A SPECTRAL MAPPING THEOREM FOR REPRESENTATIONS
OF ONE-PARAMETER GROUPS

H. SEFEROĞLU

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Abstract. In this paper we present some generalization (at the same time
a new and a short proof in the Banach algebra context) of the Weak Spect-
tral Mapping Theorem (WSMT) for non-quasianalytic representations of one-
parameter groups.

1. Introduction

Let $X$ be a complex Banach space and let $B(X)$ be the algebra of all bounded
linear operators on $X$. A family $T = \{T(t)\}_{t \in \mathbb{R}}$ in $B(X)$ is called a $C_0$-group if the
following properties are satisfied:

(i) $T(0) = 1_X$, the identity operator on $X$;
(ii) $T(t+s) = T(t)T(s)$, for every $t, s \in \mathbb{R}$;
(iii) $\lim_{t \to 0} \|T(t)x - x\| = 0$, for all $x \in X$.

The generator of $T = \{T(t)\}_{t \in \mathbb{R}}$ is the linear operator $A$ defined by

$$Ax = \lim_{t \to 0} \frac{1}{t}(T(t)x - x), \quad x \in D(A).$$

The generator is always a closed, densely defined operator.

For many applications it is a fundamental problem to decide whether the spec-
trum of each operator $T(t)$ can be obtained from the spectrum $\sigma(A)$ of $A$.

If $A$ is bounded, by the Dunford functional calculus for all $t \in \mathbb{R}$ we have the spectral mapping theorem

$$\sigma(\exp(tA)) = \exp(t\sigma(A)).$$

In general, however, a generator $A$ is unbounded and the spectral mapping theorem in the above form does not hold. One hopes that the Weak Spectral Mapping Theorem (WSMT) holds, i.e.,

$$\sigma(T(t)) = \exp(t\sigma(A)) \quad \text{for all } t \in \mathbb{R}.$$  

For a systematic treatment (and references) of the WSMT, we refer to the mono-
graphas [3], [10]. From Hille-Phillips [4, p. 665] one already knows that the WSMT may fail.
We recall that a measurable locally bounded function $\omega : \mathbb{R} \to \mathbb{R}$ is called a weight if $1 \leq \omega(t)$ and $\omega(t+s) \leq \omega(t)\omega(s)$ for all $t, s \in \mathbb{R}$. A weight function $\omega(t)$ is said to be non-quasianalytic if

$$\int_{\mathbb{R}} \ln \omega(t) \frac{1}{1+t^2} dt < \infty.$$  

In [2] it was shown that for the $C_0$-groups $T = \{T(t)\}_{t \in \mathbb{R}}$ satisfying $\|T(t)\| \leq \omega(t)$ ($t \in \mathbb{R}$), the WSMT holds, where $\omega(t)$ is a non-quasianalytic weight function on $\mathbb{R}$. It is also known that the growth cannot be weakened in general.

Now let $T = \{T(t)\}_{t \in \mathbb{R}}$ be a uniformly bounded $C_0$-group. If the generator $A$ is bounded, we can deduce even more:

$$\sigma(\hat{\mu}(T)) = \hat{\mu}(\sigma(iA)),$$

where $\mu$ is a bounded regular complex Borel measure on $\mathbb{R}$, $\hat{\mu}$, the Fourier-Stieltjes transform of $\mu$ and $\hat{\mu}(T) = \int T(t)d\mu(t)$. To see this let $B$ be the maximal uniformly closed commutative subalgebra of $B(X)$ containing $A$. Let $M_B$ be the maximal ideal space of $B$. Denote by $\sigma_B(S)$ the spectrum of any $S \in B$ with respect to $B$. Since $\sigma_B(iA)$ is real, $\sigma_B(iA) = \sigma(iA)$. Also since $B$ is a unital full subalgebra of $B(X)$ and $\hat{\mu}(T) \in B$, we have

$$\sigma(\hat{\mu}(T)) = \sigma_B(\hat{\mu}(T)) = \{\phi(\hat{\mu}(T)) : \phi \in M_B\}$$

$$= \left\{ \int \exp(tA)d\mu(t) : \phi \in M_B \right\} = \left\{ \int \exp(t\phi(A))d\mu(t) : \phi \in M_B \right\}$$

$$= \hat{\mu}(\sigma_B(iA)) = \hat{\mu}(\sigma(iA)).$$

When the generator $A$ is unbounded, this raises the natural question for which measures the equality

$$\sigma(\hat{\mu}(T)) = \hat{\mu}(\sigma(iA))$$

holds. In general, this does not hold for every measure (see Remark 3.4).

In this paper we show that if $T = \{T(t)\}_{t \in \mathbb{R}}$ is a $C_0$-group dominated by some non-quasianalytic weight $\omega$ on $\mathbb{R}$, then for the measures from the largest regular weight measure algebra $regM_\omega(\mathbb{R})$, the equality (1.1) holds. $regM_\omega(\mathbb{R})$ contains weight group algebras $L^1_w(\mathbb{R})$ and $M_\omega(\mathbb{R}_d)$. By taking in (1.1) the special case $\mu = \delta_t$, the Dirac measure concentrated in $t \in \mathbb{R}$, we obtain the WSMT. Hence, the equality (1.1) generalizes the WSMT.

2. Preliminaries

Later on we will need some preliminary results with which we now proceed. Let $A$ be a complex commutative Banach algebra, and let $\Delta_A$ be the structure space of $A$. By $\hat{a}$ we denote the Gelfand transform of the element $a \in A$. The hull of any ideal $I \subset A$ is $h(I) = \{ \phi \in \Delta_A : \hat{a}(\phi) = 0, a \in I \}$. $A$ is regular if for each closed $S \subset \Delta_A$ and each $\phi \in \Delta_A \setminus S$ there exists $a \in A$ such that $\hat{a}(\phi) = 1$ and $\hat{a}(S) = \{0\}$. If $A$ is regular and semisimple, then $I_S = \{ a \in A : \hat{a}(\phi) = 0, \phi \in S \}$ is the largest closed ideal in $A$ whose hull is $S$. $J_S$ is closure $\{ a \in A : supp \hat{a} \cap S = \emptyset \}$ and $supp \hat{a}$ is compact is the smallest closed ideal in $A$ whose hull is $S$.
Lemma 2.1. Let $A$ be a regular semisimple Banach algebra with identity $1_A$ and let $\psi : A \to B(X)$ be a continuous homomorphism such that $\psi(1_A) = 1_X$. Then

$$\sigma(\psi(a)) = \hat{a}(h(\ker \psi)) \text{ for each } a \in A.$$  

**Proof.** Put $C = \overline{\psi(A)}$. Then $C$ is a commutative Banach algebra with identity $1_X$. We see that the mapping $\psi^* : \Delta_C \to h(\ker \psi)$ is injective. First we claim that $\psi^*$ is surjective. Suppose on the contrary that there exists $\phi_0 \in h(\ker \psi)$ but $\phi_0 \notin \psi^* \Delta_C$. Let $U$ and $V$ be disjoint neighborhoods of $\phi_0$ and $\psi^* \Delta_C$, respectively. Then there exist some elements $a, b \in A$, such that $\hat{a}(\phi_0) = 1$, $\hat{a}(\Delta_C \setminus U) = \{0\}$, $\hat{b}(\psi^* \Delta_C) = \{1\}$ and $\hat{b}(\Delta_C \setminus V) = \{0\}$. It is easy to see that $\hat{a}(\phi) \hat{b}(\phi) = 0$ for all $\phi \in M_A$. Since $A$ is semisimple we have $ab = 0$ which implies $\psi(a)\psi(b) = 0$. Now since $\psi(b) \in C^{-1}$ we obtain $\psi(a) = 0$, and so $\hat{a}(\phi_0) = 0$. This is a contradiction, proving our claim. Hence we have $\psi^* \Delta_C = h(\ker \psi)$, which implies

$$\sigma_C(\psi(a)) = \hat{a}(h(\ker \psi)) \text{ for all } a \in A,$$

where $\sigma_C(\psi(a))$ is the spectrum of $\psi(a)$ with respect to the algebra $C$. It remains to show that $C$ is a full subalgebra of $B(X)$. For this purpose let $c \in C$ be such that $c \in B(X)^{-1}$ and let $D$ be the smallest closed subalgebra of $B(X)$ that contains $c^{-1}$ and $C$. We can see that $D$ is a commutative Banach algebra with the identity $1_X$ and $C$ is a regular subalgebra of $D$. By the Shilov theorem [5 Corollary 9.5.5] any $\phi \in \Delta_C$ can be extended to some $\hat{\phi} \in \Delta_D$. Since $c \in D^{-1}$ we have $\phi(c) = \hat{\phi}(c) \neq 0$ for all $\phi \in \Delta_C$. This implies that $C \subseteq C^{-1}$. The proof is complete. \hfill $\square$

The Beurling algebra with weight $\omega$ on $\mathbb{R}$ is

$$L^1_\omega(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) : \|f\|_\omega = \int_{\mathbb{R}} |f(t)| \omega(t) dt < \infty \right\}.$$  

$L^1_\omega(\mathbb{R})$ is a Banach algebra with respect to convolution. The following result [8 chap. 5, sect. 1] is well known.

**Theorem 2.2.** If $\omega$ is a non-quasianalytic weight on $\mathbb{R}$, then:

(i) the algebra $L^1_\omega(\mathbb{R})$ has structure space $\mathbb{R}$ and is regular on $\mathbb{R}$;
(ii) the Gelfand transform of $f \in L^1_\omega(\mathbb{R})$ is just $\hat{f}$, the Fourier transform of $f$;
(iii) $L^1_\omega(\mathbb{R})$ is Tauberian, i.e., the set of all $f \in L^1_\omega(\mathbb{R})$ whose Fourier transform has compact support is dense in $L^1_\omega(\mathbb{R})$.

As is known [10 chap. 2], if $T = \{T(t)\}_{t \in \mathbb{R}}$ is a $C_0$-group dominated by some a non-quasianalytic weight on $\mathbb{R}$, then $\sigma(iA)$ is located on the real line and every $\lambda \in \sigma(iA)$ is an approximate point spectrum for $iA$. Moreover, 

$$R(z, A) = \begin{cases} \int_0^\infty \exp(-zt)T(t)dt, & \Re z > 0, \\ -\int_{-\infty}^0 \exp(-zt)T(t)dt, & \Re z < 0, \end{cases}$$  

(2.1)

where $R(z, A)$ is the resolvent of $A$. Let

$$\tilde{f}(\lambda) = \int_{\mathbb{R}} f(t) \exp(-i\lambda t) dt$$
be the Fourier transform of \( f \in L^1_\omega(\mathbb{R}) \). For every \( f \in L^1_\omega(\mathbb{R}) \) we can define \( \hat{f}(T) \in B(X) \) by
\[
\hat{f}(T) = \int_\mathbb{R} f(t)T(t)dt.
\]

**Lemma 2.3.** Let \( \omega \) be a non-quasianalytic weight on \( \mathbb{R} \) and let \( f \in L^1_\omega(\mathbb{R}) \).

a) If \( \hat{f}(T) = 0 \), then \( \hat{f}(\lambda) = 0 \) for every \( \lambda \in \sigma(iA) \).

b) If \( \hat{f} \) is compactly supported and vanishes in a neighborhood of \( \sigma(iA) \), then \( \hat{f}(T) = 0 \).

**Proof.** a) If \( \lambda \in \sigma(iA) \), then there is a sequence \((x_n)_{n \in \mathbb{N}}\) of norm one vectors in \( X \), such that \( \|Ax_n - (-\lambda)x_n\| \to 0 \) as \( n \to \infty \). This implies [10, Proposition 2.1.6] that \( \|T(t)x_n - \exp(-i\lambda t)x_n\| \to 0 \). Consequently, we have
\[
|\hat{f}(\lambda)| = \|\hat{f}(T)x_n - \hat{f}(\lambda)x_n\| \leq \int_\mathbb{R} \|T(t)x_n - \exp(-i\lambda t)x_n\||f(t)|dt \to 0.
\]

b) has been proved in [10, Lemma 2.4.3]. We include a proof for completeness.

Let \( f \in L^1_\omega(\mathbb{R}) \) be such that \( \text{supp} \hat{f} \) is compact and \( \hat{f} \) vanishes in a neighborhood of \( \sigma(iA) \). Applying the formula for the inverse Fourier transform and the formula (2.1), we can write
\[
\hat{f}(T) = \lim_{\delta \to 0^+} \int_\mathbb{R} \exp(-\delta|t|)f(t)T(t)dt = \lim_{\delta \to 0^+} \int_\mathbb{R} \exp(-\delta|t|) \left( \int_\mathbb{R} \exp(i\lambda t)\hat{f}(\lambda)d\lambda \right)T(t)dt = \lim_{\delta \to 0^+} \int_\mathbb{R} \hat{f}(\lambda)(R(\delta - i\lambda, A) - R(-\delta - i\lambda, A))d\lambda = 0.
\]

\( \square \)

3. The main result

Before stating the main result we need some notation. Let \( M(\mathbb{R}) \) be the convolution measure algebra of \( \mathbb{R} \) and let \( \omega \) be a weight function on \( \mathbb{R} \). \( M_\omega(\mathbb{R}) \) is the set of all \( \mu \in M(\mathbb{R}) \) such that
\[
\|\mu\|_\omega = \int_\mathbb{R} \omega(t)d|\mu|(t) < \infty.
\]

With respect to convolution \( M_\omega(\mathbb{R}) \) is a commutative Banach algebra with the identity \( \delta_0 \), where \( \delta_0 \) is the Dirac measure concentrated in zero. As is known [7, Theorem 4.3.6, \( M_\omega(\mathbb{R}) \) contains a greatest regular subalgebra, which we will denote by \( \text{reg}M_\omega(\mathbb{R}) \). If \( \omega \) is a non-quasianalytic weight on \( \mathbb{R} \), then \( L^1_\omega(\mathbb{R}) \) and \( M_\omega(\mathbb{R}_d) (= L^1_\omega(\mathbb{R}_d)) \) are subalgebras of \( \text{reg}M_\omega(\mathbb{R}) \) (here \( \mathbb{R}_d \) denotes \( \mathbb{R} \) with the discrete topology), and consequently, \( L^1_\omega(\mathbb{R}) + M_\omega(\mathbb{R}_d) \subset \text{reg}M_\omega(\mathbb{R}) \). But \( L^1_\omega(\mathbb{R}) + M_\omega(\mathbb{R}_d) \subset \text{reg}M_\omega(\mathbb{R}) \). To see this let \( \tau \) be the locally compact group topology on \( \mathbb{R} \) which is equal to or stronger than the usual topology on \( \mathbb{R} \) and let \( \mathbb{R}_\tau = (\mathbb{R}, \tau) \). Then the algebra \( L^1_\omega(\mathbb{R}_\tau) \) also belongs to \( \text{reg}M_\omega(\mathbb{R}) \).
Lemma 3.2. Under the above notation, \( h(\ker \psi) = \overline{\sigma(iA)^{\Delta_{\text{reg}}}} \).

Proof. First we show that \( \overline{\sigma(iA)^{\Delta_{\text{reg}}}} \subseteq h(\ker \psi) \). Since \( h(\ker \psi) \) is closed, it is sufficient to show that \( \sigma(iA) \subseteq h(\ker \psi) \). Suppose on the contrary that there exists a \( \lambda_0 \in \sigma(iA) \) but \( \lambda_0 \notin h(\ker \psi) \). Then there exists a \( \mu \in \text{reg}M_\omega(\mathbb{R}) \) such that \( \hat{\mu}(\lambda_0) \neq 0 \) and \( \mu^\vee \) vanishes in some neighborhood of \( h(\ker \psi) \). Hence \( \mu \) belongs to the smallest ideal in \( \text{reg}M_\omega(\mathbb{R}) \), whose hull is \( h(\ker \psi) \). Consequently, \( \mu \in \ker(\psi) \) and so \( \hat{\mu}(\chi_0) = 0 \). It follows that \( (\mu * f)^\vee(\chi_0) = \hat{\mu}(\chi_0)\hat{f}(\chi_0) = 0 \) for every \( f \in L^1(\mathbb{R}) \). Since \( \mu * f \in L^1(\mathbb{R}) \), by Lemma 2.3b we have \( 0 = (\mu * f)^\vee(\lambda) = \hat{\mu}(\lambda)\hat{f}(\lambda) \) for all \( \lambda \in \sigma(iA) \) and \( f \in L^1(\mathbb{R}) \). Thus since \( \lambda_0 \in \sigma(iA) \), this clearly implies that \( \hat{\mu}(\lambda_0) = 0 \). This is a contradiction.

For the reverse inclusion assume that there exists a \( \phi_0 \in h(\ker \psi) \) but \( \phi_0 \notin \overline{\sigma(iA)^{\Delta_{\text{reg}}}} \). Then there exists a neighborhood (in \( \Delta_{\text{reg}} \)) \( V \) of \( \sigma(iA) \) and a \( \mu \in \text{reg}M_\omega(\mathbb{R}) \) such that \( \mu^\vee(\phi_0) \neq 0 \) and \( \mu^\vee \) vanishes on \( V \). It follows that \( \hat{\mu} \) vanishes on \( V \cap \mathbb{R} \). We see that \( V \cap \mathbb{R} \) is a neighborhood of \( \sigma(iA) \) for the relative topology induced in \( \mathbb{R} \) by \( \Delta_{\text{reg}} \). Let us show that the usual topology of \( \mathbb{R} \) is a base for the relative topology.
topology induced in $\mathbb{R}$ by $\Delta_{reg}$. For this fix $\lambda_0 \in \mathbb{R}$ and \{\(\mu_1, ..., \mu_n\) \(\subset reg M_\omega(\mathbb{R})\).

It suffices to show that

$$U = \left\{ \lambda \in \mathbb{R} : \sup_{\lambda \in K} |\exp(i\lambda t) - \exp(i\lambda_0 t)| < \delta \right\}$$

$$\subset \left\{ \lambda \in \mathbb{R} : |\hat{\mu}_i(\lambda) - \hat{\mu}_i(\lambda_0)| < \varepsilon, i = 1, ..., n \right\},$$

for some compact $K \subset \mathbb{R}$ and $\delta > 0$. Choose a compact set $K$ in $\mathbb{R}$, so that $|\mu_i|(\mathbb{R} \setminus K) < \varepsilon/4$ and $0 < \delta < \varepsilon/2 \max_i |\mu_i|(\mathbb{R})$, $i = 1, ..., n$. If $\lambda \in U$, then

$$|\hat{\mu}_i(\lambda) - \hat{\mu}_i(\lambda_0)| \leq \int_K \left| \exp(i\lambda t) - \exp(i\lambda_0 t) \right| |d\mu_i| + \int_{\mathbb{R} \setminus K} \left| \exp(i\lambda t) - \exp(i\lambda_0 t) \right| |d\mu_i|$$

$$\leq \sup_{\lambda \in K} \left| \exp(i\lambda t) - \exp(i\lambda_0 t) \right| (\max_i |\mu_i|(\mathbb{R})) + 2 |\mu_i|(\mathbb{R} \setminus K) < \varepsilon, \quad i = 1, ..., n.$$

Hence $V \cap \mathbb{R}$ is an open set in the usual topology of $\mathbb{R}$. Thus we obtain that $\hat{\mu}$ vanishes in a (real) neighborhood of $\sigma(iA)$. Let $f \in L^1(\mathbb{R})$ be such that $supp \hat{f}$ is compact. Then $(\mu * f)^\gamma$ vanishes in a neighborhood of $\sigma(iA)$ and $supp(\mu * f)^\gamma$ is compact. By Lemma 2.3b) we have $\hat{\mu}(T) f(T) = 0$. On the other hand, by Theorem 2.2ii) the set of all $f \in L^1(\mathbb{R})$ whose Fourier transform has compact support is dense in $L^1(\mathbb{R})$. It follows that $\hat{\mu}(T) f(T) = 0$ for all $f \in L^1(\mathbb{R})$.

Let $f_n = 2n \chi_{[-1/n, 1/n]}$, where $\chi_{[-1/n, 1/n]}$ is the characteristic function of $[-1/n, 1/n]$. It is easy to see that $f_n(T) \to 1_X$ strongly. Now from the identity $\hat{\mu}(T) f_n(T) = 0$ we deduce that $\hat{\mu}(T) = 0$. Hence, we have $\mu \in ker(\psi)$. Since $\phi_0 \in h(ker \psi)$ we obtain $\mu^\gamma(\phi_0) = 0$. This is a contradiction. The proof is complete. 

\textbf{Proof of Theorem 3.1} As we already noted, the map $\psi : reg M_\omega(\mathbb{R}) \to B(X)$ defined by $\psi(\mu) = \hat{\mu}(T)$ is a continuous homomorphism. Note also that $\psi(\delta_0) = 1_X$. Let $\mu \in reg M_\omega(\mathbb{R})$ be given. Then by Lemma 2.1 we have $\sigma(\hat{\mu}(T)) = \mu^\gamma(h(ker \psi)).$

By Lemma 3.2 since $h(ker \psi) = \sigma(iA)^\Delta_{reg}$ we obtain $\sigma(\hat{\mu}(T)) = \mu^\gamma(\sigma(iA)^\Delta_{reg})$. Also since $\mu(\sigma(iA)) = \mu^\gamma(\sigma(iA)) \subset \mu^\gamma(\sigma(iA)^\Delta_{reg})$ and $\mu^\gamma(\sigma(iA)^\Delta_{reg})$ is closed, we have $\overline{\mu(\sigma(iA))} \subset \mu^\gamma(\sigma(iA)^\Delta_{reg})$. On the other hand, from the continuity $\mu^\gamma$ on $\Delta_{reg}$ we deduce that $\mu^\gamma(\sigma(iA)^\Delta_{reg}) \subset \mu^\gamma(\sigma(iA)) = \overline{\mu(\sigma(iA))}$. Thus we obtain that $\sigma(\hat{\mu}(T)) = \mu^\gamma(\sigma(iA))$. The proof is complete. 

Let us record a consequence of this theorem.

\textbf{Corollary 3.3.} Let $w$ be a non-quasianalytic weight on $\mathbb{R}$ and let $T = \{T(t)\}_{t \in \mathbb{R}}$ be a $C_0$-group on a Banach space $X$ with generator $A$, such that

$$||T(t)|| \leq \omega(t), \quad t \in \mathbb{R}.$$

If $E$ is a $T$-invariant closed subspace of $X$, then

$$(3.2) \quad \sigma(\hat{\mu}(T)|_E) = \hat{\mu}(\sigma(iAE))$$

for every $\mu \in reg M_\omega(\mathbb{R})$.

\textbf{Remark 3.4.} Theorem 3.1 does not hold for every measure. To see this let $X = L^1(\mathbb{R})$ and let $T(t)f(s) = f(s - t), f \in L^1(\mathbb{R})$. Then $T = \{T(t)\}_{t \in \mathbb{R}}$ is a $C_0$-group and $\hat{\mu}(T)f = \mu * f$ for every $\mu \in M(\mathbb{R})$. Hence, the algebra $\{\hat{\mu}(T) : \mu \in M(\mathbb{R})\}$ is
the multiplier algebra of $L^1(\mathbb{R})$. We know [6, p. 15] that the multiplier algebra of $L^1(\mathbb{R})$ is a full subalgebra of $B(L^1(\mathbb{R}))$. It follows that $\sigma(\hat{\mu}(T)) = \mu^\vee(\Delta_M(\mathbb{R}))$ (here $\mu^\vee$ is the Gelfand transform of $\mu \in M(\mathbb{R})$). Note also that $Af = f'$, where $f$ is a rapidly decreasing function on $\mathbb{R}$. It is easy to see that $\sigma(iA) = \mathbb{R}$. Since $\mathbb{R}$ is not dense in $\Delta_M(\mathbb{R})$ [11, p. 107], there exists a $\mu \in M(\mathbb{R})$ such that $\hat{\mu}(\mathbb{R}) = \{0\}$ and $\mu^\vee(\Delta_M(\mathbb{R})) \neq \{0\}$ (see also [2, Remark 1]).

References


Faculty of Arts and Sciences, Department of Mathematics, Yuzuncu Yil University, 65080, Van, Turkey

E-mail address: seferoglu2003@yahoo.com