PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 8, Pages 2457–2463
S 0002-9939(06)08210-4
Article electronically published on March 20, 2006

A SPECTRAL MAPPING THEOREM FOR REPRESENTATIONS OF ONE-PARAMETER GROUPS

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(Communicated by David R. Larson)

Abstract. In this paper we present some generalization (at the same time a new and a short proof in the Banach algebra context) of the Weak Spectral Mapping Theorem (WSMT) for non-quasianalytic representations of one-parameter groups.

1. Introduction

Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. A family $\mathbf{T} = \{T(t)\}_{t\in\mathbb{R}}$ in $B(X)$ is called a C_0-group if the following properties are satisfied:

(i) $T(0) = 1_X$, the identity operator on X;
(ii) $T(t+s) = T(t)T(s)$, for every $t, s \in \mathbb{R}$;
(iii) $\lim_{t\to 0} \|T(t)x - x\| = 0$, for all $x \in X$.

The generator of $\mathbf{T} = \{T(t)\}_{t\in\mathbb{R}}$ is the linear operator A defined by

$$Ax = \lim_{t\to 0} \frac{1}{t}(T(t)x - x),\ x \in D(A).$$

The generator is always a closed, densely defined operator.

For many applications it is a fundamental problem to decide whether the spectrum of each operator $T(t)$ can be obtained from the spectrum $\sigma(A)$ of A.

If A is bounded, by the Dunford functional calculus for all $t \in \mathbb{R}$ we have the spectral mapping theorem

$$\sigma(\exp(tA)) = \exp(t\sigma(A)).$$

In general, however, a generator A is unbounded and the spectral mapping theorem in the above form does not hold. One hopes that the Weak Spectral Mapping Theorem (WSMT) holds, i.e.,

$$\sigma(T(t)) = \overline{\exp(t\sigma(A))} \text{ for all } t \in \mathbb{R}.$$

For a systematic treatment (and references) of the WSMT, we refer to the monographs [3], [10]. From Hille-Phillips [4, p. 665] one already knows that the WSMT may fail.

Received by the editors February 3, 2005.
2000 *Mathematics Subject Classification.* Primary 22D15, 22D20, 46J05, 47A10.
Key words and phrases. Representation group, spectrum, Banach algebra, group algebra.

We recall that a measurable locally bounded function $\omega : \mathbb{R} \to \mathbb{R}$ is called a weight if $1 \leq \omega(t)$ and $\omega(t+s) \leq \omega(t)\omega(s)$ for all $t, s \in \mathbb{R}$. A weight function $\omega(t)$ is said to be non-quasianalytic if

$$\int_{\mathbb{R}} \frac{\ln \omega(t)}{1+t^2} dt < \infty.$$

In [9] it was shown that for the C_0-groups $\mathbf{T} = \{T(t)\}_{t\in\mathbb{R}}$ satisfying $\|T(t)\| \leq \omega(t)$ $(t \in \mathbb{R})$, the WSMT holds, where $\omega(t)$ is a non-quasianalytic weight function on $\mathbb{R}$. It is also known that the growth cannot be weakened in general.

Now let $\mathbf{T} = \{T(t)\}_{t\in\mathbb{R}}$ be a uniformly bounded C_0-group. If the generator A is bounded, we can deduce even more:

$$\sigma(\hat{\mu}(\mathbf{T})) = \hat{\mu}(\sigma(iA)),$$

where μ is a bounded regular complex Borel measure on $\mathbb{R}$, $\hat{\mu}$, the Fourier-Stieltjes transform of μ and $\hat{\mu}(\mathbf{T}) = \int_{\mathbb{R}} T(t)d\mu(t)$. To see this let B be the maximal uniformly closed commutative subalgebra of $B(X)$ containing A. Let M_B be the maximal ideal space of B. Denote by $\sigma_B(S)$ the spectrum of any $S \in B$ with respect to B. Since $\sigma_B(iA)$ is real, $\sigma_B(iA) = \sigma(iA)$. Also since B is a unital full subalgebra of $B(X)$ and $\hat{\mu}(\mathbf{T}) \in B$, we have

$$\sigma(\hat{\mu}(\mathbf{T})) = \sigma_B(\hat{\mu}(\mathbf{T})) = \{\phi(\hat{\mu}(\mathbf{T})) : \phi \in M_B\}$$
$$= \left\{ \int_{\mathbb{R}} \phi(\exp(tA))d\mu(t) : \phi \in M_B \right\} = \left\{ \int_{\mathbb{R}} \exp(t\phi(A))d\mu(t) : \phi \in M_B \right\}$$
$$= \hat{\mu}(\sigma_B(iA)) = \hat{\mu}(\sigma(iA)).$$

When the generator A is unbounded, this raises the natural question for which measures the equality

$$\sigma(\hat{\mu}(\mathbf{T})) = \overline{\hat{\mu}(\sigma(iA))} \tag{1.1}$$

holds. In general, this does not hold for every measure (see Remark 3.4).

In this paper we show that if $\mathbf{T} = \{T(t)\}_{t\in\mathbb{R}}$ is a C_0-group dominated by some non-quasianalytic weight ω on $\mathbb{R}$, then for the measures from the largest regular weight measure algebra $regM_\omega(\mathbb{R})$, the equality (1.1) holds. $regM_\omega(\mathbb{R})$ contains weight group algebras $L^1_\omega(\mathbb{R})$ and $M_\omega(\mathbb{R}_d)$. By taking in (1.1) the special case $\mu = \delta_t$, the Dirac measure concentrated in $t \in \mathbb{R}$, we obtain the WSMT. Hence, the equality (1.1) generalizes the WSMT.

2. Preliminaries

Later on we will need some preliminary results with which we now proceed. Let A be a complex commutative Banach algebra, and let Δ_A be the structure space of A. By $\widehat{a}$ we denote the Gelfand transform of the element $a \in A$. The hull of any ideal $I \subset A$ is $h(I) = \{\phi \in \Delta_A : \widehat{a}(\phi) = 0, a \in I\}$. A is regular if for each closed $S \subset \Delta_A$ and each $\phi \in \Delta_A \backslash S$ there exists $a \in A$ such that $\widehat{a}(\phi) = 1$ and $\widehat{a}(S) = \{0\}$. If A is regular and semisimple, then $I_S = \{a \in A : \widehat{a}(\phi) = 0, \phi \in S\}$ is the largest closed ideal in A whose hull is S. $J_S =$ closure $- \{a \in A : supp\, \widehat{a} \cap S = \emptyset$ and $supp$ $\widehat{a}$ is compact$\}$ is the smallest closed ideal in A whose hull is S.

Lemma 2.1. *Let A be a regular semisimple Banach algebra with identity 1_A and let $\psi : A \to B(X)$ be a continuous homomorphism such that $\psi(1_A) = 1_X$. Then*

$$\sigma(\psi(a)) = \widehat{a}(h(\ker\psi)) \text{ for each } a \in A.$$

Proof. Put $C = \overline{\psi(A)}$. Then C is a commutative Banach algebra with identity 1_X. We see that the mapping $\psi^* : \Delta_C \to h(\ker\psi)$ is injective. First we claim that ψ^* is surjective. Suppose on the contrary that there exists a $\phi_0 \in h(\ker\psi)$ but $\phi_0 \notin \psi^*\Delta_C$. Let U and V be a disjoint neighborhoods of ϕ_0 and $\psi^*\Delta_C$, respectively. Then there exist some elements $a, b \in A$, such that $\widehat{a}(\phi_0) = 1$, $\widehat{a}(\Delta_C \backslash U) = \{0\}$, $\widehat{b}(\psi^*\Delta_C) = \{1\}$ and $\widehat{b}(\Delta_C \backslash V) = \{0\}$. It is easy to see that $\widehat{a}(\phi)\widehat{b}(\phi) = 0$ for all $\phi \in M_A$. Since A is semisimple we have $ab = 0$ which implies $\psi(a)\psi(b) = 0$. Now since $\psi(b) \in C^{-1}$ we obtain $\psi(a) = 0$, and so $\widehat{a}(\phi_0) = 0$. This is a contradiction, proving our claim. Hence we have $\psi^*\Delta_C = h(\ker\psi)$, which implies

$$\sigma_C(\psi(a)) = \widehat{a}(h(\ker\psi)) \text{ for all } a \in A,$$

where $\sigma_C(\psi(a))$ is the spectrum of $\psi(a)$ with respect to the algebra C. It remains to show that C is a full subalgebra of $B(X)$. For this purpose let $c \in C$ be such that $c \in B(X)^{-1}$ and let D be the smallest closed subalgebra of $B(X)$ that contains c^{-1} and C. We can see that D is a commutative Banach algebra with the identity 1_X and C is a regular subalgebra of D. By the Shilov theorem [5, Corollary 9.5.5] any $\phi \in \Delta_C$ can be extended to some $\bar{\phi} \in \Delta_D$. Since $c \in D^{-1}$ we have $\phi(c) = \bar{\phi}(c) \neq 0$ for all $\phi \in \Delta_C$. This implies that $c \in C^{-1}$. The proof is complete. □

The Beurling algebra with weight ω on $\mathbb{R}$ is

$$L^1_\omega(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) : \|f\|_\omega = \int_{\mathbb{R}} |f(t)|\,\omega(t)dt < \infty \right\}.$$

$L^1_\omega(\mathbb{R})$ is a Banach algebra with respect to convolution. The following result [8, chap. 5, sect. 1] is well known.

Theorem 2.2. *If ω is a non-quasianalytic weight on $\mathbb{R}$, then:*

(i) *the algebra $L^1_\omega(\mathbb{R})$ has structure space $\mathbb{R}$ and is regular on $\mathbb{R}$;*
(ii) *the Gelfand transform of $f \in L^1_\omega(\mathbb{R})$ is just $\widehat{f}$, the Fourier transform of f;*
(iii) *$L^1_\omega(\mathbb{R})$ is Tauberian, i.e., the set of all $f \in L^1_\omega(\mathbb{R})$ whose Fourier transform has compact support is dense in $L^1_\omega(\mathbb{R})$.*

As is known [10, chap. 2], if $\mathbf{T} = \{T(t)\}_{t\in\mathbb{R}}$ is a C_0-group dominated by some a non-quasianalytic weight on $\mathbb{R}$, then $\sigma(iA)$ is located on the real line and every $\lambda \in \sigma(iA)$ is an approximate point spectrum for iA. Moreover,

$$R(z, A) = \begin{cases} \int_0^\infty \exp(-zt)T(t)dt, & Rez > 0, \\ -\int_{-\infty}^0 \exp(-zt)T(t)dt, & Rez < 0, \end{cases} \tag{2.1}$$

where $R(z, A)$ is the resolvent of A. Let

$$\widehat{f}(\lambda) = \int_{\mathbb{R}} f(t)\exp(-i\lambda t)dt$$

be the Fourier transform of $f \in L^1_\omega(\mathbb{R})$. For every $f \in L^1_\omega(\mathbb{R})$ we can define $\widehat{f}(\mathbf{T}) \in B(X)$ by

$$\widehat{f}(\mathbf{T}) = \int_{\mathbb{R}} f(t)T(t)dt.$$

Lemma 2.3. *Let ω be a non-quasianalytic weight on $\mathbb{R}$ and let $f \in L^1_\omega(\mathbb{R})$.*

a) *If $\widehat{f}(\boldsymbol{T}) = 0$, then $\widehat{f}(\lambda) = 0$ for every $\lambda \in \sigma(iA)$.*

b) *If $\widehat{f}$ is compactly supported and vanishes in a neighborhood of $\sigma(iA)$, then $\widehat{f}(\boldsymbol{T}) = 0$.*

Proof. a) If $\lambda \in \sigma(iA)$, then there is a sequence $(x_n)_{n\in\mathbb{N}}$ of norm one vectors in X, such that $\|Ax_n - (-i\lambda)x_n\| \to 0$ as $n \to \infty$. This implies [10, Proposition 2.1.6] that $\|T(t)x_n - \exp(-i\lambda t)x_n\| \to 0$. Consequently, we have

$$|\widehat{f}(\lambda)| = ||\widehat{f}(\mathbf{T})x_n - \widehat{f}(\lambda)x_n|| \leq \int_{\mathbb{R}} ||T(t)x_n - \exp(-i\lambda t)x_n|||f(t)|dt \to 0.$$

b) has been proved in [10, Lemma 2.4.3]. We include a proof for completeness.

Let $f \in L^1_\omega(\mathbb{R})$ be such that $supp\widehat{f}$ is compact and $\widehat{f}$ vanishes in a neighborhood of $\sigma(iA)$. Applying the formula for the inverse Fourier transform and the formula (2.1), we can write

$$\begin{aligned}\widehat{f}(\mathbf{T}) &= \lim_{\delta\to 0^+} \int_{\mathbb{R}} \exp(-\delta|t|)f(t)T(t)dt \\ &= \lim_{\delta\to 0^+} \int_{\mathbb{R}} \exp(-\delta|t|)\left(\int_{\mathbb{R}} \exp(i\lambda t)\widehat{f}(\lambda)d\lambda\right)T(t)dt \\ &= \lim_{\delta\to 0^+} \int_{\mathbb{R}} \widehat{f}(\lambda)(R(\delta - i\lambda, A) - R(-\delta - i\lambda, A))d\lambda = 0.\end{aligned}$$

□

3. The main result

Before stating the main result we need some notation. Let $M(\mathbb{R})$ be the convolution measure algebra of $\mathbb{R}$ and let ω be a weight function on $\mathbb{R}$. $M_\omega(\mathbb{R})$ is the set of all $\mu \in M(\mathbb{R})$ such that

$$\|\mu\|_\omega = \int_{\mathbb{R}} \omega(t)d\,|\mu|(t) < \infty.$$

With respect to convolution $M_\omega(\mathbb{R})$ is a commutative Banach algebra with the identity δ_0, where δ_0 is the Dirac measure concentrated in zero. As is known [7, Theorem 4.3.6], $M_\omega(\mathbb{R})$ contains a greatest regular subalgebra, which we will denote by $regM_\omega(\mathbb{R})$. If ω is a non-quasianalytic weight on $\mathbb{R}$, then $L^1_\omega(\mathbb{R})$ and $M_\omega(\mathbb{R}_d)$ $(= L^1_\omega(\mathbb{R}_d))$ are subalgebras of $regM_\omega(\mathbb{R})$ (here $\mathbb{R}_d$ denotes $\mathbb{R}$ with the discrete topology), and consequently, $L^1_\omega(\mathbb{R}) + M_\omega(\mathbb{R}_d) \subset regM_\omega(\mathbb{R})$. But $L^1_\omega(\mathbb{R}) + M_\omega(\mathbb{R}_d) \subset_{\neq} regM_\omega(\mathbb{R})$. To see this let τ be the locally compact group topology on $\mathbb{R}$ which is equal to or stronger than the usual topology on $\mathbb{R}$ and let $\mathbb{R}_\tau = (\mathbb{R}, \tau)$. Then the algebra $L^1_\omega(\mathbb{R}_\tau)$ also belongs to $regM_\omega(\mathbb{R})$.

Let $\mathbf{T} = \{T(t)\}_{t\in\mathbb{R}}$ be a C_0-group dominated by some weight ω on $\mathbb{R}$. Let

$$\hat{\mu}(\lambda) = \int\limits_{\mathbb{R}} \exp(-i\lambda t) d\mu(t)$$

be the Fourier-Stieltjes transform of $\mu \in M_\omega(\mathbb{R})$. For every $\mu \in M_\omega(\mathbb{R})$ we can define $\hat{\mu}(\mathbf{T}) \in B(X)$ by

$$\hat{\mu}(\mathbf{T}) = \int\limits_{\mathbb{R}} T(t) d\mu(t).$$

One easily checks that the map $\psi : reg(M_\omega(\mathbb{R})) \to B(X)$ defined by $\psi(\mu) = \hat{\mu}(\mathbf{T})$ is a continuous homomorphism.

The main result of this paper is the following theorem.

Theorem 3.1. *Let w be a non-quasianalytic weight on $\mathbb{R}$ and let $\boldsymbol{T}=\{T(t)\}_{t\in\mathbb{R}}$ be a C_0-group on a Banach space X with generator A, such that*

$$\|T(t)\| \le \omega(t), \quad t \in \mathbb{R}.$$

Then

$$\sigma(\hat{\mu}(\boldsymbol{T})) = \overline{\hat{\mu}(\sigma(iA))} \text{ for every } \mu \in regM_\omega(\mathbb{R}). \tag{3.1}$$

By taking in the equality (3.1) $\mu = \delta_t$, the Dirac measure concentrated in $t \in \mathbb{R}$, Theorem 3.1 gives the WSMT.

We also need the following notation. For brevity, the structure space of $regM_\omega(\mathbb{R})$ will be denoted by Δ_{reg}. Also the symbol $\mu^\vee$ will be used to denote the Gelfand transform of $\mu \in regM_\omega(\mathbb{R})$. The real line $\mathbb{R}$ (and consequently, $\sigma(iA)$) can be considered as a subset of the structure space of $regM_\omega(\mathbb{R})$, and the restriction of the Gelfand transform of $\mu \in regM_\omega(\mathbb{R})$ to $\mathbb{R}$ coincides with the Fourier-Stieltjes transform of $\mu \in M_\omega(\mathbb{R})$. Thus $regM_\omega(\mathbb{R})$ is a regular semisimple Banach algebra with the identity δ_0. $\overline{\sigma(iA)}^{\Delta_{reg}}$ will be denoted the closure of $\sigma(iA))$ in the usual topology of Δ_{reg}.

Another result we need for the proof of the main result is the following lemma.

Lemma 3.2. *Under the above notation, $h(\ker\psi) = \overline{\sigma(iA)}^{\Delta_{reg}}$.*

Proof. First we show that $\overline{\sigma(iA)}^{\Delta_{reg}} \subset h(\ker\psi)$. Since $h(\ker\psi)$ is closed, it is sufficient to show that $\sigma(iA) \subset h(\ker\psi)$. Suppose on the contrary that there exists a $\lambda_0 \in \sigma(iA)$ but $\lambda_0 \notin h(\ker\psi)$. Then there exists a $\mu \in regM_\omega(\mathbb{R})$ such that $\hat{\mu}(\lambda_0) \neq 0$ and $\mu^\vee$ vanishes in some neighborhood of $h(\ker\psi)$. Hence μ belongs to the smallest ideal in $regM_\omega(\mathbb{R})$, whose hull is $h(\ker\psi)$. Consequently, $\mu \in \ker(\psi)$ and so $\hat{\mu}(\mathbf{T})$=0. It follows that $(\mu * f)\hat{\ }(\mathbf{T}) = \hat{\mu}(\mathbf{T})\hat{f}(\mathbf{T}) = 0$ for every $f \in L^1_\omega(\mathbb{R})$. Since $\mu * f \in L^1_\omega(\mathbb{R})$, by Lemma 2.3a) we have $0 = (\mu * f)\hat{\ }(\lambda) = \hat{\mu}(\lambda)\hat{f}(\lambda)$ for all $\lambda \in \sigma(iA)$ and $f \in L^1_\omega(\mathbb{R})$. Also since $\lambda_0 \in \sigma(iA)$, this clearly implies that $\hat{\mu}(\lambda_0) = 0$. This is a contradiction.

For the reverse inclusion assume that there exists a $\phi_0 \in h(\ker\psi)$ but $\phi_0 \notin \overline{\sigma(iA)}^{\Delta_{reg}}$. Then there exists a neighborhood (in Δ_{reg}) V of $\sigma(iA)$ and a $\mu \in regM_\omega(\mathbb{R})$ such that $\mu^\vee(\phi_0) \neq 0$ and $\mu^\vee$ vanishes on V. It follows that $\hat{\mu}$ vanishes on $V \cap \mathbb{R}$. We see that $V \cap \mathbb{R}$ is a neighborhood of $\sigma(iA)$ for the relative topology induced in $\mathbb{R}$ by Δ_{reg}. Let us show that the usual topology of $\mathbb{R}$ is a base for the relative topology induced in $\mathbb{R}$ by Δ_{reg}. For this fix $\lambda_0 \in \mathbb{R}$ and $\{\mu_1, ..., \mu_n\} \subset regM_\omega(\mathbb{R})$. It suffices to show that

$$U = \left\{ \lambda \in \mathbb{R} : \sup_{\lambda \in K} |\exp(i\lambda t) - \exp(i\lambda_0 t)| < \delta \right\}$$
$$\subset \{\lambda \in \mathbb{R} : |\hat{\mu}_i(\lambda) - \hat{\mu}_i(\lambda_0)| < \varepsilon, i = 1, ..., n\},$$

for some compact $K \subset \mathbb{R}$ and $\delta > 0$. Choose a compact set K in $\mathbb{R}$, so that $|\mu_i|(\mathbb{R}\backslash K) < \varepsilon/4$ and $0 < \delta < \varepsilon/2 \max_i |\mu_i|(\mathbb{R})$, $i = 1, ..., n$. If $\lambda \in U$, then

$$|\hat{\mu}_i(\lambda) - \hat{\mu}_i(\lambda_0)| \leq \int_K |\exp(i\lambda t) - \exp(i\lambda_0 t)| d|\mu_i| + \int_{\mathbb{R}\backslash K} |\exp(i\lambda t) - \exp(i\lambda_0 t)| d|\mu_i|$$
$$\leq \sup_{\lambda \in K} |\exp(i\lambda t) - \exp(i\lambda_0 t)| (\max_i |\mu_i|(\mathbb{R})) + 2|\mu_i|(\mathbb{R}\backslash K) < \varepsilon, \quad i = 1, ..., n.$$

Hence $V \cap \mathbb{R}$ is an open set in the usual topology of $\mathbb{R}$. Thus we obtain that $\hat{\mu}$ vanishes in a (real) neighborhood of $\sigma(iA)$. Let $f \in L^1_\omega(\mathbb{R})$ be such that $supp\hat{f}$ is compact. Then $(\mu * f)\hat{}$ vanishes in a neighborhood of $\sigma(iA)$ and $supp(\mu * f)\hat{}$ is compact. By Lemma 2.3b) we have $\hat{\mu}(\mathbf{T})\hat{f}(\mathbf{T})$=0. On the other hand, by Theorem 2.2iii) the set of all $f \in L^1_\omega(\mathbb{R})$ whose Fourier transform has compact support is dense in $L^1_\omega(\mathbb{R})$. It follows that $\hat{\mu}(\mathbf{T})\hat{f}(\mathbf{T})$=0 for all $f \in L^1_\omega(\mathbb{R})$. Let $f_n = 2n\chi_{[-1/n,1/n]}$, where $\chi_{[-1/n,1/n]}$ is the characteristic function of $[-1/n, 1/n]$. It is easy to see that $\hat{f}_n(\mathbf{T}) \to 1_X$ strongly. Now from the identity $\hat{\mu}(\mathbf{T})\hat{f}_n(\mathbf{T})$=0 we deduce that $\hat{\mu}(\mathbf{T})$=0. Hence, we have $\mu \in \ker(\psi)$. Since $\phi_0 \in h(\ker \psi)$ we obtain $\mu^\vee(\phi_0) = 0$. This is a contradiction. The proof is complete. □

Proof of Theorem 3.1. As we already noted, the map $\psi : reg(M_\omega(\mathbb{R})) \to B(X)$ defined by $\psi(\mu) = \hat{\mu}(\mathbf{T})$ is a continuous homomorphism. Note also that $\psi(\delta_0) = 1_X$. Let $\mu \in regM_\omega(\mathbb{R})$ be given. Then by Lemma 2.1 we have $\sigma(\hat{\mu}(\mathbf{T}))= \mu^\vee(h(\ker \psi))$. By Lemma 3.2 since $h(\ker \psi) = \overline{\sigma(iA)}^{\Delta_{reg}}$ we obtain $\sigma(\hat{\mu}(\mathbf{T}))=\mu^\vee(\overline{\sigma(iA)}^{\Delta_{reg}})$. Also since $\hat{\mu}(\sigma(iA)) = \mu^\vee(\sigma(iA)) \subset \mu^\vee(\overline{\sigma(iA)}^{\Delta_{reg}})$ and $\mu^\vee(\overline{\sigma(iA)}^{\Delta_{reg}})$ is closed, we have $\overline{\hat{\mu}(\sigma(iA))} \subset \mu^\vee(\overline{\sigma(iA)}^{\Delta_{reg}})$. On the other hand, from the continuity $\mu^\vee$ on Δ_{reg} we deduce that $\mu^\vee(\overline{\sigma(iA)}^{\Delta_{reg}}) \subset \overline{\mu^\vee(\sigma(iA))} = \overline{\hat{\mu}(\sigma(iA))}$. Thus we obtain that $\sigma(\hat{\mu}(\mathbf{T}))= \overline{\hat{\mu}(\sigma(iA))}$. The proof is complete. □

Let us record a consequence of this theorem.

If $T = \{T(t)\}_{t \in \mathbb{R}}$ is a C_0-group on a Banach space X with generator A and E is a $\mathbf{T}$-invariant closed subspace of X, then $T(t)|_E$, the restriction of the operators $T(t)$ to E, define a C_0-group $\mathbf{T}_E$ on E. Its generator A_E is precisely the part of A in E; i.e., $D(A_E) = \{x \in D(A) \cap E : Ax \in E\}$ and $A_E x = Ax, x \in D(A_E)$.

Corollary 3.3. *Let w be a non-quasianalytic weight on $\mathbb{R}$ and let $\boldsymbol{T}$=$\{T(t)\}_{t \in \mathbb{R}}$ be a C_0-group on a Banach space X with generator A, such that*

$$\|T(t)\| \leq \omega(t), \quad t \in \mathbb{R}.$$

If E is a $\mathbf{T}$-invariant closed subspace of X, then

$$\sigma(\hat{\mu}(\boldsymbol{T})|_E) = \overline{\hat{\mu}(\sigma(iA_E))} \textit{ for every } \mu \in regM_\omega(\mathbb{R}). \tag{3.2}$$

Remark 3.4. Theorem 3.1 does not hold for every measure. To see this let $X = L^1(\mathbb{R})$ and let $T(t)f(s) = f(s-t)$, $f \in L^1(\mathbb{R})$. Then $\mathbf{T}$=$\{T(t)\}_{t \in \mathbb{R}}$ is a C_0-group and $\hat{\mu}(\mathbf{T})f = \mu * f$ for every $\mu \in M(\mathbb{R})$. Hence, the algebra $\{\hat{\mu}(\mathbf{T})$: $\mu \in M(\mathbb{R})\}$ is the multiplier algebra of $L^1(\mathbb{R})$. We know [6, p. 15] that the multiplier algebra of $L^1(\mathbb{R})$ is a full subalgebra of $B(L^1(\mathbb{R}))$. It follows that $\sigma(\hat{\mu}(\mathbf{T}))=\mu^\vee(\Delta_{M(\mathbb{R})})$ (here $\mu^\vee$ is the Gelfand transform of $\mu \in M(\mathbb{R})$). Note also that $Af = f'$, where f is a rapidly decreasing function on $\mathbb{R}$. It is easy to see that $\sigma(iA) = \mathbb{R}$. Since $\mathbb{R}$ is not dense in $\Delta_{M(\mathbb{R})}$ [11, p. 107], there exists a $\mu \in M(\mathbb{R})$ such that $\hat{\mu}(\mathbb{R}) = \{0\}$ and $\mu^\vee(\Delta_{M(\mathbb{R})}) \neq \{0\}$ (see also [2, Remark 1]).

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