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# SIMPLE REAL RANK ZERO ALGEBRAS WITH LOCALLY HAUSDORFF SPECTRUM

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Abstract. Let $\mathcal{A}$ be a unital, simple, separable $C^*$-algebra with real rank zero, stable rank one, and weakly unperforated ordered $K_0$ group. Suppose, also, that $\mathcal{A}$ can be locally approximated by type I algebras with Hausdorff spectrum and bounded irreducible representations (the bound being dependent on the local approximating algebra). Then $\mathcal{A}$ is tracially approximately finite dimensional (i.e., $\mathcal{A}$ has tracial rank zero).

Hence, $\mathcal{A}$ is an $AH$-algebra with bounded dimension growth and is determined by $K$-theoretic invariants.

The above result also gives the first proof for the locally $AH$ case.

## 1. Introduction

In the $K$-theoretic classification program for simple unital separable stably finite nuclear $C^*$-algebras, a great deal of progress has been made for those algebras which have stable rank one, real rank zero, and weak unperforation in the ordered $K_0$-group (see, for example, [7], [13], [4], [1] and the last paragraph of [10]). One of the fundamental results in this direction is the work in [7], where Elliott and Gong classified (using $K$-theoretic invariants) all simple unital $AH$-algebras with bounded dimension growth and real rank zero.

We note that the class of algebras in [7] exhausts the current invariant for simple unital stably finite real rank zero nuclear $C^*$-algebras. Much work to date has been done to give classification results, for simple, nuclear, stably finite, real rank zero algebras, that do not assume that the $C^*$-algebras involved are $AH$-algebras (see, for example, [13] and the references therein).

**Definition 1.1.** Let $\mathcal{A}$ be a simple unital $C^*$-algebra. Then $\mathcal{A}$ is said to be tracially approximately finite dimensional (abbreviated by "TAF") if for every $\epsilon > 0$, for every finite subset $\mathcal{F}$ of $\mathcal{A}$ and for every strictly positive element $a \in \mathcal{A}$, there is a projection $p$ which is Murray-von Neumann equivalent to a subprojection in the hereditary subalgebra generated by $a$ and there exists a finite dimensional $C^*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that: (a) $1_{\mathcal{A}} - p = 1_{\mathcal{B}}$ where $1_{\mathcal{B}}$ is the unit of $\mathcal{B}$, (b) $\|xp - px\| < \epsilon$ for every $x \in \mathcal{F}$, and (c) $(1_{\mathcal{A}} - p)x(1_{\mathcal{A}} - p)$ is within $\epsilon$ of an element of $\mathcal{B}$, for every $x \in \mathcal{F}$.

The term "tracial rank zero" is often used in place of "tracially approximately finite dimensional" (see, for instance, [12] and [13]). Hence, by definition, a simple,

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unital $C^*$-algebra is $TAF$ if and only if it has tracial rank zero. (Indeed, Lin has a notion of *tracial rank*, which takes on values other than zero. For example, all nonreal rank zero, simple, unital $AH$-algebras, with bounded dimension growth, have tracial rank one; see [12].)

Lin has shown that the class of simple unital separable nuclear $TAF$ algebras which satisfy the universal coefficient theorem is exactly the class of [7] (see [13]).

**Definition 1.2.** Let $\mathcal{A}$ be a $C^*$-algebra. (a) Then $\mathcal{A}$ is said to be locally type I if for every $\epsilon > 0$, for every finite subset $\mathcal{F}$ of $\mathcal{A}$, there is a separable type I $C^*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that every element of $\mathcal{F}$ is within $\epsilon$ of an element of $\mathcal{B}$. (b) If in (a), every (local approximating) type I $C^*$-algebra $\mathcal{B}$ has Hausdorff spectrum and there exists an integer $L$ (dependent on $\mathcal{B}$) such that every irreducible representation of $\mathcal{B}$ has dimension less than $L$, then $\mathcal{A}$ is said to have locally Hausdorff spectrum. (c) If in (a), every (locally approximating) type I $C^*$-algebra has the form $\bigoplus_{i=1}^N p_i \mathbb{M}_{n_i}(C(X_i))p_i$, where each $X_i$ is a compact metric space and each $p_i$ is a projection in $\mathbb{M}_{n_i}(C(X_i))$, then $\mathcal{A}$ is said to be locally $AH$.

We note that Dadarlat and Eilers have given an example of a (nonsimple) separable, unital, locally $AH$ $C^*$-algebra which has real rank zero and stable rank one, but is not an $AH$-algebra (see [3]).

We also note that in [2], Dadarlat has shown that if $\mathcal{A}$ is a separable nuclear $C^*$-algebra which can be locally approximated by $C^*$-algebras which satisfy the universal coefficient theorem, then $\mathcal{A}$ also satisfies the universal coefficient theorem. Hence, every locally type I $C^*$-algebra satisfies the universal coefficient theorem.

In [15], Lin proved the following very interesting result (there are several proofs in the literature; other proofs can be found in [1, Corollary 7.11], [14] and [17, Theorem 5.16]):

**Theorem 1.3.** *Let $\mathcal{A}$ be a unital separable simple locally type I $C^*$-algebra with real rank zero, stable rank one, weak unperforation in the $K_0$-group. Suppose also that the tracial simplex has countably many extreme points. Then $\mathcal{A}$ is TAF. By a theorem of Lin, this implies that $\mathcal{A}$ is an $AH$-algebra with bounded dimension growth and is determined by $K$-theoretic invariants.*

We note that Lin's result requires a restriction on the tracial simplex of $\mathcal{A}$ (countably many extreme points). There have also been other interesting results in the literature which require this restriction on the tracial simplex (see, for example [1], the last paragraph of [10], [14], [15] and [17]).

In this paper, we remove the unique trace condition in Lin's result provided that the (local) type I algebras have Hausdorff spectrum and bounded irreducible representations.

**Definition 1.4.** $\mathcal{LCH}^+$ is the class of simple unital separable $C^*$-algebras with real rank zero, stable rank one, weak unperforation in the ordered $K_0$-group, and having locally Hausdorff spectrum.

**Theorem 1.5.** *Let $\mathcal{A}$ be a $C^*$-algebra in $\mathcal{LCH}^+$. Then $\mathcal{A}$ is TAF. Hence, by a theorem of Lin, $\mathcal{A}$ is an $AH$-algebra with bounded dimension growth and is determined by $K$-theoretic invariants.*

Our result gives the first proof that a simple unital separable locally $AH$ $C^*$-algebra with real rank zero, stable rank one, weak unperforation in the $K_0$ group is $TAF$, without any restriction on the tracial simplex.

A modification of our argument gives a short alternative proof of the following result of Lin (which also follows from our result).

**Theorem 1.6** (see [16]). *Let $\mathcal{A}$ be a simple unital AH-algebra which has stable rank one, real rank zero and weakly unperforated $K_0$ group. Then $\mathcal{A}$ is TAF.*

Note that in the hypothesis of the above result, it is not assumed that $\mathcal{A}$ has bounded dimension growth. Also, Lin's argument does not generalize to the locally *AH* case.

In what follows, if $\mathcal{A}$ is a unital $C^*$-algebra, then $T(\mathcal{A})$ is the simplex of unital traces on $\mathcal{A}$.

## 2. Main result

*Proof of Theorem* 1.5. Let $\{\mathcal{G}_m^{(1)}\}_{m=1}^{\infty}$ be an increasing sequence of finite subsets of $\mathcal{A}$ such that $\mathcal{A} = \overline{\bigcup_{m=1}^{\infty} \mathcal{G}_m^{(1)}}$. Let $f$ be the function on the unit interval $[0,1]$ given by $f(t) = 0$ for $t < 1/2$ and $f(t) = 1$ for $t \geq 1/2$. For each $m$, let $\mathcal{G}_m^{(2)}$ be the (finite) set of elements of $\mathcal{A}$ given by $\mathcal{G}_m^{(2)} =_{df} \{f|a|/\|a\|) : a \neq 0, a \in \mathcal{G}_m^{(1)}$, $f$ is continuous on the spectrum of $|a|/\|a\|\}$ (here, given $a \in \mathcal{A}$, $|a|$ is the absolute value of $a$ and $\|a\|$ is the norm of $a$). Note that $\bigcup_{m=1}^{\infty} \mathcal{G}_m^{(2)}$ is dense in the set of projections of $\mathcal{A}$. Now for each $m$, let $\mathcal{G}_m =_{df} \mathcal{G}_m^{(1)} \cup \mathcal{G}_m^{(2)}$. Since $\mathcal{A}$ is in $\mathcal{LCH}^+$, let $\{\mathcal{A}_m\}_{m=1}^{\infty}$ be a sequence of unital separable subalgebras of $\mathcal{A}$, with Hausdorff spectrum and bounded irreducible representations, such that for each $m$, $a$ is within a distance $1/2m$ of an element, say $\phi_m(a)$, of $\mathcal{A}_m$ for every $a \in \mathcal{G}_m$. If $a$ is a projection in $\mathcal{G}_m^{(2)}$, we further require that $\phi_m(a)$ be a projection. Now, for each $m$, $\mathcal{A}_m$ need not be a continuous trace $C^*$-alegbra, but by [19, Theorem 4], $\mathcal{A}_m$ is "continuous trace" with respect to the normalized trace; that is, for each $a \in \mathcal{A}_m$, the map $\widehat{\mathcal{A}_m} \to \mathbb{R}$ given by $\pi \mapsto tr(\pi(a))$ is continuous (where $tr$ is the unital, normalized trace on the image of $\pi$, and $\widehat{\mathcal{A}_m}$ is the spectrum space of irreducible representations of $\mathcal{A}_m$). But for each $m$, for each $p \in \mathcal{G}_m^{(2)}$, the map $\pi \mapsto tr(\pi(\phi_m(p)))$ can take on only finitely many (rational) values. Hence, $\widehat{\mathcal{A}_m}$ is the disjoint union of finitely many clopen sets such that for each $p \in \mathcal{G}_m^{(2)}$, the map $\pi \mapsto tr(\pi(\phi_m(p)))$ has constant value on each clopen set. Hence, for each $m$, $\mathcal{A}_m$ can be realized as a finite direct sum $\mathcal{A}_m = \bigoplus_{i=1}^{N_m} \mathcal{A}_{m,i}$ where each summand $\mathcal{A}_{m,i}$ has spectrum being one of the clopen sets. In particular, this means that for every $m$, for every projection $p \in \mathcal{G}_m^{(2)}$, for $1 \leq i \leq N_m$, the map $\pi \mapsto tr(\pi(1_{\mathcal{A}_{m,i}} \phi_m(p) 1_{\mathcal{A}_{m,i}}))$ is constant on the spectrum $\widehat{\mathcal{A}_{m,i}}$ (where "$tr$", as always, denotes the unital normalized trace on the image of $\pi$). We may assume that $1_{\mathcal{A}_m} = 1_{\mathcal{A}}$ for every $m$. Let $\mathcal{B} =_{df} \sum^{\oplus}{}_{m=1}^{\infty} \sum^{\oplus}{}_{i=1}^{N_m} \mathcal{A}_{m,i} = \sum^{\oplus}{}_{l=1}^{\infty} \mathcal{B}_l$. Then the multiplier algebra of $\mathcal{B}$ is $\mathcal{M}(\mathcal{B}) = \prod_{l=1}^{\infty} \mathcal{B}_l$ such that each $\mathcal{B}_l$ is one of the $\mathcal{A}_{m,i}$s.

For $a \in \bigcup_{m=1}^{\infty} \mathcal{G}_m$ and for each strictly positive integer $l$, let $(a,l)$ be an element $b \in \mathcal{B}_l$ defined in the following manner: Suppose that $\mathcal{B}_l$ is the summand $\mathcal{A}_{m,i}$ of $\mathcal{M}(\mathcal{B})$. If $a$ is in $\mathcal{G}_m$, then let $(a,l)$ be $1_{\mathcal{B}_l} \phi_m(a) 1_{\mathcal{B}_l}$. Otherwise, let $(a,l)$ be zero.

We may assume that for every integer $l$, $(1_{\mathcal{A}}, l) = 1_{\mathcal{B}_l}$. We have an $*$-homomorphism $\Gamma\colon\ \mathcal{A} \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ which is defined as follows: suppose that $a \in \mathcal{A}$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\bigcup_{m=1}^{\infty} \mathcal{G}_m$ which converges to $a$. Then we let $\Gamma(a) =_{df} \lim_{n\to\infty}(a_n, l)/\mathcal{B}$. One can check that $\Gamma$ is indeed a well-defined $*$-homomorphism.

Now since $\mathcal{A}$ is simple, $\Gamma$ is either injective or the zero map. Since $(1_{\mathcal{A}}, l) = 1_{\mathcal{B}_l}$ for every $l, \Gamma$ is unital and hence must be injective.

For each $l$, let $\tau_l$ be a unital trace on $\mathcal{B}_l$ obtained by a point evaluation on $\hat{\mathcal{B}}_l$, the spectrum of $\mathcal{B}_l$ (that is, $\tau_l$ is obtained by composing an irreducible representation of $\mathcal{B}_l$ with the usual unital trace on matrices).

Let $\epsilon > 0$ and a finite subset $\mathcal{F}$ of $\mathcal{A}$ be given. To show that $\mathcal{A}$ is $TAF$, we need to prove that there is a projection $p \in \mathcal{A}$ and there is a finite dimensional $C^*$-subalgebra $\mathcal{C}$ of $\mathcal{A}$, with $1_{\mathcal{C}} = 1 - p$, such that:

(1) $\sup_{r \in T(\mathcal{A})} \tau(p) < \epsilon$,

(2) $\|pf - fp\| < \epsilon$ for every $f \in \mathcal{F}$, and

(3) $(1-p)f(1-p)$ is within $\epsilon$ of an element of $\mathcal{C}$ for every $f \in \mathcal{F}$.

So, let $\epsilon$ and $\mathcal{F}$ be given as above. To simplify notation, we may assume that each element of $\mathcal{F}$ has norm less than or equal to one (adjust $\epsilon$ if necessary).

**Claim.** There is a strictly positive integer $L$ such that for each $l \geq L$, there is a projection $p_l$ in $\mathcal{B}_l$ and a finite dimensional $C^*$-subalgebra $\mathcal{C}_l$ of $\mathcal{B}_l$ with $1_{\mathcal{C}_l} = 1_{\mathcal{B}_l} - p_l$ such that:

(1) if $\mathcal{B}_l$ is the summand $\mathcal{A}_{m,i}$, then $p_l$ has the form $1_{\mathcal{B}_l}\phi_m(p)1_{\mathcal{B}_l}$ for some projection $p \in \mathcal{G}_m^{(2)}$,

(2) $\tau_l(p_l) < \epsilon/100$,

(3) $\|p_l f - f_{p_l}\| < \epsilon/100$ for every $f \in \mathcal{F}$, and

(4) $1_{\mathcal{C}_l} f 1_{\mathcal{C}_l}$ is within $\epsilon/100$ of an element of $\mathcal{C}_l$ for every $f \in \mathcal{F}$.

Now suppose, to the contrary, that the claim is not true. Let $\{l_\alpha\}_{\alpha \in I}$ be a subnet of the sequence of positive integers such that for each $\alpha \in I$, the statement of the claim does not hold for $l = l_\alpha$. Now for each integer $k$, let $\tilde{\tau}_k$ be the trace on $\mathcal{M}(\mathcal{B}) = \prod_{l=1}^{\infty} \mathcal{B}_l$ given by $\tilde{\tau}_k((a_l)_{l=1}^{\infty}) = \tau_k(a_k)$ ($\tau_k$ is defined two paragraphs before the claim, and $a_l \in \mathcal{B}_l$ for every $l$). Now since $T(\mathcal{M}(\mathcal{B}))$ is $w^*$-compact, the net $\{\tilde{\tau}_{l_\alpha}\}_{\alpha \in I}$ has a converging subnet. For simplicity, let us assume that $\{\tilde{\tau}_{l_\alpha}\}_{\alpha \in I}$ converges to, say $\tilde{\tau}$. Note that $\tilde{\tau}$ induces a trace on $\mathcal{M}(\mathcal{B})/\mathcal{B}$, which we also denote by "$\tilde{\tau}$". Since $\Gamma\colon\ \mathcal{A} \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is a unital $*$-embedding, $\tilde{\tau} \circ \Gamma$ is a tracial state on $\mathcal{A}$. For simplicity, we will also denote $\tilde{\tau} \circ \Gamma$ by "$\tilde{\tau}$".

Note that the argument of Theorem 1.3 actually works for any (arbitrary) *single* trace (see either [1, Corollary 7.11], [15] or [17, Theorem 5.16], and in the locally $AH$ case, an elementary proof can be obtained using the argument in the last section of [11]). Hence, we have that there exists a projection $q \in \mathcal{A}$ and a finite dimensional $C^*$-subalgebra $\mathcal{D}$ of $\mathcal{A}$ with $1_{\mathcal{D}} = 1 - q$ such that:

(1) $\tilde{\tau}(q) < \epsilon/1000$,

(2) $\|qf - fq\| < \epsilon/1000$ for every $f \in \mathcal{F}$, and

(3) $(1-q)f(1-q)$ is within $\epsilon/1000$ of an element of $\mathcal{D}$ for every $f \in \mathcal{F}$.

Now by our choices of the $\mathcal{A}_m$s and $\mathcal{G}_m^{(2)}$s, there is a positive integer $M > 0$, and there is a sequence $\{\epsilon_m\}_{m=1}^{\infty}$ of positive real numbers converging to zero, such that for each $m \geq M$, we have the following:

(a) There is a matrix algebra, say $\mathcal{D}_m$, which is a subalgebra of $\mathcal{A}_m$, and there is a unitary element $U$ of $\mathcal{A}$ such that (i) $\mathcal{D}_m = U\mathcal{D}U^*$, and (ii) $U$ is within $\epsilon_m$ of $1_{\mathcal{A}}$.

(b) $UqU^* = 1_{\mathcal{A}} - 1_{\mathcal{D}_m}$, and $1_{\mathcal{A}} - 1_{\mathcal{D}_m}$ is an element of $\mathcal{G}_m^{(2)}$. (Recall that $1_{\mathcal{A}} = 1_{\mathcal{A}_m}$.)

(c) $(1_{\mathcal{A}} - 1_{\mathcal{D}_m})a$ is within $\epsilon/500$ of $a(1_{\mathcal{A}} - 1_{\mathcal{D}_m})$, for every $a \in \mathcal{F}$.

Now suppose that for each $m$, $\mathcal{A}_m = \bigoplus_{l=L_m}^{L_{m+1}-1} \mathcal{B}_l$ (so $\mathcal{B}_{L_m+k} = \mathcal{A}_{m,k+1}$ and $L_{m+1} - L_m = N_m$). And suppose that for $m \geq M$, $1_{\mathcal{A}} - 1_{\mathcal{D}_m} = q_{L_m} \oplus q_{L_m+1} \oplus q_{L_m+2} \oplus \cdots \oplus q_{L_{m+1}-1}$, where $q_{L_m+k}$ is a projection in $\mathcal{B}_{L_m+k}$, for each $k$. Then $\Gamma(q) = (0, 0, \ldots, 0, q_{L_M}, q_{L_M+1}, q_{L_M+2}, \ldots)/\mathcal{B}$, where $q_{L_M}$ is in the $L_M$th position, and where we view $\mathcal{B}$ as $\mathcal{B} = \sum^{\oplus}{}_{l=1}^{\infty} \mathcal{B}_l$.

By the definition of $\tilde{\tau}$, and since $\tilde{\tau}(q) < \epsilon/1000$, we must have that $\lim_\alpha \tau_{l_\alpha}(ql_\alpha) < \epsilon/1000$. Choose $\alpha_0$ such that for $\alpha \geq \alpha_0$, $\tau_{l_\alpha}(q_{l_\alpha}) < \epsilon/1000$. Let $m_0$ be the integer such that $\mathcal{B}_{l_{\alpha_0}}$ comes from $\mathcal{A}_{m_0}$. Choosing $\alpha_0$ "large" enough if necessary, we may assume that $m_0 \geq M$ and $\epsilon_{m_0} < \epsilon/1000$. Hence, taking $l = l_{\alpha_0}$, $m = m_0$, $\mathcal{C}_l = 1_{\mathcal{B}_{l_{\alpha_0}}} \mathcal{D}_{m_0} 1_{\mathcal{B}_{l_{\alpha_0}}}$, and $p_l = q_{l_{\alpha_0}}$, we have that clauses (1)–(4) of the Claim are satisfied for $l = l_{\alpha_0}$. This is a contradiction. Hence, the Claim must be true.

So let $L$ and $\mathcal{C}_l$ and $p_l \forall l \geq L$ be as in the Claim. Recalling our definition of $\mathcal{B}$ and the $\mathcal{B}_l$s, suppose that $m, N_m$ are integers such that $\mathcal{B}_{L+j} = \mathcal{A}_{m,j}$ for $1 \leq j \leq N_m$ and $\mathcal{A}_m = \bigoplus_{j=1}^{N_m} \mathcal{A}_{m,j} = \bigoplus_{j=1}^{N_m} \mathcal{B}_{L+j}$. Let $r =_{df} \bigoplus_{j=1}^{N_m} p_{L+j}$ and $\mathcal{E} =_{df} \bigoplus_{j=1}^{N_m} \mathcal{C}_{L+j}$. Then (a) $1_{\mathcal{E}} = 1 - r$, (b) $\|rf - fr\| < \epsilon$ for every $f \in \mathcal{F}$, and (c) $(1-r)f(1-r)$ is within $\epsilon$ of an element of (the finite-dimensional $C^*$-algebra) $\mathcal{E}$ for every $f \in \mathcal{F}$.

Also, by clause (1) of the Claim, it follows that for $1 \leq j \leq N_m$, the map on $\widehat{\mathcal{B}_{L+j}}$ (the spectrum of $\mathcal{B}_{L+j}$), given by $\pi \mapsto tr(\pi(p_{L+j}))$, is a constant rational-valued function (here, $tr$ is the unital trace on matrices). Hence, since $\tau_{L+j}(p_{L+j}) < \epsilon/100$ for $1 \leq j \leq N_m$, it follows that $\tau(r) < \epsilon$ for every $\tau \in T(\mathcal{A})$.

So since $\epsilon$ and $\mathcal{F}$ are arbitrary, $\mathcal{A}$ is $TAF$. $\square$

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