{2n-2}-c^{2n}-c^{6n-4}-c^{6n-2} > 0.\end{aligned}$$

Then we get

$$\beta(\rho,c) \geq \beta(1,c) = \beta(c,c) > 0.$$

The desired inequality (6) now follows.

Next, a long and tedious calculation shows that

$$\prod_{n=1}^{4} g_n(\rho,c,\theta) \leq \prod_{n=1}^{4} g_n(\rho,c,\pi). \tag{7}$$

Thus we have

$$\begin{aligned}c^*_{A(c,1)}(\rho,\rho e^{i\theta}) &\leq \rho\prod_{n=1}^{\infty} f_n(\rho,c)g_n(\rho,c,\pi)\\ &= \rho\prod_{n=1}^{\infty}\left[\frac{(1+\rho^2c^{2n-1})(1+\rho^{-2}c^{2n-1})}{(1+c^{2n-1})^2}\cdot\frac{(1+c^{2n-2})(1+c^{2n})}{(1+\rho^2c^{2n-2})(1+\rho^{-2}c^{2n})}\right]\\ &= 2\rho(1+\rho^{-2}c)\prod_{n=1}^{\infty}\frac{(1+\rho^2c^{2n-1})(1+\rho^{-2}c^{2n+1})(1+c^{2n})^2}{(1+\rho^2c^{2n-2})(1+\rho^{-2}c^{2n})(1+c^{2n-1})^2}.\end{aligned}$$

Note that every term in the last infinite product is less than 1. To use *Mathematica*, we choose

$$K = 2\rho(1+\rho^{-2}c)\prod_{n=1}^{10}\frac{(1+\rho^2c^{2n-1})(1+\rho^{-2}c^{2n+1})(1+c^{2n})^2}{(1+\rho^2c^{2n-2})(1+\rho^{-2}c^{2n})(1+c^{2n-1})^2}.$$

3. Estimates for $\gamma(\rho)$

In the previous section it has been proved that

$$d(\omega(z),\omega_0) = \left|\frac{\omega(z)-\omega_0}{1-\omega_0\omega(z)}\right| \leq K < 1$$

for $|z| = \rho$. Write

$$\frac{\omega(z)-\omega_0}{1-\omega_0\omega(z)} = te^{i\theta},$$

where $0 \leq t \leq K, \theta \in [0,2\pi]$. It is easy to check that $|\omega(z)| \leq \omega_0$ is equivalent to

$$\cos\theta \leq -\frac{t(1+\omega_0^2)}{2\omega_0} < 0.$$

To get a bound for $\gamma(\rho)$, we choose $a = \omega_0^\alpha$, where

$$\alpha = \sqrt{\frac{1+3K^2}{1-K^2}}.$$

Then for $|z| = \rho$,

$$\begin{aligned} & \frac{|\omega(z)-a|^2}{(1-a^2)(1-|\omega(z)|^2)} \\ = \quad & \frac{(\omega_0-\omega_0^\alpha)^2 + t^2(1-\omega_0^{\alpha+1})^2 + 2\omega_0 t(1-\omega_0^{\alpha-1})(1-\omega_0^{\alpha+1})\cos\theta}{(1-\omega_0^{2\alpha})(1-\omega_0^2)(1-t^2)} \\ \leq \quad & \frac{(\omega_0^2-\omega_0^\alpha)^2(1-t^2) + t^2\omega_0^{\alpha-1}(1-\omega_0^2)^2}{(1-\omega_0^{2\alpha})(1-\omega_0^2)(1-t^2)} \\ \triangleq \quad & H(\rho, t, \omega_0). \end{aligned}$$

To estimate $H(\rho, t, \omega_0)$, we need the following two lemmas.

Lemma 1. *For any $0 \leq u \leq v < 1, x \in (0,1)$, we have*

$$\frac{x^u - x^v}{1 - x^{u+v}} \leq \frac{v-u}{v+u}. \tag{8}$$

It can be easily checked. So we omit the details.

Lemma 2. *For any $u \geq 1, x \in (0,1)$, we have*

$$\frac{(x-x^u)^2}{(1-x^2)(1-x^{2u})} \leq \frac{(u-1)^2}{4u}. \tag{9}$$

Proof. For any $u \geq 1, x \in (0,1)$, define

$$\Phi(x) = \frac{(x-x^u)^2}{(1-x^2)(1-x^{2u})}.$$

It is easy to check that

$$\begin{aligned} \Phi'(x) \quad \sim \quad & 2(x-x^u)(1-ux^{u-1})(1-x^2)(1-x^{2u}) \\ & -(x-x^u)^2(-2x-2ux^{2u-1}+(2u+2)x^{2u+1}) \\ \sim \quad & (1-x^{u+1})[1-x^{2u}-ux^{u-1}(1-x^2)] \\ \sim \quad & 1 - u\frac{x^{u-1}-x^{u+1}}{1-x^{2u}}. \end{aligned}$$

Applying Lemma 1, we know $\Phi'(x) > 0$. Hence $\Phi(x)$ is an increasing function of x. The desired inequality (9) then follows. □

Note that $\alpha \geq 1$. We use Lemma 2 and Lemma 1 to obtain

$$\begin{aligned} H(\rho, t, \omega_0) \quad = \quad & \frac{(\omega_0-\omega_0^\alpha)^2}{(1-\omega_0^{2\alpha})(1-\omega_0^2)} + \frac{t^2(\omega_0^{\alpha-1}-\omega_0^{\alpha+1})}{(1-t^2)(1-\omega_0^{2\alpha})} \\ \leq \quad & \frac{(\alpha-1)^2}{4\alpha} + \frac{t^2}{\alpha(1-t^2)} \\ \leq \quad & \frac{(\alpha-1)^2}{4\alpha} + \frac{K^2}{\alpha(1-K^2)} \\ = \quad & \frac{1}{2}\left(\sqrt{\frac{1+3K^2}{1-K^2}} - 1\right). \end{aligned}$$

Putting things together, we obtain

$$\gamma(\rho) \leq \frac{1}{2}\left(\sqrt{\frac{1+3K^2}{1-K^2}}-1\right). \tag{10}$$

Using *Mathematica*, we obtain that (5) holds with $c = 0.25018$.

4. Remarks

First, write

$$u_n(\rho,c,\theta) = \frac{\left|\exp\{\frac{\pi}{\log(1/c)}(\theta+2n\pi)\}-1\right|}{\left\{1+\exp\{\frac{2\pi}{\log(1/c)}(\theta+2n\pi)\}-2\exp\{\frac{\pi}{\log(1/c)}(\theta+2n\pi)\}\cos\left(2\pi\frac{\log\rho}{\log c}\right)\right\}^{\frac{1}{2}}}.$$

In fact, Hinkkanen gave an estimate for K(take a closer look at the proof in [2]): $K = \prod_{n=-\infty}^{\infty} u_n(\rho,c,\pi)$, which can be proved by a calculus argument. If we use the above estimate for K and Hinkkanen's estimate for $\gamma(\rho)$ (also use (4) and (5) here), a better constant than the original one can be obtained: $c = 0.1921\cdots$. Thus Korenblum's maximum principle holds for general Bergman space $A^p(\mathbb{D})$ $(p \geq 1)$ with an absolute constant $c = 0.1921\cdots$. Also, this is true with $c = 0.21$. However, we cannot prove that is true with $c = 0.25018$.

Next, following Schuster [5] and using the estimate (10) in the Fock space setting, we can prove that Theorem 2 in [5] holds with $c = 0.7248$. Namely, we have

Theorem 2. *Suppose that $f(z)$ and $g(z)$ are two entire functions satisfying $|f(z)| \leq |g(z)|$ for $|z| > c$, where $c = 0.7248$. Then $\|f\|_F \leq \|g\|_F$.*

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