SIMPLE REAL RANK ZERO ALGEBRAS
WITH LOCALLY HAUSDORFF SPECTRUM

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Abstract. Let $\mathcal{A}$ be a unital, simple, separable $C^*$-algebra with real rank zero, stable rank one, and weakly unperforated ordered $K_0$ group. Suppose, also, that $\mathcal{A}$ can be locally approximated by type I algebras with Hausdorff spectrum and bounded irreducible representations (the bound being dependent on the local approximating algebra). Then $\mathcal{A}$ is tracially approximately finite dimensional (i.e., $\mathcal{A}$ has tracial rank zero).

Hence, $\mathcal{A}$ is an $AH$-algebra with bounded dimension growth and is determined by $K$-theoretic invariants.

The above result also gives the first proof for the locally $AH$ case.

1. Introduction

In the $K$-theoretic classification program for simple unital separable stably finite nuclear $C^*$-algebras, a great deal of progress has been made for those algebras which have stable rank one, real rank zero, and weak unperforation in the ordered $K_0$-group (see, for example, [7], [13], [4], [11] and the last paragraph of [10]). One of the fundamental results in this direction is the work in [7], where Elliott and Gong classified (using $K$-theoretic invariants) all simple unital $AH$-algebras with bounded dimension growth and real rank zero.

We note that the class of algebras in [7] exhausts the current invariant for simple unital stably finite real rank zero nuclear $C^*$-algebras. Much work to date has been done to give classification results, for simple, nuclear, stably finite, real rank zero algebras, that do not assume that the $C^*$-algebras involved are $AH$-algebras (see, for example, [13] and the references therein).

Definition 1.1. Let $\mathcal{A}$ be a simple unital $C^*$-algebra. Then $\mathcal{A}$ is said to be tracially approximately finite dimensional (abbreviated by “TAF”) if for every $\epsilon > 0$, for every finite subset $\mathcal{F}$ of $\mathcal{A}$ and for every strictly positive element $a \in \mathcal{A}$, there is a projection $p$ which is Murray-von Neumann equivalent to a subprojection in the hereditary subalgebra generated by $a$ and there exists a finite dimensional $C^*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that: (a) $1_\mathcal{A} - p = 1_\mathcal{B}$ where $1_\mathcal{B}$ is the unit of $\mathcal{B}$, (b) $\|xp - px\| < \epsilon$ for every $x \in \mathcal{F}$, and (c) $(1_\mathcal{A} - p)x(1_\mathcal{A} - p)$ is within $\epsilon$ of an element of $\mathcal{B}$, for every $x \in \mathcal{F}$.

The term “tracial rank zero” is often used in place of “tracially approximately finite dimensional” (see, for instance, [12] and [13]). Hence, by definition, a simple,
unital $C^*$-algebra is TAF if and only if it has tracial rank zero. (Indeed, Lin has a notion of *tracial rank*, which takes on values other than zero. For example, all nonreal rank zero, simple, unital AH-algebras, with bounded dimension growth, have tracial rank one; see [12].)

Lin has shown that the class of simple unital separable nuclear TAF algebras which satisfy the universal coefficient theorem is exactly the class of [7] (see [13]).

**Definition 1.2.** Let $A$ be a $C^*$-algebra. (a) Then $A$ is said to be locally type I if for every $\epsilon > 0$, for every finite subset $\mathcal{F}$ of $A$, there is a separable type I $C^*$-subalgebra $B$ of $A$ such that every element of $\mathcal{F}$ is within $\epsilon$ of an element of $B$. (b) If in (a), every (local approximating) type I $C^*$-algebra $B$ has Hausdorff spectrum and there exists an integer $L$ (dependent on $B$) such that every irreducible representation of $B$ has dimension less than $L$, then $A$ is said to have locally Hausdorff spectrum. (c) If in (a), every (locally approximating) type I $C^*$-algebra has the form $\bigoplus_{i=1}^N \pi_i \mathcal{M}_{n_i}(C(X_i))\pi_i$, where each $X_i$ is a compact metric space and each $\pi_i$ is a projection in $\mathcal{M}_{n_i}(C(X_i))$, then $A$ is said to be locally AH.

We note that Dadarlat and Eilers have given an example of a (nonsimple) separable, unital, locally AH $C^*$-algebra which has real rank zero and stable rank one, but is not an AH-algebra (see [3]).

We also note that in [2], Dadarlat has shown that if $A$ is a separable nuclear $C^*$-algebra which can be locally approximated by $C^*$-algebras which satisfy the universal coefficient theorem, then $A$ also satisfies the universal coefficient theorem. Hence, every locally type I $C^*$-algebra satisfies the universal coefficient theorem.

In [15], Lin proved the following very interesting result (there are several proofs in the literature; other proofs can be found in [1, Corollary 7.11], [14] and [17, Theorem 5.16]):

**Theorem 1.3.** Let $A$ be a unital separable simple locally type I $C^*$-algebra with real rank zero, stable rank one, weak unperforation in the $K_0$-group. Suppose also that the tracial simplex has countably many extreme points. Then $A$ is TAF. By a theorem of Lin, this implies that $A$ is an AH-algebra with bounded dimension growth and is determined by $K$-theoretic invariants.

We note that Lin’s result requires a restriction on the tracial simplex of $A$ (countably many extreme points). There have also been other interesting results in the literature which require this restriction on the tracial simplex (see, for example [1], the last paragraph of [10, 14, 15 and 17]).

In this paper, we remove the unique trace condition in Lin’s result provided that the (local) type I algebras have Hausdorff spectrum and bounded irreducible representations.

**Definition 1.4.** $\mathcal{LCH}^+$ is the class of simple unital separable $C^*$-algebras with real rank zero, stable rank one, weak unperforation in the ordered $K_0$-group, and having locally Hausdorff spectrum.

**Theorem 1.5.** Let $A$ be a $C^*$-algebra in $\mathcal{LCH}^+$. Then $A$ is TAF. Hence, by a theorem of Lin, $A$ is an AH-algebra with bounded dimension growth and is determined by $K$-theoretic invariants.

Our result gives the first proof that a simple unital separable locally AH $C^*$-algebra with real rank zero, stable rank one, weak unperforation in the $K_0$ group is TAF, without any restriction on the tracial simplex.
A modification of our argument gives a short alternative proof of the following result of Lin (which also follows from our result).

**Theorem 1.6** (see [16]). Let \( \mathcal{A} \) be a simple unital AH-algebra which has stable rank one, real rank zero and weakly unperforated \( K_0 \) group. Then \( \mathcal{A} \) is TAF.

Note that in the hypothesis of the above result, it is not assumed that \( \mathcal{A} \) has bounded dimension growth. Also, Lin’s argument does not generalize to the locally AH case.

In what follows, if \( \mathcal{A} \) is a unital \( C^* \)-algebra, then \( T(\mathcal{A}) \) is the simplex of unital traces on \( \mathcal{A} \).

2. **Main result**

**Proof of Theorem 1.6.** Let \( \{G_m^{(1)}\}_{m=1}^\infty \) be an increasing sequence of finite subsets of \( \mathcal{A} \) such that \( \mathcal{A} = \bigcup_{m=1}^\infty \overline{G_m^{(1)}} \). Let \( f \) be the function on the unit interval \([0, 1]\) given by \( f(t) = 0 \) for \( t < 1/2 \) and \( f(t) = 1 \) for \( t \geq 1/2 \). For each \( m \), let \( G_m^{(2)} \) be the (finite) set of elements of \( \mathcal{A} \) given by \( G_m^{(2)} = \{ f(\|a\|)/\|a\|) : a \neq 0, a \in G_m^{(1)}, f \) is continuous on the spectrum of \( |a|/\|a\| \} \) (here, given \( a \in \mathcal{A} \), \( |a| \) is the absolute value of \( a \) and \( \|a\| \) is the norm of \( a \)). Note that \( \bigcup_{m=1}^\infty G_m^{(2)} \) is dense in the set of projections of \( \mathcal{A} \). Now for each \( m \), let \( G_m = G_m^{(1)} \cup G_m^{(2)} \). Since \( \mathcal{A} \) is in \( \mathcal{LCH}^+ \), let \( \{A_m\}_{m=1}^\infty \) be a sequence of unital separable subalgebras of \( \mathcal{A} \), with Hausdorff spectrum and bounded irreducible representations, such that for each \( m \), \( a \) is within a distance \( 1/2m \) of an element, say \( \phi_m(a) \), of \( A_m \) for every \( a \in G_m \). If \( a \) is a projection in \( G_m^{(2)} \), we further require that \( \phi_m(a) \) be a projection. Now, for each \( m \), \( A_m \) need not be a continuous trace \( C^* \)-algebra, but by [19] Theorem 4, \( A_m \) is “continuous trace” with respect to the normalized trace; that is, for each \( a \in A_m \), the map \( \overline{A_m} \rightarrow \mathbb{R} \) given by \( \pi \mapsto tr(\pi(a)) \) is continuous (where \( tr \) is the unital, normalized trace on the image of \( \pi \), and \( \overline{A_m} \) is the spectrum space of irreducible representations of \( A_m \)). But for each \( m \), for each \( p \in G_m^{(2)} \), the map \( \pi \mapsto tr(\pi(\phi_m(p))) \) can take on only finitely many (rational) values. Hence, \( \overline{A_m} \) is the disjoint union of finitely many clopen sets such that for each \( p \in G_m^{(2)} \), the map \( \pi \mapsto tr(\pi(\phi_m(p))) \) has constant value on each clopen set. Hence, for each \( m \), \( A_m \) can be realized as a finite direct sum \( A_m = \bigoplus_{i=1}^{N_m} A_{m,i} \) where each summand \( A_{m,i} \) has spectrum being one of the clopen sets. In particular, this means that for every \( m \), for every projection \( p \in G_m^{(2)} \), for \( 1 \leq i \leq N_m \), the map \( \pi \mapsto tr(\pi(1_{A_{m,i}} \phi_m(p) 1_{A_{m,i}})) \) is constant on the spectrum \( \overline{A_{m,i}} \) (where “\( tr \)”, as always, denotes the unital normalized trace on the image of \( \pi \)). We may assume that \( 1_{A_{m,i}} = 1_A \) for every \( m \). Let \( \mathcal{B} = \bigoplus_{i=1}^\infty \bigoplus_{m=1}^\infty \bigoplus_{i=1}^{N_m} A_{m,i} = \bigoplus_{i=1}^\infty B_i \). Then the multiplier algebra of \( \mathcal{B} \) is \( \mathcal{M}(\mathcal{B}) = \prod_{i=1}^\infty B_i \) such that each \( B_i \) is one of the \( A_{m,i} \).

For \( a \in \bigcup_{m=1}^\infty G_m \) and for each strictly positive integer \( l \), let \( (a, l) \) be an element \( b \in B_l \) defined in the following manner: Suppose that \( B_i \) is the summand \( A_{m,i} \) of \( \mathcal{M}(\mathcal{B}) \). If \( a \) is in \( G_m \), then let \( (a, l) = 1_{B_l} \phi_m(a) 1_{B_l} \). Otherwise, let \( (a, l) \) be zero.

We may assume that for every integer \( l \), \( (1_A, l) = 1_{B_l} \). We have an \(*\)-homomorphism \( \Gamma: \mathcal{A} \rightarrow \mathcal{M}(\mathcal{B})/\mathcal{B} \) which is defined as follows: suppose that \( a \in \mathcal{A} \). Let \( \{a_n\}_{n=1}^\infty \) be a sequence in \( \bigcup_{m=1}^\infty G_m \) which converges to \( a \). Then we let \( \Gamma(a) = \bigoplus \lim_{n \rightarrow \infty} (a_n,l)/\mathcal{B} \). One can check that \( \Gamma \) is indeed a well-defined \(*\)-homomorphism.
Now since $\mathcal{A}$ is simple, $\Gamma$ is either injective or the zero map. Since $(1, \mathcal{A}, l) = B_l$ for every $l$, $\Gamma$ is unital and hence must be injective.

For each $l$, let $\tau_l$ be a unital trace on $B_l$ obtained by a point evaluation on $B_l$, the spectrum of $B_l$ (that is, $\tau_l$ is obtained by composing an irreducible representation of $B_l$ with the usual unital trace on matrices).

Let $\epsilon > 0$ and a finite subset $F$ of $\mathcal{A}$ be given. To show that $\mathcal{A}$ is $TAF$, we need to prove that there is a projection $p \in \mathcal{A}$ and there is a finite dimensional $C^*$-subalgebra $\mathcal{C}$ of $\mathcal{A}$, with $1_{\mathcal{C}} = 1 - p$, such that:

1. $\sup_{r \in T(\mathcal{A})} \tau(r) < \epsilon$,
2. $\|pf - fp\| < \epsilon$ for every $f \in F$,
3. $(1 - p)f(1 - p)$ is within $\epsilon$ of an element of $\mathcal{C}$ for every $f \in F$.

So, let $\epsilon$ and $F$ be given as above. To simplify notation, we may assume that each element of $F$ has norm less than or equal to one (adjust $\epsilon$ if necessary).

**Claim.** There is a strictly positive integer $L$ such that for each $l \geq L$, there is a projection $p_l$ in $B_l$ and a finite dimensional $C^*$-subalgebra $\mathcal{C}_l$ of $B_l$ with $1_{\mathcal{C}_l} = B_l - p_l$ such that:

1. if $B_l$ is the summand $A_{m,i}$, then $p_l$ has the form $1_{B_l} \phi(m)(1_{B_l})$ for some projection $\phi \in \mathcal{G}^{(2)}_m$,
2. $\tau_l(p_l) < \epsilon/100$,
3. $\|pf_l - fp_l\| < \epsilon/100$ for every $f \in F$,
4. $L_{\mathcal{C}_l}f(1 - p)$ is within $\epsilon/100$ of an element of $\mathcal{C}_l$ for every $f \in F$.

Now suppose, to the contrary, that the claim is not true. Let $\{l_\alpha\}_{\alpha \in I}$ be a subnet of the sequence of positive integers such that for each $\alpha \in I$, the statement of the claim does not hold for $l = l_\alpha$. Now for each integer $k$, let $\tilde{\tau}_k$ be the trace on $\mathcal{M}(B) = \prod_{k=1}^{\infty} B_{l_k}$ given by $\tilde{\tau}_k((a_i)_{i=1}^{\infty}) = \tau_k(a_k)$ ($\tilde{\tau}_k$ is defined two paragraphs before the claim, and $a_i \in B_{l_i}$ for every $i$. Now since $T(\mathcal{M}(B))$ is $w^*$-compact, the net $\{\tilde{\tau}_\alpha\}_{\alpha \in I}$ has a converging subnet. For simplicity, let us assume that $\{\tilde{\tau}_\alpha\}_{\alpha \in I}$ converges to, say $\tilde{\tau}$. Note that $\tilde{\tau}$ induces a trace on $\mathcal{M}(B)/B$, which we also denote by $\tilde{\tau}$.

Since $\Gamma : A \to \mathcal{M}(B)/B$ is a unital $*$-embedding, $\tilde{\tau} \circ \Gamma$ is a tracial state on $A$. For simplicity, we will also denote $\tilde{\tau} \circ \Gamma$ by $\tilde{\tau}$.

Note that the argument of Theorem 1.3 actually works for any (arbitrary) single trace (see either [1, Corollary 7.11], [15] or [17, Theorem 5.16], and in the locally AH case, an elementary proof can be obtained using the argument in the last section of [11]). Hence, we have that there exists a projection $q \in A$ and a finite dimensional $C^*$-subalgebra $D$ of $A$ with $1_D = 1 - q$ such that:

1. $\tilde{\tau}(q) < \epsilon/1000$,
2. $\|qf - fq\| < \epsilon/1000$ for every $f \in F$, and
3. $(1 - q)f(1 - q)$ is within $\epsilon/1000$ of an element of $D$ for every $f \in F$.

Now by our choices of the $A_{m,s}$ and $\mathcal{G}^{(2)}_{m,s}$, there is a positive integer $M > 0$, and there is a sequence $\{\epsilon_m\}_{m=1}^{\infty}$ of positive real numbers converging to zero, such that for each $m \geq M$, we have the following:

(a) There is a matrix algebra, say $D_m$, which is a subalgebra of $A_m$, and there is a unitary element $U$ of $A$ such that (i) $D_m = UDU^*$, and (ii) $U$ is within $\epsilon_m$ of $1_A$.

(b) $UqU^* = 1_A - 1_{D_m}$, and $1_A - 1_{D_m}$ is an element of $\mathcal{G}^{(2)}_m$. (Recall that $1_A = 1_{A_{m,s}}$.)

(c) $(1_A - 1_{D_m})a$ is within $\epsilon/500$ of $a(1_A - 1_{D_m})$, for every $a \in F$. 


Now suppose that for each $m$, $A_m = \bigoplus_{l=L_m}^{L_m+1} B_l$ (so $B_{L_m+k} = A_{m,k+1}$ and $L_{m+1} - L_m = N_m$). And suppose that for $m \geq M$, $1_A - q_{L_m} \leq q_{L_m+1} \odot q_{L_m+2} \oplus \cdots \oplus q_{L_m+1}$, where $q_{L_m+k}$ is a projection in $B_{L_m+k}$, for each $k$. Then $\Gamma \left( q \right) = (0, 0, \ldots, 0, q_{L_m}, q_{L_m+1}, q_{L_m+2}, \ldots)/B$, where $q_{L_m}$ is in the $L_m$th position, and where we view $B$ as $B = \bigoplus_{l=1}^{\infty} B_l$.

By the definition of $\hat{r}$, and since $\hat{r}(q) < \epsilon/1000$, we must have that $\lim_{m} \tau_{m}(q_{L_m}) < \epsilon/1000$. Choose $m_0$ such that for $\alpha \geq m_0$, $\tau_{m}(q_{L_m}) < \epsilon/1000$. Let $m_0$ be the integer such that $B_{L_m}$ comes from $A_{m_0}$. Choosing $m_0$ “large” enough if necessary, we may assume that $m_0 \geq M$ and $\epsilon_{m_0} < \epsilon/1000$. Hence, taking $l = l_{m_0}, m = m_0, \mathcal{C} = 1_{B_{L_m}} \bigoplus_{l=1}^{m_0} 1_{B_{L_m}}$, and $\nu_l = q_{L_m}$, we have that clauses (1)–(4) of the Claim are satisfied for $l = l_{m_0}$. This is a contradiction. Hence, the Claim must be true.

So let $L$ and $\mathcal{T}$ and $p/\nu l \geq \epsilon$ be as in the Claim. Recalling our definition of $B$ and the $B_l$s, suppose that $m, N_m$ are integers such that $B_{L+j} = A_{m,j}$ for $1 \leq j \leq N_m$ and $A_m = \bigoplus_{j=1}^{N_m} A_{m,j} = \bigoplus_{j=1}^{N_m} B_{L+j}$. Let $r = df \bigoplus_{j=1}^{N_m} p_{L+j}$ and $\mathcal{E} = df \bigoplus_{j=1}^{N_m} \mathcal{C}_{L+j}$. Then (a) $\| f - r \| < \epsilon$ for every $f \in \mathcal{F}$, and (c) $(1 - r)f(1 - r)$ is within $\epsilon$ of an element of the (finite-dimensional $C^*$-algebra) $\mathcal{E}$ for every $f \in \mathcal{F}$.

Also, by clause (1) of the Claim, it follows that for $1 \leq j \leq N_m$, the map on $B_{l_{m_0}}$ (the spectrum of $B_{L+1}$), given by $\pi \mapsto \tau_r(\pi(p_{L+j}))$, is a constant rational-valued function (here, $\tau_r$ is the unital trace on matrices). Hence, since $\tau_{L+j}(p_{L+j}) < \epsilon/100$ for $1 \leq j \leq N_m$, it follows that $\tau(r) < \epsilon$ for every $\tau \in T(A)$.

So since $\epsilon$ and $\mathcal{F}$ are arbitrary, $A$ is $TAF$. \hfill \Box

References


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