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A SHORT PROOF OF THE ZEILBERGER-BRESSOUD q-DYSON THEOREM

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ABSTRACT. We give a formal Laurent series proof of Andrews's q-Dyson Conjecture, first proved by Zeilberger and Bressoud.

1. INTRODUCTION

Freeman Dyson, motivated by a problem in particle physics, conjectured the following identity in 1962:

Theorem 1.1 (Dyson's Conjecture). *For nonnegative integers* $a_0, a_1, \ldots, a_n$,

$$\operatorname*{CT}_{\mathbf{x}} \prod_{0 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^{a_j} = \frac{(a_0 + a_1 + \cdots + a_n)!}{a_0!\, a_1! \cdots a_n!}, \tag{1.1}$$

where $\mathrm{CT}_{\mathbf{x}}$ *denotes the constant term in* $x_0, \ldots, x_n$.

Dyson's conjecture was quickly proved by Wilson [12] and independently by Gunson [6]. An elegant recursive proof was published by Good [5] in 1970.

A q-analog of Theorem 1.1 was conjectured by George Andrews [1] in 1975:

Theorem 1.2 (Zeilberger-Bressoud). *For any nonnegative integers* $a_0, a_1, \ldots, a_n$,

$$\operatorname*{CT}_{\mathbf{x}} \prod_{0 \le i < j \le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i} q\right)_{a_j} = \frac{(q)_{a_0 + \cdots + a_n}}{(q)_{a_0} \cdots (q)_{a_n}}, \tag{1.2}$$

where $(z)_m = (1 - z)(1 - qz) \cdots (1 - q^{m-1} z)$.

Andrews's q-Dyson conjecture attracted much interest, but was not proved until 1985, in a combinatorial *tour de force* by Zeilberger and Bressoud [16]. In related work, Stanley [9, 10] reformulated the conjecture in terms of symmetric functions and proved a limiting form of the conjecture, and Kadell [7] proved the 4-variable case using an approach similar to Good's. Bressoud and Goulden [3] extended the Zeilberger-Bressoud approach to some generalizations of (1.2), and Stembridge [11] gave an elegant recursive proof of the equal parameter case. Cherednik [4] proved the Macdonald constant term conjecture for root systems [8], which generalizes the equal parameter case.

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Zeilberger and Bressoud's combinatorial proof is, so far, the only proof of Andrews's q-Dyson conjecture. Here we give a very different and shorter proof, using properties of the formal Laurent series.

The idea behind the proof is the well-known fact that to prove the equality of two polynomials of degree at most d, it is sufficient to prove that they are equal at $d+1$ points. As is often the case, points at which the polynomials vanish are most easily dealt with.

It is not difficult to show that for fixed nonnegative integers $a_1, \dots, a_n$, both sides of (1.2) are polynomials in q^{a_0} of degree at most $a_1+\cdots+a_n$ and that the polynomial corresponding to the right side of (1.2) vanishes for $a_0 = -1, -2, \dots, -(a_1+\cdots+a_n)$. The main part of our proof is to show that the polynomial corresponding to the left-hand side of (1.2) also vanishes at these points. We do this by expanding the left-hand side in partial fractions in such a way that we can show that each summand has zero constant term. The proof is completed by observing that the case $a_0 = 0$ of (1.2) is equivalent to the n-variable case.

2. Basic facts

We use the following standard notation:

$$(z)_n = \frac{(z)_\infty}{(zq^n)_\infty} = \prod_{m=0}^{\infty} \frac{(1-zq^m)}{(1-zq^{m+n})}. \tag{2.1}$$

Note that if p is a nonnegative integer, then

$$(z)_p = (1-z)(1-zq)\cdots(1-zq^{p-1}),$$
$$(z)_{-p} = \frac{1}{(1-zq^{-1})(1-zq^{-2})\cdots(1-zq^{-p})}.$$

The q-binomial coefficients are defined for all integers n and nonnegative integers m by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q^{n-m+1})_m}{(q)_m} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q)(1-q^2)\cdots(1-q^m)}. \tag{2.2}$$

For $0 \le m \le n$, we have

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q)_n}{(q)_m (q)_{n-m}}. \tag{2.3}$$

The well-known q-binomial theorem [2, Theorem 2.1] is the identity

$$\frac{(az)_\infty}{(z)_\infty} = \sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} z^k. \tag{2.4}$$

Setting $z = uq^n$ and $a = q^{-n}$ in (2.4), and using the definition (2.1), we obtain

$$(u)_n = \sum_{k=0}^{\infty} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} (-u)^k \tag{2.5}$$

for all integers n.

We will also need the easily-proved identity

$$\left(\frac{x_i}{x_j}\right)_l \left(\frac{x_j}{x_i} q\right)_m = q^{\binom{m+1}{2}} \left(-\frac{x_j}{x_i}\right)^m \left(\frac{x_i}{x_j} q^{-m}\right)_{l+m}. \tag{2.6}$$

3. The proof

Let us fix $\mathbf{a} = (a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are nonnegative integers. Let $a = a_1 + \cdots + a_n$ and let

$$P_{\mathbf{a}}(q^b) = \frac{(q)_a}{(q)_{a_1} \cdots (q)_{a_n}} \begin{bmatrix} b+a \\ a \end{bmatrix} = \frac{(1-q^{b+a})(1-q^{b+a-1}) \cdots (1-q^{b+1})}{(q)_{a_1} \cdots (q)_{a_n}}. \tag{3.1}$$

Then the right-hand side of (1.2) is equal to $P_{\mathbf{a}}(q^{a_0})$. We observe that

(i) $P_{\mathbf{a}}(q^b)$ is a polynomial in q^b of degree at most a.
(ii) $P_{\mathbf{a}}(q^b) = 0$ for $q^b = q^{-1}, q^{-2}, \ldots, q^{-a}$.

Moreover, $P_{\mathbf{a}}(q^b)$ is uniquely determined by these two properties up to a constant factor (which may depend on q but not on b).

Let $Q_{\mathbf{a}}(q^b)$ be defined by

$$Q_{\mathbf{a}}(q^b) = \operatorname*{CT}_{\mathbf{x}} \prod_{j=1}^{n} \left(\frac{x_0}{x_j}\right)_b \left(\frac{x_j}{x_0}q\right)_{a_j} \prod_{1 \le i < j \le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i}q\right)_{a_j}. \tag{3.2}$$

Then the left-hand side of (1.2) equals $Q_{\mathbf{a}}(q^{a_0})$.

In fact $Q_{\mathbf{a}}(q^b)$ is well defined for negative integers b if we treat the rational function in (3.2) as a Laurent series in x_0. The following two lemmas show that $Q_{\mathbf{a}}(q^b)$ equals $P_{\mathbf{a}}(q^b)$ up to a constant multiple.

Lemma 3.1. *For fixed $\mathbf{a} \in \mathbb{N}^n$, $Q_{\mathbf{a}}(q^b)$ is a polynomial in q^b of degree at most a.*

Lemma 3.2 (Main Lemma)**.** *For any $\mathbf{a} \in \mathbb{N}^n$, $Q_{\mathbf{a}}(q^b) = 0$ for $b = -1, -2, \ldots, -a$.*

Lemma 3.2 is the heart of our proof of Theorem 1.2. We will prove it in Sections 4 and 5. Here we give the rest of the proof of Theorem 1.2.

Proof of Lemma 3.1. By (2.6) we have

$$\left(\frac{x_0}{x_j}\right)_b \left(\frac{x_j}{x_0}q\right)_{a_j} = q^{\binom{a_j+1}{2}} \left(-\frac{x_j}{x_0}\right)^{a_j} \left(\frac{x_0}{x_j}q^{-a_j}\right)_{b+a_j},$$

for all integers b, where both sides are regarded as Laurent series in x_0.

Thus $Q_{\mathbf{a}}(q^b)$ can be rewritten as

$$\operatorname*{CT}_{\mathbf{x}} L(x_1, \ldots, x_n, \mathbf{a}) \prod_{j=1}^{n} q^{\binom{a_j+1}{2}} \left(-\frac{x_j}{x_0}\right)^{a_j} \left(\frac{x_0}{x_j}q^{-a_j}\right)_{b+a_j}, \tag{3.3}$$

where $L(x_1, \ldots, x_n, \mathbf{a})$ is a Laurent polynomial in $x_1, \ldots, x_n$ independent of x_0 and b.

Using the q-binomial theorem (2.5), we see that for $1 \le j \le n$,

$$q^{\binom{a_j+1}{2}} \left(-\frac{x_j}{x_0}\right)^{a_j} \left(\frac{x_0}{x_j}q^{-a_j}\right)_{b+a_j} = \sum_{k_j \ge 0} C(k_j) \begin{bmatrix} b+a_j \\ k_j \end{bmatrix} x_0^{k_j - a_j} x_j^{a_j - k_j},$$

where $C(k_j) = (-1)^{k_j} q^{\binom{a_j+1}{2} + \binom{k_j}{2} - k_j a_j}$.

Expanding the product in (3.3) and taking the constant term in x_0, we get

$$Q_{\mathbf{a}}(q^b) = \sum_{\mathbf{k}} \begin{bmatrix} b+a_1 \\ k_1 \end{bmatrix} \begin{bmatrix} b+a_2 \\ k_2 \end{bmatrix} \cdots \begin{bmatrix} b+a_n \\ k_n \end{bmatrix} \operatorname*{CT}_{x_1, \ldots, x_n} L'(x_1, \ldots, x_n, \mathbf{a}, \mathbf{k}), \tag{3.4}$$

where the sum ranges over all sequences $\mathbf{k} = (k_1, \ldots, k_n)$ of nonnegative integers such that $k_1 + k_2 + \cdots + k_n = a_1 + a_2 + \cdots + a_n$, and $L'(x_1, \ldots, x_n, \mathbf{a}, \mathbf{k})$ is a Laurent polynomial in $x_1, \dots, x_n$ independent of b. Since $\begin{bmatrix} b+a_i \\ k_i \end{bmatrix}$ is a polynomial in q^b of degree k_i, each summand in (3.4) is a polynomial in q^b of degree $k_1 + k_2 + \cdots + k_n = a_1 + a_2 + \cdots + a_n$, and so is the sum. □

Proof of Theorem 1.2. We proceed by induction on n. Theorem 1.2 is trivial for $n = 0$ and reduces to the q-binomial theorem for $n = 1$. Suppose the theorem is true for n variables. We may call these variables $x_1, \dots, x_n$ rather than $x_0, \dots x_{n-1}$, so our induction hypothesis implies that (1.2) holds when $a_0 = 0$.

We will show that $P_{\mathbf{a}}(q^{a_0}) = Q_{\mathbf{a}}(q^{a_0})$ for all nonnegative integers a_0. We know that (i) $P_{\mathbf{a}}(q^0) = Q_{\mathbf{a}}(q^0)$ by the induction hypothesis; (ii) by Lemma 3.1, both $P_{\mathbf{a}}(q^b)$ and $Q_{\mathbf{a}}(q^b)$ define polynomials in q^b of degree no greater than a; (iii) by Lemma 3.2, $P_{\mathbf{a}}(q^b) = Q_{\mathbf{a}}(q^b) = 0$ for $q^b = q^{-1}, q^{-2}, \dots, q^{-a}$. So $P_{\mathbf{a}}(q^b)$ and $Q_{\mathbf{a}}(q^b)$ are equal as polynomials in q^b. □

4. Constant term evaluations

We will evaluate the constant term $Q_{\mathbf{a}}(q^b)$ defined by (3.2), where b is a negative integer, by partial fraction expansion. Although we are taking the constant term of a Laurent series in x_0 with coefficients that are Laurent polynomials in $x_1, \dots, x_n$, when we expand by partial fractions, we get terms that are not of this form, and in order to evaluate their constant terms we need to work in a larger ring: the field of iterated Laurent series $K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle = K((x_n))((x_{n-1}))\dots((x_0))$, where $K = \mathbb{C}(q)$, in which all series are regarded first as Laurent series in x_0, then as Laurent series in x_1, and so on. For a more detailed account of the properties of this field, with other applications, see [13] and [14].

Every element of $K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle$ has a unique Laurent series expansion. The series expansions of $1/(1 - q^k x_i/x_j)$ will be especially important. If $i < j$, then

$$\frac{1}{1 - q^k x_i/x_j} = \sum_{l=0}^{\infty} q^{kl} x_i^l x_j^{-l}.$$

However, if $i > j$, then this expansion is not valid and instead we have the expansion

$$\frac{1}{1 - q^k x_i/x_j} = \frac{1}{-q^k x_i/x_j(1 - q^{-k} x_j/x_i)} = \sum_{l=0}^{\infty} -q^{-k(l+1)} x_i^{-l-1} x_j^{l+1}.$$

Let $F(\mathbf{x})$ be in $K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle$. The *constant term* of $F(\mathbf{x})$ in x_i, denoted by $\mathrm{CT}_{x_i} F(\mathbf{x})$, is defined to be the sum of those terms in the series expansion of $F(\mathbf{x})$ that are free of x_i. This definition clearly extends the constant term operators used earlier. It follows that

$$\operatorname*{CT}_{x_i} \frac{1}{1 - q^k x_i/x_j} = \begin{cases} 1, & \text{if } i < j, \\ 0, & \text{if } i > j. \end{cases} \tag{4.1}$$

We shall call the monomial $M = q^k x_i/x_j$ *small* if $i < j$ and *large* if $i > j$. Thus the constant term in x_i of $1/(1 - M)$ is 1 if M is small and 0 if M is large.

An important property of the constant term operators defined in this way is their commutativity:

$$\operatorname*{CT}_{x_i} \operatorname*{CT}_{x_j} F(\mathbf{x}) = \operatorname*{CT}_{x_j} \operatorname*{CT}_{x_i} F(\mathbf{x}).$$

Commutativity implies that the constant term in a set of variables is well defined, and this property will be used in our proof of the Main Lemma. (Note that, by contrast, the constant term operators in [15] do not commute.)

The *degree* of a rational function of x is the degree in x of the numerator minus the degree in x of the denominator. For example, if $i \neq j$, then the degree of $1 - x_j/x_i = (x_i - x_j)/x_i$ is 0 in x_i and 1 in x_j. A rational function is called *proper* in x if its degree in x is negative. The following lemma gives a formula for the constant term in x_k of certain elements of $K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle$ which are proper rational functions of x_k.

Lemma 4.1. *Let*

$$R = \frac{p(x_k)}{x_k^d \prod_{i=1}^m (1 - x_k/\alpha_i)}$$

be a proper rational function of x_k, where $p(x_k)$ is a polynomial in x_k, and the α_i are distinct monomials, each of the form $x_t q^s$. Then

$$\operatorname*{CT}_{x_k} R = \sum_j \big(R\,(1 - x_k/\alpha_j)\big)\Big|_{x_k=\alpha_j}, \tag{4.2}$$

where the sum ranges over all j such that x_k/α_j is small.

Proof. The field $K\langle\langle x_n, \dots, x_0\rangle\rangle$ contains the polynomial ring $K[x_0, \dots, x_n]$ as a subring and hence contains the field $K(x_0, \dots, x_n)$ of rational functions as a subfield. Thus any identity in $K(x_0, \dots, x_n)$ is also an identity in $K\langle\langle x_n, \dots, x_0\rangle\rangle$.

The partial fraction decomposition of R with respect to x_k is

$$\frac{p(x_k)}{x_k^d \prod_{i=1}^m (1 - x_k/\alpha_i)} = \frac{p_0(x_k)}{x_k^d} + \sum_{j=1}^m \frac{1}{1 - x_k/\alpha_j} \left(\frac{p(x_k)}{x_k^d \prod_{i=1, i\neq j}^m (1 - x_k/\alpha_i)} \right)\Bigg|_{x_k=\alpha_j},$$

where $p_0(x_k)$ is a polynomial in x_k of degree less than d. The term $p_0(x_k)/x_k^d$ contributes nothing to the constant term in x_k, and $1/(1 - x_k/\alpha_j)$ contributes to the constant term in x_k only if x_k/α_j is small. The result of the lemma then follows easily. □

The following lemma plays an important role in our argument.

Lemma 4.2. *Let $A_1, \dots, A_s$ be nonnegative integers. Then for any positive integers $k_1, \dots, k_s$ with $1 \le k_i \le A_1 + \cdots + A_s$ for all i, either $1 \le k_i \le A_i$ for some i or $-A_j \le k_i - k_j \le A_i - 1$ for some $i < j$.*

Proof. We prove by contradiction that there is no $k_1, \dots, k_s$ such that for all i, $A_i < k_i \le A_1 + \cdots + A_s$, and for all $i < j$, either $k_i - k_j \ge A_i$ or $k_i - k_j \le -A_j - 1$. Suppose $k_1, \dots, k_s$ do satisfy these conditions. We construct a tournament on $1, 2, \dots, s$ with numbers on the arcs as follows: For $i < j$, if $k_i - k_j \ge A_i$, then we draw an arc $i \xleftarrow{A_i} j$ from j to i and if $k_i - k_j \le -1 - A_j$, then we draw an arc $i \xrightarrow{A_j+1} j$ from i to j.

We call an arc from u to v an *ascending arc* if $u < v$ and a *descending arc* if $u > v$. We note two facts: (i) the number on an arc from u to v is less than or equal to $k_v - k_u$, and (ii) the number on an ascending arc is always positive.

A consequence of (i) is that for any directed path from e to f, the sum along the arcs is less than or equal to $k_f - k_e$. It follows that the sum along a cycle is nonpositive. But any cycle must have at least one ascending arc, and by (ii) the number on this arc is positive, and so the sum along the cycle is positive. Thus there can be no cycles.

Therefore the tournament we have constructed is transitive, and hence defines a total ordering $\to$ on $1, 2, \ldots, s$. Assume the total ordering is given by $i_1 \to i_2 \to \cdots \to i_{s-1} \to i_s$. Then $k_{i_s} - k_{i_1} \geq A_{i_2} + A_{i_3} + \cdots + A_{i_s}$. This implies that

$$\begin{aligned} k_{i_s} &\geq k_{i_1} + A_{i_2} + A_{i_3} + \cdots + A_{i_s} \\ &> A_{i_1} + A_{i_2} + A_{i_3} + \cdots + A_{i_s} \\ &= A_1 + A_2 + \cdots + A_s, \end{aligned}$$

a contradiction. □

5. Proof of the Main Lemma

Let

$$\mathcal{Q}(b) = \prod_{j=1}^{n} \frac{(x_j q/x_0)_{a_j}}{(1 - x_0/x_j q)(1 - x_0/x_j q^2) \cdots (1 - x_0/x_j q^b)} \prod_{1 \leq i < j \leq n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i} q\right)_{a_j},$$

so that in the notation of the previous section, $\mathrm{CT}_{\mathbf{x}}\ \mathcal{Q}(b) = Q_{\mathbf{a}}(q^{-b})$. Our goal is to show that $\mathrm{CT}_{\mathbf{x}}\, \mathcal{Q}(b) = 0$ for $b = 1, 2, \ldots, a$.

Since the degree in x_0 of $1 - x_j q^i/x_0 = (x_0 - x_j q^i)/x_0$ is 0, $\mathcal{Q}(b)$ is proper in x_0, with degree $-nb$.

Applying Lemma 4.1, we have

$$\underset{x_0}{\mathrm{CT}}\, \mathcal{Q}(b) = \sum_{\substack{0 < r_1 \leq n \\ 1 \leq k_1 \leq b}} \mathcal{Q}(b \,|\, r_1; k_1), \tag{5.1}$$

where

$$\mathcal{Q}(b \,|\, r_1; k_1) = \mathcal{Q}(b) \left(1 - \frac{x_0}{x_{r_1} q^{k_1}}\right)\Bigg|_{x_0 = x_{r_1} q^{k_1}}.$$

For each term in (5.1) we will extract the constant term in x_{r_1}, and then perform further constant term extractions, eliminating one variable at each step. In order to keep track of the terms we obtain, we introduce some notation.

For any rational function F of $x_0, x_1, \ldots, x_n$, and for sequences of integers $\mathbf{k} = (k_1, k_2, \ldots, k_s)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_s)$, let $E_{\mathbf{r},\mathbf{k}} F$ be the result of replacing x_{r_i} in F with $x_{r_s} q^{k_s - k_i}$ for $i = 0, 1, \ldots, s-1$, where we set $r_0 = k_0 = 0$. Then for $0 < r_1 < r_2 < \cdots < r_s \leq n$ and $0 < k_i \leq b$, we define

$$\mathcal{Q}(b \,|\, \mathbf{r}; \mathbf{k}) = \mathcal{Q}(b \,|\, r_1, \ldots, r_s; k_1, \ldots, k_s) = E_{\mathbf{r},\mathbf{k}} \left[\mathcal{Q}(b) \prod_{i=1}^{s} \left(1 - \frac{x_0}{x_{r_i} q^{k_i}}\right) \right]. \tag{5.2}$$

Note that the product on the right-hand side of (5.2) cancels all the factors in the denominator of $\mathcal{Q}$ that would be taken to zero by $E_{\mathbf{r},\mathbf{k}}$.

Lemma 5.1. *The rational functions $\mathcal{Q}(b \,|\, \mathbf{r}; \mathbf{k})$ have the following two properties:*

(i) *If $1 \leq k_i \leq a_{r_1} + \cdots + a_{r_s}$ for all i with $1 \leq i \leq s$, then $\mathcal{Q}(b \,|\, \mathbf{r}; \mathbf{k}) = 0$.*

(ii) *If $k_i > a_{r_1} + \cdots + a_{r_s}$ for some i with $1 \leq i \leq s$ and $n > s$, then*

$$\underset{x_s}{\mathrm{CT}}\, \mathcal{Q}(b \,|\, \mathbf{r}; \mathbf{k}) = \sum_{\substack{r_s < r_{s+1} \leq n \\ 1 \leq k_{s+1} \leq b}} \mathcal{Q}(b \,|\, r_1, \ldots, r_s, r_{s+1}; k_1, \ldots, k_s, k_{s+1}). \tag{5.3}$$

Proof of property (i). By Lemma 4.2, either $1 \le k_i \le a_{r_i}$ for some i with $1 \le i \le s$, or $-a_{r_j} \le k_i - k_j \le a_{r_i} - 1$ for some $i < j$. If $1 \le k_i \le a_{r_i}$, then $\mathcal{Q}(b \mid \mathbf{r}; \mathbf{k})$ has the factor

$$E_{\mathbf{r},\mathbf{k}}\left[\left(\frac{x_{r_i}}{x_0}q\right)_{a_{r_i}}\right] = \left(\frac{x_{r_s}q^{k_s-k_i}}{x_{r_s}q^{k_s}}q\right)_{a_{r_i}} = (q^{1-k_i})_{a_{r_i}} = 0.$$

If $-a_{r_j} \le k_i - k_j \le a_{r_i} - 1$ where $i < j$, then $\mathcal{Q}(b \mid \mathbf{r}; \mathbf{k})$ has the factor

$$E_{\mathbf{r},\mathbf{k}}\left[\left(\frac{x_{r_i}}{x_{r_j}}\right)_{a_{r_i}}\left(\frac{x_{r_j}}{x_{r_i}}q\right)_{a_{r_j}}\right],$$

which by (2.6) is equal to

$$\begin{aligned} &q^{\binom{a_{r_j}+1}{2}}\left(-\frac{x_{r_j}}{x_{r_i}}\right)^{a_{r_j}}\left(\frac{x_{r_i}}{x_{r_j}}q^{-a_{r_j}}\right)_{a_{r_i}+a_{r_j}} \\ &\quad = q^{\binom{a_{r_j}+1}{2}}(-q^{k_i-k_j})^{a_{r_j}}(q^{k_j-k_i-a_{r_j}})_{a_{r_i}+a_{r_j}} = 0. \end{aligned}$$

□

Proof of property (ii). Note that since $b \ge k_i$ for all i, the hypothesis implies that $b > a_{r_1} + \cdots + a_{r_s}$.

We first show that $\mathcal{Q}(b \mid \mathbf{r}; \mathbf{k})$ is proper in x_{r_s}. To do this we write $\mathcal{Q}(b \mid \mathbf{r}; \mathbf{k})$ as N/D, in which N (the "numerator") is

$$E_{\mathbf{r},\mathbf{k}}\left[\prod_{j=1}^{n}\left(\frac{x_j}{x_0}q\right)_{a_j} \cdot \prod_{\substack{1\le i,j\le n\\ j\ne i}}\left(\frac{x_i}{x_j}\,q^{\chi(i>j)}\right)_{a_i}\right],$$

where $\chi(S)$ is 1 if the statement S is true, and 0 otherwise, and D (the "denominator") is

$$E_{\mathbf{r},\mathbf{k}}\left[\prod_{j=1}^{n}\left(\frac{x_0}{x_jq^b}\right)_b \Big/ \prod_{i=1}^{s}\left(1-\frac{x_0}{x_{r_i}q^{k_i}}\right)\right].$$

Now let $R = \{r_0, r_1, \ldots, r_s\}$. Then the degree in x_s of

$$E_{\mathbf{r},\mathbf{k}}\left[\left(1-\frac{x_i}{x_j}q^l\right)\right]$$

is 1 if $i \in R$ and $j \notin R$, and is 0 otherwise, as is easily seen by checking the four cases. Thus the part of N contributing to the degree in x_{r_s} is

$$E_{\mathbf{r},\mathbf{k}}\left[\prod_{i=1}^{s}\prod_{j\ne r_0,\ldots,r_s}\left(\frac{x_{r_i}}{x_j}q^{\chi(r_i>j)}\right)_{a_{r_i}}\right],$$

which has degree $(n-s)(a_{r_1} + \cdots + a_{r_s})$, and the part of D contributing to the degree in x_{r_s} is

$$E_{\mathbf{r},\mathbf{k}}\left[\prod_{j\ne r_0,\ldots,r_s}\left(\frac{x_0}{x_jq^b}\right)_b\right],$$

which has degree $(n-s)b$.

Thus the total degree of $\mathcal{Q}(b \mid \mathbf{r}; \mathbf{k})$ in x_{r_s} is $(n-s)(a_{r_1} + \cdots + a_{r_s} - b) < 0$, so $\mathcal{Q}(b \mid \mathbf{r}; \mathbf{k})$ is proper in x_{r_s}.

Next we apply Lemma 4.1. For any rational function F of x_{r_s} and integers j and k, let $T_{j,k}F$ be the result of replacing x_{r_s} with $x_j q^{k-k_s}$ in F. Since $x_{r_s} q^{k_s}/(x_j q^k)$ is small when $j > r_s$ and is large when $j < r_s$, Lemma 4.1 gives

$$\operatorname*{CT}_{x_s} \mathcal{Q}(b \mid \mathbf{r}; \mathbf{k}) = \sum_{\substack{r_s < r_{s+1} \leq n \\ 1 \leq k_{s+1} \leq b}} T_{r_{s+1},k_{s+1}} \left[\mathcal{Q}(b \mid \mathbf{r}; \mathbf{k}) \left(1 - \frac{x_{r_s} q^{k_s}}{x_{r_{s+1}} q^{k_{s+1}}} \right) \right]. \tag{5.4}$$

We must show that the right-hand side of (5.4) is equal to the right-hand side of (5.3). Let us set $\mathbf{r}' = (r_1, \ldots, r_s, r_{s+1})$ and $\mathbf{k}' = (k_1, \ldots, k_s, k_{s+1})$. Then the equality follows easily from the identity

$$T_{r_{s+1},k_{s+1}} \circ E_{\mathbf{r},\mathbf{k}} = E_{\mathbf{r}',\mathbf{k}'}. \tag{5.5}$$

To see that (5.5) holds, we have

$$(T_{r_{s+1},k_{s+1}} \circ E_{\mathbf{r},\mathbf{k}})\, x_{r_i} = T_{r_{s+1},k_{s+1}} \left[x_{r_s} q^{k_s - k_i} \right] = x_{r_{s+1}} q^{k_{s+1} - k_i} = E_{\mathbf{r}',\mathbf{k}'}\, x_{r_i},$$

and if $j \notin \{r_0, \ldots, r_s\}$, then $(T_{r_{s+1},k_{s+1}} \circ E_{\mathbf{r},\mathbf{k}})\, x_j = x_j = E_{\mathbf{r}',\mathbf{k}'}\, x_j$. □

Proof of the Main Lemma. We prove by induction on $n - s$ that

$$\operatorname*{CT}_{\mathbf{x}} \mathcal{Q}(b \mid \mathbf{r}; \mathbf{k}) = 0;$$

the Main Lemma is the case $s = 0$. (Note that taking the constant term with respect to a variable that does not appear has no effect.)

We may assume that $s \leq n$ and $0 < r_1 < \cdots < r_s \leq n$, since otherwise $\mathcal{Q}(b \mid \mathbf{r}; \mathbf{k})$ is not defined. If $s = n$, then r_i must equal i for $i = 1, \ldots, n$ and thus $\mathcal{Q}(b \mid \mathbf{r}; \mathbf{k}) = \mathcal{Q}(b \mid 1, 2, \ldots, n; k_1, k_2, \ldots, k_n)$, which is 0 by part (i) of Lemma 5.1, since $k_i \leq b \leq a_1 + \cdots + a_n$ for each i.

Now suppose that $0 \leq s < n$. If part (i) of Lemma 5.1 applies, then $\mathcal{Q}(b|\mathbf{r}; \mathbf{k}) = 0$. Otherwise, part (ii) of Lemma 5.1 applies, and (5.3) holds. Applying $\mathrm{CT}_{\mathbf{x}}$ to both sides of (5.3) gives

$$\operatorname*{CT}_{\mathbf{x}} \mathcal{Q}(b \mid \mathbf{r}; \mathbf{k}) = \sum_{\substack{r_s < r_{s+1} \leq n \\ 1 \leq k_{s+1} \leq b}} \operatorname*{CT}_{\mathbf{x}} \mathcal{Q}(b \mid r_1, \ldots, r_s, r_{s+1}; k_1, \ldots, k_s, k_{s+1}),$$

and by induction, every term on the right is 0. □

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