ISOTOPIC FAMILIES OF CONTACT MANIFOLDS
FOR ELLIPTIC PDE

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Abstract. A test for a function to be a solution of an elliptic PDE is given
in terms of extensions, as solutions, from the boundaries inside the domains
belonging to an isotopic family. It generalizes a result of Ehrenpreis for spheres
moved along a straight line.

1. Introduction

The following problem has circulated among a certain group of mathematicians
for a long time: given a family of closed Jordan curves in the complex plane and
a continuous function admitting a holomorphic extension inside each curve, when
is the function holomorphic in the union of the curves? We refer the reader to
the articles [11], [12] by L. Zalcman, where this and closely related problems are
discussed.

The case of rotation-invariant families was carefully studied in [1], [5]–[8]. How-
ever, even for simple families, for instance, circles of constant radius centered on
a segment of a straight line, the question remained open until recently, when sig-
nificant progress was made. Namely, in [2], a complete description of arbitrary
continuous one-parameter families of circles, detecting holomorphicity in the above
sense, was given for rational functions, and for real-analytic functions. Independ-
ently, the real-analytic case, though for special families of circles (constant radius,
centers on a segment), was treated by Ehrenpreis in [3]. Soon afterwards, Tumanov
[10] solved the case of circles of constant radius, centered on a segment (the strip-
problem) for continuous functions. In both [2] and [10] the problem is solved by
methods of several complex variables.

In the recently published book [4] (see Section 9.5), Ehrenpreis proposed consid-
ering the above question from the point of view of PDE, by replacing the Cauchy-
Riemann equations with more general PDE. The idea is to characterize solutions
of a PDE in terms of their restrictions to a family of closed hypersurfaces (contact
manifolds). The condition is that the restriction to each hypersurface must coincide
with the boundary value of a solution in the domain bounded by this hypersurface.

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This extension depends, in general, on the domain, and the contacts on the hypersurfaces are assumed of sufficiently high order, i.e. along with higher-order normal derivatives constituting overdetermined Dirichlet-Neumann boundary data.

Counting the number of parameters involved in the problem shows that the families may be taken to be one-parametric and therefore each family can be viewed as a curve in the space of domains. Thus, in a sense, one would like to integrate along a curve of domains the condition of tangency to a solution at each “point” of the curve to recover a global solution. We refer to the book [4] for a more detailed explanation of the concept of contact manifolds.

In the framework of this concept, Ehrenpreis proved a theorem on characterization of solutions of the elliptic PDE

\[ P(x, D)f = 0 \]

by their restrictions to spheres \( S_t = \{(x_1 - t)^2 + x_2^2 + \cdots + x_n^2 = 1\} \) of radius 1 centered on the \( x_1\)-axis ([4, Th. 9.5]). Essentially, the result in [4] says that if a smooth function in the strip \( \{x_1 \in (-\infty, \infty) : x_2^2 + \cdots + x_n^2 \leq 1\} \) agrees to order \( m \), on each sphere \( S_t \) with a solution \( F_t \) of \( PF_t = 0 \) in a neighborhood of the ball \( B_t \) bounded by \( S_t \), then \( f \) itself is a solution, \( Pf = 0 \). We omit here some additional technical conditions and refer the reader to [4] for an exact formulation of the theorem.

The goal of this article is to generalize Ehrenpreis' result from the specific family of spheres, \( S_t \), to quite general families of closed hypersurfaces. We also simplify the conditions for agreement of solutions, formulated in [4] in terms of approximations and bounds of derivatives, by requiring the solvability of an overdetermined Dirichlet-Neumann problem on each contact manifold.

Our result can be regarded as a response to Problem 9.8 ([4, p. 579]), which reads “Develop a complete theory of PDE contact manifolds”.

2. MAINE RESULT

We fix a \( C^1 \)-isotopy of domains \( D_t \in \mathbb{R}^n, \ t \in I = (-1, 1) \). This family can be exhibited as follows: there is an initial domain \( D = D_0 \); then \( D_t = \omega_t(D) \), where \( \omega_t : D \mapsto D_t \) is a diffeomorphism, \( \omega_0 = id \) and the family \( \omega_t \) is continuously differentiable in the parameter \( t \). We assume also that each domain \( D_t \) has \( C^1 \)-boundary \( \partial D_t \) and the diffeomorphism \( \omega_t \) admits a \( C^1 \)-extension to the closed domain \( D_t \).

Denote by \( \nu_t \) the outward unit normal vector on \( \partial D_t \). We say that the isotopy \( D_t \) is transversal if for each \( t \in I \), the inner product

\[ \rho_t(u) = \langle \partial_t \omega_t(u), \nu_t(\omega_t(u)) \rangle \neq 0 \]

for a dense set of \( u \in \partial D \).

This means that the set of points where the direction of the transformation \( \omega_t \) is tangent to the boundary \( \partial D_t \) is nowhere dense. A simple example of non-transversal isotopy is rotation of a ball because in this case the vector of the transformation is tangent to the boundary sphere at each point. On the other hand any family of translations in a constant direction \( e \) of a strictly convex domain \( D, D_t = te + D \) (in particular, translation of a ball as in Ehrenpreis’ theorem), is a transversal isotopy. Indeed, in this case, the direction vector \( \partial_t \omega_t(u) = e \) can be tangent only to the surface on a subset of lower dimension.

We make the following additional assumption of regularity.
Each solution \( u \in C^m(\Omega) \) to the equation \( L^* u = 0 \) in the closed domain \( \Omega \) can be approximated with respect to the \( C^m \)-norm by solutions in a neighborhood of \( \Omega \). Here \( L^* \) is the formal adjoint operator. Sufficient conditions for \((*)\) to be fulfilled are given in the book of Tarkhanov [9, Ch. 6]. For instance, \((*)\) holds if the domain \( D \) is smooth and the operator \( L \) has real-analytic coefficients.

Our main result is the following.

**Theorem 2.1.** Let \( D_t = \omega_t(D) \) be a smooth transversal isotopy of domains in \( \mathbb{R}^n \), \( D = D_0 \). Set

\[
\Omega = \bigcup_{t \in I} \partial D_t,
\]

and let

\[
L = P(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha
\]

be an elliptic partial differential operator of order \( 2m \) with smooth coefficients defined in the domain \( \Omega \) and satisfying the condition \((*)\). Suppose that the function \( f \in C^{2m}(\Omega) \) has the property that on each boundary \( \partial D_t \), it is tangent to order \( m \) to a solution in the domain \( D_t \), meaning that for each \( t \in I \), the Dirichlet-Neumann boundary problem:

\[
LF_t(x) = 0, \quad x \in D_t,
\]

with the boundary conditions

\[
\partial_{\nu_t}^j F_t(x) = \partial_{\nu_t}^j f(x), \quad x \in \partial D_t, \quad j = 0, 1, \ldots, (m - 1), m,
\]

has a solution \( F_t \in C^m(\overline{D_t}) \).

Then \( f \) is a global solution, \( Lf(x) = 0 \), for \( x \in \Omega \).

Note that the well-determined boundary data for the elliptic operator \( L \) of order \( 2m \) include \( m \) derivatives of orders from 0 to \( m - 1 \). However the boundary data in Theorem 2.1 are overdetermined, as they contain the extra \( m \)-th derivative; therefore requiring solvability of the Cauchy problem in the above theorem imposes a nontrivial condition on \( f \).

Here is a particular case of Theorem 2.1, for the Laplace operator:

**Corollary 2.2.** Let \( D_t \) and \( \Omega \) be as in Theorem 2.1 and let \( f \) be a \( C^2 \)-function in \( \Omega \). If for each \( t \), the function \( f \) coincides to order 1 on \( \partial D_t \) with its Poisson integral in \( D_t \), then \( f \) is harmonic in \( \Omega \).

3. **Proof of the main result**

We fix a transversal smooth isotopy of domains in \( \mathbb{R}^n \), \( D_t = \omega_t(D) \), \( t \in I, \omega_0 = id \). Sometimes we will find it to be more convenient writing \( \omega(t, x) \) instead of \( \omega_t(x) \). We also fix a function \( f \in C^{2m}(\Omega) \) which has for each \( t \) an extension \( F_t \) in \( D_t \) such that \( \partial_{\nu_t}^j (f - F_t) = 0 \) on \( \partial D_t \) for \( j = 0, \ldots, m \).

We need the following lemma.

**Lemma 3.1.** Let \( G \) be a smooth function in \( \Omega = \bigcup_{t \in I} D_t \). Then

\[
\frac{d}{dt} \left| \int_{D_t} G(x) dV(x) \right|_{t=0} = \int_{\partial D_0} \frac{d}{dt} \rho_0(x) dS(x),
\]

where \( dS \) is the surface measure on \( \partial D_0 \) and the function \( \rho_t \) is defined in Section 2.
Proof. The change of variables $x = \omega_t(u)$ in $\int_{D_t} G(x) \, dx$ gives

$$\int_D G(\omega_t(u)) |J_t(u)| dV(u),$$

where $J_t(u) = J(t, u)$ is the Jacobian $\det(\partial \omega_t/\partial u)$. Since the Jacobians do not vanish and therefore preserve the sign, we can omit the absolute value in (3.2).

Differentiating (3.2) in $t$ at $t = 0$ and taking into account that $J_t(0, x) = 1$, we obtain

$$\frac{d}{dt} \int_{D_t} G(x) dV(x)|_{t=0} = \int_D \sum_{k=1}^n \partial_k G(u) \partial_t \omega_k(0, u) dV(u) + \int_D G(u) \partial_t J(0, u) dV(u).$$

Rewrite the integrand in the first integral in the right-hand side of (3.3) as

$$\sum_{k=1}^n \partial_k[G(u) \partial_t \omega_k(0, u)] - G(u) \text{div}[\partial_t \omega(0, u)].$$

Now, since the Jacobian matrix of $\omega_0$ is the identity matrix, we have for the derivative of the Jacobian in the second integral in (3.3)

$$\partial_t J(0, u) = \partial_t \det[\nabla \omega_1(t, u), \ldots, \nabla \omega_n(t, u)]$$

$$= \sum_{k=1}^n \det[\nabla \omega_1(0, u), \ldots, \partial_t \nabla k(0, u), \ldots, \nabla \omega_n(0, u)]$$

$$= \sum_{k=1}^n \partial_k \partial_t \omega_k(0, u) = \text{div}[\partial_t \omega(0, u)].$$

Plugging (3.4) and (3.5) into (3.3) and canceling (3.5) with the div term in (3.4) yields that the left-hand side in (3.3) equals

$$\int_D \left\{ \sum_{k=1}^n \partial_k[G(u) \partial_t \omega_k(0, u)] \right\} dV(u),$$

which by Green’s formula is equal to

$$\int_{\partial D} \left\{ \sum_{k=1}^n \nu_k(u) G(u) \partial_t \omega_k(0, u) \right\} dS(u)$$

$$= \int_{\partial D} G(u) < \nu(u), \partial_t \omega(0, u) > dS(u)$$

where $\nu = \nu_0$ is the unit normal vector on the boundary $\partial D$ but this is just the right-hand side in (3.1). The lemma is proved.

Let

$$(L^\ast \psi)(x) = \sum_{|\alpha| \leq 2m} (-1)^{|\alpha|} D^\alpha(a_\alpha(x) \psi(x))$$

be the formally adjoint operator, which is also elliptic. It is well known that the well-posed Cauchy boundary problem involves $m$ Cauchy data; that is, the equation $L^\ast \psi = 0$ has in $D$ a unique solution $u$ satisfying any $m$ prescribed boundary conditions $\partial^j u = \psi_j$, $j = 0, \cdots, m - 1$, on $\partial D.$
Denote by $C_j$ the Dirichlet system of boundary operators (see, e.g. [9, p. 298]), for which Green’s formula holds:

$$
\int_D (L\phi \psi - \phi L^*\psi) dV = \int_{\partial D} (\sum_{j=0}^{2m-1} C_j \phi \partial_j^\nu \psi) dS.
$$

When the function $\phi$ is smooth in a neighborhood of the domain $D$, $C_j$ acts on $\phi$ as a differential operator of order not greater than $2m - 1 - j$.

**Lemma 3.2.** On the boundary $\partial D_t$, the identities $C_j(f - F_t) = 0$ hold for $j = 0, \ldots, m - 2$.

**Proof.** First we assume a condition, slightly stronger than the condition of approximation, namely we assume that every solution in $D$ extends to a solution in a neighborhood of $D$. This occurs, for instance, when both the boundary of the domain $D$ and the coefficients of the operator $L$ are real-analytic.

By changing the parameter $t$, it is clear that it suffices to prove the identity for $t = 0$, i.e., for the domain $D$. Take a function $\psi$ which solves the equation $L^*\psi = 0$ in a neighborhood of the domain $D$. Apply Green’s formula for the domain $D_t$ and the functions $\phi = f - F_t$ and $\psi$. Since $LF_t = L^*\psi = 0$

in $D_t$, we obtain

$$
(3.6) \quad \int_{D_t} (Lf)\psi dV = \int_{\partial D_t} (\sum_{j=0}^{m-2} C_j(f - F_t) \partial_j^\nu \psi) dS.
$$

We only need to explain why the index $j$ in the sum on the right-hand side of (3.6) does not exceed $m - 2$. Indeed, by the main condition of Theorem 2.1, $f - F_t$ vanishes on $\partial D_t$ with all derivatives up to the order $m$. Since ord $C_j \leq 2m - 1 - j$, we have $C_j(f - F_t) = 0$ as long as $2m - 1 - j \leq m$, i.e., when $j \geq m - 1$.

Choose $\psi$ to be the solution to the Dirichlet-Neumann problem:

$$
L^*\psi = 0
$$

with the boundary data on $\partial D$:

$$
(3.7) \quad \psi = \partial_\nu \psi = \cdots = \partial_{\nu}^{m-2}\psi = 0, \quad \partial_{\nu}^{m-1}\psi = h,
$$

where $h$ is arbitrary.

Now differentiate both sides of (3.6) with respect to $t$ at the point $t = 0$. By Lemma 1, the left-hand side becomes after differentiation

$$
\int_{\partial D} (Lf)\psi \rho_0 dS = 0,
$$

since $\psi = 0$ on $\partial D$. As far as the right-hand side is concerned, we first change the variable $u = \omega_t(x)$ in the surface integral and then differentiate the integrand in $t$.

The operator of differentiation in $t$ at $t = 0$ is defined by the vector field

$$
X = (\partial_t\omega(0, u), \nabla),
$$

where $\nabla$ is the gradient in the variable $u$. This vector field can be decomposed as

$$
X = T + \lambda \partial_\nu,
$$

where $T$ is a tangential vector field on $\partial D$, $\nu$ is the unit normal vector to $\partial D$ and $\lambda$ is the inner product of $X$ and $\nu$, that is, $\lambda = \rho_0(x)$.
Differentiation with respect to the variable \( t \) throws up many terms, but only one of them remains. Indeed, since all the derivatives of \( \psi \) up to order \( m - 2 \) vanish on \( \partial D \), the only nonzero term after differentiation in the surface integral in (3.6) may be \( C_{m-2}(f - F_0) X \partial_\nu^{m-2} \psi \). Moreover, since the tangential derivative \( T \partial_\nu^{m-2} \psi = 0 \) we have

\[
X \partial_\nu^{m-2} \psi = \rho \partial_\nu^{m-1} \psi = \rho h.
\]

Thus, we have

\[
(3.8) \quad \int_{\partial D} C_{m-2}(f - F_0) \rho_0 h dS = 0,
\]

and since \( h \) is arbitrary and \( \rho \) vanishes on a nowhere dense set, we conclude that \( C_{m-2}(f - F_0) = 0 \) on \( \partial D \).

This argument applies to any domain \( D_t \); therefore the summation in the formula (3.6) extends to \( m - 3 \).

Now we can repeat the argument, choosing \( \psi \), a solution to the adjoint equation

\[
L^* \psi = 0
\]

with the boundary conditions on \( \partial D \):

\[
\partial_j \nu \psi = 0, \quad j = 0, \ldots, m - 3, \quad \partial_\nu^{m-2} \psi = h,
\]

where \( h \) is arbitrary. Such a solution exists, and is non-unique as the \((m - 1)\)-th normal derivative is not specified. Differentiating in \( t \) and repeating the above argument we obtain that

\[
C_{m-3}(f - F_t) = 0 \text{ on } \partial D_t.
\]

Proceeding this way, we complete the proof of Lemma 3.2 under the assumption that any solution of the adjoint equation in \( \overline{D} \) extends as a solution in a neighborhood of \( \overline{D} \). If only an approximation (condition \((*)\)) is assumed, then in the proof above we choose a sequence \( \psi_k \) of solutions to \( L^* \psi_k = 0 \) in a neighborhood of \( \overline{D} \), approximating with derivatives a solution \( \psi \) of the boundary problem (3.7) in \( D \). Then after differentiation in the parameter \( t \) we let \( k \) tend to \( \infty \) and arrive at the identity (3.8) which, in turn, leads to the desired conclusions. Lemma 3.2 is proved.

**Proof of Theorem 2.1.** By Lemma 3.2 and the definition of the adjoint operator we have

\[
\int_D (Lf) \psi dV = \int_D f L^* \psi dV.
\]

Choose \( \psi \) in the kernel of the adjoint operator, \( L^* \psi = 0 \). Then the right-hand side vanishes, so we have

\[
\int_D (Lf) \psi dV = 0.
\]

Apply Lemma 3.1 to \( G = (Lf) \psi \). Then we obtain

\[
\int_{\partial D} Lf(x) \psi(x) \rho_0(x) dS(x) = 0.
\]

However, the boundary value \( \psi \) on \( \partial D \) can be chosen arbitrarily; therefore \( (Lf) \rho_0 = 0 \) on \( \partial D \), and the condition of transversality implies that \( Lf = 0 \) on \( \partial D \). Clearly, any domain \( D_t \) in the family can be chosen as the initial one; therefore the same is true for any domain \( D_t \), so \( Lf = 0 \) on the union \( \Omega = \bigcup_{t \in I} D_t \). The proof is complete.

**Remark.** Examination of the proof shows that if we give up the condition of transversality, what we can claim in Theorem 2.1 is that \( f \) is a solution, \( Lf = 0 \), in the closure of the domain \( \{ x = \omega_t(u) : u \in D, \ t \in I, \ \rho_t(u) \neq 0 \} \), that is, in the domain where the isotopy transformation \( \omega_t \) acts on the boundary in a non-tangential direction.
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